Computing Subschemes of the Border Basis Scheme

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This is joint work with

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Lorenzo Robbiano (University of Genova)
1 – Border Bases

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$K$ field

$P = K[x_1, \ldots, x_n]$ polynomial ring
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\( I \subset P \) 0-dimensional ideal (i.e. \( \dim_K(P/I) < \infty \))

\( X = \text{Spec}(P/I) \) 0-dimensional subscheme of \( \mathbb{A}^n \) of length

\( \mu = \dim_K(P/I) \)
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\[ \mu = \dim_K(P/I) \]
\[ R = P/I \text{ affine coordinate ring of } X \]
Definition 1.1 (a) A divisor closed set of terms $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ is called an order ideal of terms.
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(b) The border of $\mathcal{O}$ is $\partial \mathcal{O} = (x_1 \mathcal{O} \cup \cdots \cup x_n \mathcal{O}) \setminus \mathcal{O}$. We write $\partial \mathcal{O} = \{ b_1, \ldots, b_\nu \}$. 
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(c) Let $\gamma_{ij} \in K$. Then the set $G = \{g_1, \ldots, g_\nu\}$ such that $g_j = b_j - \sum_{i=1}^{\mu} \gamma_{ij} t_i$ is called an $\mathcal{O}$-border prebasis.
Definition 1.1  (a) A divisor closed set of terms $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ is called an **order ideal** of terms.

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(c) Let $\gamma_{ij} \in K$. Then the set $G = \{g_1, \ldots, g_\nu\}$ such that $g_j = b_j - \sum_{i=1}^{\mu} \gamma_{ij} t_i$ is called an $\mathcal{O}$-**border prebasis**.

(d) An $\mathcal{O}$-border prebasis $G$ is called an $\mathcal{O}$-**border basis** of $I$ if $I = \langle G \rangle$ and if $\mathcal{O}$ represents a $K$-basis of $R = P/I$. 
Picture of an order ideal and its border
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- term in the order ideal
- term in the border
Definition 1.2 Let $G$ be an $\mathcal{O}$-border prebasis as above. For $r = 1, \ldots, n$, the matrix $A_r = (a_{ij}^{(r)}) \in \text{Mat}_\mu(K)$, where

$$a_{ij}^{(r)} = \begin{cases} 
\delta_{im} & \text{if } x_r t_j = t_m \\
\gamma_{im} & \text{if } x_r t_j = b_m 
\end{cases}$$

is called the $r$-th formal multiplication matrix for $G$. 
**Definition 1.2** Let $G$ be an $O$-border prebasis as above. For $r = 1, \ldots, n$, the matrix $A_r = (a_{ij}^{(r)}) \in \text{Mat}_\mu(K)$, where

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**Theorem 1.3 (Mourrain)**

An $O$-border prebasis $G \subset I$ is an $O$-border basis of $I$ if and only if the formal multiplication matrices commute, i.e. if and only if $A_i A_j - A_j A_i = 0$ for $1 \leq i < j \leq n$. 
2 – Border Basis Schemes
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**Definition 2.1** Let \( c_{ij} \) be indeterminates. Then the set \( G = \{g_1, \ldots, g_\nu\} \) such that \( g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i \) is called the **generic** \( \mathcal{O} \)-border prebasis.
Definition 2.2 (a) For $r = 1, \ldots, n$, the matrix $A_r = (a_{ij}^{(r)}) \in \text{Mat}_\mu(K[c_{ij}])$, where

$$a_{ij}^{(r)} = \begin{cases} 
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Definition 2.2 (a) For $r = 1, \ldots, n$, the matrix
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where
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c_{im} & \text{if } x_r t_j = b_m 
\end{cases} \]
is called the $r$-th generic multiplication matrix for $\mathcal{O}$.

(b) Consider the ideal in $K[c_{ij}]$ which is generated by all entries of
the commutator matrices $A_r A_s - A_s A_r$ with $1 \leq r < s \leq n$. Then
the subscheme of $\mathbb{A}^{\mu\nu}_K = \text{Spec}(K[c_{ij}])$ defined by this ideal is called
the $\mathcal{O}$-border basis scheme. It is denoted by $\mathcal{B}_\mathcal{O}$, its vanishing ideal is denoted by $I(\mathcal{B}_\mathcal{O})$, and its affine coordinate ring is denoted by
$B_\mathcal{O} = K[c_{11}, \ldots, c_{\mu\nu}] / I(\mathcal{B}_\mathcal{O})$. 
Example 2.3  Let $\mathcal{O} = \{1, x, y, xy\} \subseteq \mathbb{T}^2$. Then we have

$$A_x = \begin{pmatrix}
0 & c_{12} & 0 & c_{14} \\
1 & c_{22} & 0 & c_{24} \\
0 & c_{32} & 0 & c_{34} \\
0 & c_{42} & 1 & c_{44}
\end{pmatrix} \quad \text{and} \quad A_y = \begin{pmatrix}
0 & 0 & c_{11} & c_{13} \\
0 & 0 & c_{21} & c_{23} \\
1 & 0 & c_{31} & c_{33} \\
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and the defining ideal of $\mathbb{B}_\mathcal{O}$ is generated by

$$\{ \begin{array}{l} c_{11}c_{32} + c_{13}c_{42} - c_{14}, \\ c_{21}c_{32} + c_{23}c_{42} - c_{24}, \\ c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23}, \\ c_{31}c_{32} + c_{33}c_{42} + c_{12} - c_{34}, \\ c_{21}c_{32} + c_{34}c_{41} - c_{33}, \\ c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}, \end{array} \quad \begin{array}{l} c_{12}c_{21} + c_{14}c_{41} - c_{13}, \\ c_{12}c_{23} - c_{11}c_{34} + c_{14}c_{43} - c_{13}c_{44}, \\ c_{23}c_{32} - c_{31}c_{34} + c_{34}c_{43} - c_{33}c_{44} - c_{14}, \\ c_{22}c_{23} - c_{21}c_{34} + c_{24}c_{43} - c_{23}c_{44} + c_{13}, \\ c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}, \\ c_{34}c_{41} - c_{23}c_{42} + c_{24} - c_{33} \end{array} \}$$
Remark 2.4 (a) The border basis scheme is an open subscheme of the Hilbert scheme $\text{Hilb}^\mu(\mathbb{A}^n)$ parametrizing all 0-dimensional subschemes of $\mathbb{A}^n$ of length $\mu$. 
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(b) Note that its affine coordinate ring is given by easily computable quadratic equations.
**Remark 2.4 (a)** The border basis scheme is an open subscheme of the Hilbert scheme $\text{Hilb}^\mu(\mathbb{A}^n)$ parametrizing all 0-dimensional subschemes of $\mathbb{A}^n$ of length $\mu$.

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**(c)** The various border basis schemes for order ideals with $\mu$ elements cover the Hilbert scheme.
Remark 2.4  (a) The border basis scheme is an open subscheme of the Hilbert scheme \( \text{Hilb}^\mu(\mathbb{A}^n) \) parametrizing all 0-dimensional subschemes of \( \mathbb{A}^n \) of length \( \mu \).

(b) Note that its affine coordinate ring is given by easily computable quadratic equations.

(c) The various border basis schemes for order ideals with \( \mu \) elements cover the Hilbert scheme.

Idea: Using the generic multiplication matrices and the algorithms given in the preceding talk, we can calculate sets of equations which define subschemes of \( \mathcal{B}_\mathcal{O} \) parametrizing 0-dimensional schemes having certain special properties such as Gorenstein schemes, CBP, strict Gorenstein schemes, strict complete intersections, etc.
3 – Computing the Locally Gorenstein Locus

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\[ \mathcal{O} = \{ t_1, \ldots, t_\mu \} \] order ideal in \( \mathbb{T}^n \)

**Definition 3.1** The set of all \( K \)-rational points \( \Gamma = (\gamma_{ij}) \in K^{\mu \nu} \) of the border basis scheme \( \mathbb{B}_\mathcal{O} \) whose associated 0-dimensional scheme \( \mathbb{X}_\Gamma \) is locally Gorenstein is called the **locally Gorenstein locus** of \( \mathbb{B}_\mathcal{O} \) and is denoted by \( \text{L Gor}(\mathcal{O}) \).
Algorithm 3.2 (The Non-Locally Gorenstein Locus in $\mathbb{B}_\mathcal{O}$)

The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathbb{B}_\mathcal{O}$.
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The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathcal{B}_\mathcal{O}$.

(1) Determine the generic multiplication matrices $A_1, \ldots, A_n$ for $\mathcal{O}$.
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The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathbb{B}_\mathcal{O}$.

(1) Determine the generic multiplication matrices $A_1, \ldots, A_n$ for $\mathcal{O}$.

(2) Calculate the commutators $A_r A_s - A_s A_r$ for $1 \leq r < s \leq n$ and form the ideal $I(\mathbb{B}_\mathcal{O})$ in $K[c_{ij}]$ generated by their entries.
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(3) Introduce new indeterminates $z_1, \ldots, z_\mu$ and construct the matrix $C$ in $\text{Mat}_\mu(K[c_{ij}][z_1, \ldots, z_\mu])$ whose $i$-th column is given by $t_i(A_{1 \text{ tr}}, \ldots, A_{n \text{ tr}}) \cdot (z_1, \ldots, z_\mu)^{\text{tr}}$. 
Algorithm 3.2 (The Non-locally Gorenstein Locus in $\mathbb{B}_O$)
The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathbb{B}_O$.

1. Determine the generic multiplication matrices $A_1, \ldots, A_n$ for $O$.
2. Calculate the commutators $A_r A_s - A_s A_r$ for $1 \leq r < s \leq n$ and form the ideal $I(\mathbb{B}_O)$ in $K[c_{ij}]$ generated by their entries.
3. Introduce new indeterminates $z_1, \ldots, z_\mu$ and construct the matrix $C$ in $\text{Mat}_\mu(K[c_{ij}][z_1, \ldots, z_\mu])$ whose $i$-th column is given by $t_i(A_1^{tr}, \ldots, A_n^{tr}) \cdot (z_1, \ldots, z_\mu)^{tr}$.
4. Compute $\det(C)$ in $K[c_{ij}][z_1, \ldots, z_\mu]$, and let $J$ be the ideal in $K[c_{ij}]$ generated by the coefficients of $\det(C)$ w.r.t. $z_1, \ldots, z_\mu$. 
Algorithm 3.2 (The Non-Locally Gorenstein Locus in $\mathbb{B}_\mathcal{O}$)

The following steps compute an ideal in $K[c_{ij}]$ which defines the complement of the locally Gorenstein locus in $\mathbb{B}_\mathcal{O}$.

(1) Determine the generic multiplication matrices $A_1, \ldots, A_n$ for $\mathcal{O}$.

(2) Calculate the commutators $A_r A_s - A_s A_r$ for $1 \leq r < s \leq n$ and form the ideal $I(\mathbb{B}_\mathcal{O})$ in $K[c_{ij}]$ generated by their entries.

(3) Introduce new indeterminates $z_1, \ldots, z_\mu$ and construct the matrix $C$ in $\text{Mat}_\mu(K[c_{ij}][z_1, \ldots, z_\mu])$ whose $i$-th column is given by $t_i(A_1^{tr}, \ldots, A_n^{tr}) \cdot (z_1, \ldots, z_\mu)^{tr}$.

(4) Compute $\det(C)$ in $K[c_{ij}][z_1, \ldots, z_\mu]$, and let $J$ be the ideal in $K[c_{ij}]$ generated by the coefficients of $\det(C)$ w.r.t. $z_1, \ldots, z_\mu$.

(5) Return the ideal $I(\mathbb{B}_\mathcal{O}) + J$. 
Example 3.3 Let us compute the locally Gorenstein locus of $\mathbb{B}_\mathcal{O}$ in the above example $\mathcal{O} = \{1, x, y, xy\}$. 
Example 3.3 Let us compute the locally Gorenstein locus of $B_\mathcal{O}$ in the above example $\mathcal{O} = \{1, x, y, xy\}$.

Let $Z = (z_1, z_2, z_3, z_4)^{tr}$ and form the matrix $C = (Z, A_x Z, A_y Z, A_x A_y Z)$. Its four columns are

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}Z,
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}Z,
\begin{pmatrix}
p_1 & p_2 & p_3 & p_4 \\
q_1 & q_2 & q_3 & q_4 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}Z
\]

where $p_1 = c_{11}c_{32} + c_{13}c_{42}$, $p_2 = c_{21}c_{32} + c_{23}c_{42}$,

$p_3 = c_{31}c_{32} + c_{33}c_{42} + c_{12}$, $p_4 = c_{32}c_{41} + c_{42}c_{43} + c_{22}$,

$q_1 = c_{11}c_{34} + c_{13}c_{44}$, $q_2 = c_{21}c_{34} + c_{23}c_{44}$, $q_3 = c_{31}c_{34} + c_{33}c_{44} + c_{14}$, and $q_4 = c_{34}c_{41} + c_{43}c_{44} + c_{24}$. 
The determinant of $C$ is a polynomial

$$\det(C) = (-c_{11}^2 c_{14} c_{32} + c_{11}^2 c_{12} c_{34} - c_{11} c_{13} c_{41} c_{42} + c_{11} c_{12} c_{13} c_{44}$$

$$- c_{12} c_{13}^2) z_1^4 + \cdots + (-c_{41} c_{42} + 1) z_4^4$$

in $K[c_{ij}][z_1, z_2, z_3, z_4]$ which is homogeneous of degree 4 with respect to $z_1, \ldots, z_4$ and has 35 non-zero coefficients in $K[c_{ij}]$. Let $J$ be the ideal generated by these coefficients. Then the Non-Locally Gorenstein Locus NonLGor($\mathcal{O}$) is defined by the ideal $I(\mathcal{B}_\mathcal{O}) + J$.

Via the isomorphism $\mathcal{B}_\mathcal{O} \cong \tilde{\mathcal{P}} = K[c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}]$, we can examine NonLGor($\mathcal{O}$) further. Let $\tilde{J}$ be the image of $J$ in $\tilde{\mathcal{P}}$. Then we can compute a Gröbner basis of $\tilde{J}$ and check that $\dim(\tilde{\mathcal{P}}/\tilde{J}) = 4$. Hence NonLGor($\mathcal{O}$) is the set of closed points of a 4-dimensional closed subscheme of $\mathcal{B}_\mathcal{O} \cong \mathbb{A}^8$. 
4 – The Degree Filtered Border Basis Scheme
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Recall that the degree filtration of $R = P/I$ is given by $F_i R = P_{\leq i} / (I \cap P_{\leq i})$ for $i \in \mathbb{Z}$.

**Definition 4.1 (a)** A tuple $B = (\bar{t}_1, \ldots, \bar{t}_\mu) \in R^\mu$ is called a degree filtered $K$-basis of $R$ if the set $B \cap F_i R$ is a $K$-basis of $F_i R$ for every $i \in \mathbb{Z}$ and if $\text{ord}(\bar{t}_1) \leq \cdots \leq \text{ord}(\bar{t}_\mu)$. 
The first five days after the weekend are always the worst.

Recall that the degree filtration of \( R = P/I \) is given by
\[
F_i R = P_{\leq i}/(I \cap P_{\leq i}) \quad \text{for} \quad i \in \mathbb{Z}.
\]

**Definition 4.1**

(a) A tuple \( B = (\bar{t}_1, \ldots, \bar{t}_\mu) \in R^\mu \) is called a degree filtered \( K \)-basis of \( R \) if the set \( B \cap F_i R \) is a \( K \)-basis of \( F_i R \) for every \( i \in \mathbb{Z} \) and if \( \text{ord}(\bar{t}_1) \leq \cdots \leq \text{ord}(\bar{t}_\mu) \).

(b) We say that \( I \) has a degree filtered \( \mathcal{O} \)-border basis if \( \overline{\mathcal{O}} \) is a degree filtered \( K \)-basis of \( R \).
Proposition 4.2  For a $K$-rational point $\Gamma = (\gamma_{ij})$ of $\mathcal{B}_O$, the 0-dimensional scheme $\mathbb{X}_\Gamma$ associated to $\Gamma$ has a degree filtered $\mathcal{O}$-border basis if and only if $\gamma_{ij} = 0$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ such that $\deg(t_i) > \deg(b_j)$. 
Proposition 4.2  For a $K$-rational point $\Gamma = (\gamma_{ij})$ of $B_\mathcal{O}$, the 0-dimensional scheme $X_{\Gamma}$ associated to $\Gamma$ has a degree filtered $\mathcal{O}$-border basis if and only if $\gamma_{ij} = 0$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ such that $\deg(t_i) > \deg(b_j)$.

Definition 4.3  Let $I_{\mathcal{O}}^{\text{df}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$. 
Proposition 4.2 For a $K$-rational point $\Gamma = (\gamma_{ij})$ of $\mathbb{B}_\mathcal{O}$, the $0$-dimensional scheme $X_\Gamma$ associated to $\Gamma$ has a degree filtered $\mathcal{O}$-border basis if and only if $\gamma_{ij} = 0$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ such that $\text{deg}(t_i) > \text{deg}(b_j)$.

Definition 4.3 Let $I^\text{df}_\mathcal{O}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\text{deg}(t_i) > \text{deg}(b_j)$.

(a) The closed subscheme $\mathbb{B}^\text{df}_\mathcal{O}$ of $\mathbb{B}_\mathcal{O}$ defined by $I(\mathbb{B}^\text{df}_\mathcal{O}) = I(\mathbb{B}_\mathcal{O}) + I^\text{df}_\mathcal{O}$ is called the degree filtered $\mathcal{O}$-border basis scheme. Its affine coordinate ring is denoted by $B^\text{df}_\mathcal{O} = K[c_{ij}]/I(\mathbb{B}^\text{df}_\mathcal{O})$. 
Proposition 4.2 For a $K$-rational point $\Gamma = (\gamma_{ij})$ of $\mathbb{B}_\mathcal{O}$, the 0-dimensional scheme $\mathbb{X}_\Gamma$ associated to $\Gamma$ has a degree filtered $\mathcal{O}$-border basis if and only if $\gamma_{ij} = 0$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ such that $\deg(t_i) > \deg(b_j)$.

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(b) The set of polynomials $G^\text{df} = \{g_1^\text{df}, \ldots, g_\nu^\text{df}\}$ in $K[c_{ij}][x_1, \ldots, x_n]$ given by $g_j = b_j - \sum_{i: \deg(t_i) \leq \deg(b_j)} c_{ij} t_i$ is called the generic degree filtered $\mathcal{O}$-border prebasis.
Remark 4.4 (Some Properties of $\mathbb{B}_{\mathcal{O}}^{df}$)
Let $C^{\text{mondf}}$ be the set of all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$. 
Remark 4.4 (Some Properties of $\mathbb{B}_O^{df}$)

Let $C_{nondf}$ be the set of all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$.

(a) For $k = 1, \ldots, n$, let $A_k^{df}$ be the matrix obtained from $A_k$ by setting all indeterminates in $C_{nondf}$ equal to zero. Then the matrices $A_1^{df}, \ldots, A_n^{df}$ are called the generic degree filtered multiplication matrices with respect to $O$. 

17-a
Remark 4.4 (Some Properties of $\mathbb{B}^\text{df}_\mathcal{O}$)

Let $C^\text{nondf}$ be the set of all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$.

(a) For $k = 1, \ldots, n$, let $\mathcal{A}^\text{df}_k$ be the matrix obtained from $\mathcal{A}_k$ by setting all indeterminates in $C^\text{nondf}$ equal to zero. Then the matrices $\mathcal{A}^\text{df}_1, \ldots, \mathcal{A}^\text{df}_n$ are called the generic degree filtered multiplication matrices with respect to $\mathcal{O}$.

(b) When we set the indeterminates in $C^\text{nondf}$ equal to zero in $I(\mathbb{B}_\mathcal{O})$, we get an ideal $\bar{I}(\mathbb{B}^\text{df}_\mathcal{O})$ such that $B^\text{df}_\mathcal{O} \cong K[C^\text{df}] / \bar{I}(\mathbb{B}^\text{df}_\mathcal{O})$. 
Remark 4.4 (Some Properties of $\mathbb{B}^{df}_\mathcal{O}$)

Let $C^{\text{nondf}}$ be the set of all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$.

(a) For $k = 1, \ldots, n$, let $A^{\text{df}}_k$ be the matrix obtained from $A_k$ by setting all indeterminates in $C^{\text{nondf}}$ equal to zero. Then the matrices $A^{\text{df}}_1, \ldots, A^{\text{df}}_n$ are called the generic degree filtered multiplication matrices with respect to $\mathcal{O}$.

(b) When we set the indeterminates in $C^{\text{nondf}}$ equal to zero in $I(\mathbb{B}_\mathcal{O})$, we get an ideal $\bar{I}(\mathbb{B}^{df}_\mathcal{O})$ such that $B^{\text{df}}_\mathcal{O} \cong K[C^{\text{df}}]/\bar{I}(\mathbb{B}^{df}_\mathcal{O})$.

(c) If $\mathcal{O}$ has a generic Hilbert function then $\mathbb{B}_\mathcal{O} = \mathbb{B}^{df}_\mathcal{O}$. 
5 – Computing the Cayley-Bacharach Locus

Great! It is summer!
Finally I can wear short trousers
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Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal in $\mathbb{T}^n$.

**Definition 5.1** The set of all $K$-rational points $\Gamma = (\gamma_{ij}) \in K^{\mu \nu}$ of the border basis scheme $\mathbb{B}_\mathcal{O}$ whose associated 0-dimensional scheme $X_\Gamma$ is a Cayley-Bacharach scheme is called the **Cayley-Bacharach locus** of $\mathbb{B}_\mathcal{O}$ and is denoted by $\text{CB}(\mathcal{O})$. 
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**Goal:** Calculate the Cayley-Bacharach locus in $\mathbb{B}^{\text{df}}_\mathcal{O}$, i.e. the equations defining $\text{CB}(\mathcal{O}) \cap \mathbb{B}^{\text{df}}_\mathcal{O}$. 
Algorithm 5.2 (The Cayley-Bacharach Locus in $\mathbb{B}_\mathcal{O}^{df}$)
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \# \{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(\mathcal{O}) \cap \mathbb{B}_\mathcal{O}^{df}$. 

Algorithm 5.2 (The Cayley-Bacharach Locus in $\mathbb{B}_O^{df}$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \#\{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(\mathcal{O}) \cap \mathbb{B}_O^{df}$.

(1) As above, calculate $I(\mathbb{B}_O^{df}) = I(\mathbb{B}_O) + I_O^{df}$. 
Algorithm 5.2 (The Cayley-Bacharach Locus in $B^d_O$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \# \{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(\mathcal{O}) \cap B^d_O$.

(1) As above, calculate $I(B^d_O) = I(B_O) + I^d_O$.

(2) Form the generic multiplication matrices $A_1, \ldots, A_n$. For $i = 1, \ldots, \mu$, compute the multiplication matrix $M_{t_i} = t_i(A_1, \ldots, A_n)$. 
Algorithm 5.2 (The Cayley-Bacharach Locus in $\mathcal{B}^{df}_\mathcal{O}$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \#\{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(\mathcal{O}) \cap \mathcal{B}^{df}_\mathcal{O}$.

1. As above, calculate $I(\mathcal{B}^{df}_\mathcal{O}) = I(\mathcal{B}_\mathcal{O}) + I^{df}_\mathcal{O}$.

2. Form the generic multiplication matrices $A_1, \ldots, A_\mu$. For $i = 1, \ldots, \mu$, compute the multiplication matrix $M_{t_i} = t_i(A_1, \ldots, A_\mu)$.

3. For $j = 1, \ldots, \Delta$, form the matrix $V_j \in \text{Mat}_\mu(K[c_{ij}])$ whose $i$-th column is the $(\mu - \Delta + j)$-th column of $M_{t_i}^{tr}$ for $i = 1, \ldots, \mu$. 
Algorithm 5.2 (The Cayley-Bacharach Locus in $B^d_O$)
Let $O = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \#\{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(O) \cap B^d_O$.

(1) As above, calculate $I(B^d_O) = I(B_O) + I^d_O$.

(2) Form the generic multiplication matrices $A_1, \ldots, A_n$. For $i = 1, \ldots, \mu$, compute the multiplication matrix $M_{t_i} = t_i(A_1, \ldots, A_n)$.

(3) For $j = 1, \ldots, \Delta$, form the matrix $V_j \in \text{Mat}_\mu(K[c_{ij}])$ whose $i$-th column is the $(\mu - \Delta + j)$-th column of $M_{t_i}^{\text{tr}}$ for $i = 1, \ldots, \mu$.

(4) Form the block column matrix $W = \text{Col}(V_1, \ldots, V_\Delta)$ and compute the ideal $J$ generated by the maximal minors of $W$. 
Algorithm 5.2 (The Cayley-Bacharach Locus in $\mathbb{B}^\text{df}_\mathcal{O}$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\Delta = \#\{i \in \{1, \ldots, \mu\} \mid \deg(t_i) = \deg(t_\mu)\}$. The following algorithm computes the vanishing ideal of $\text{NonCB}(\mathcal{O}) \cap \mathbb{B}^\text{df}_\mathcal{O}$.

(1) As above, calculate $I(\mathbb{B}^\text{df}_\mathcal{O}) = I(\mathbb{B}_\mathcal{O}) + I^\text{df}_\mathcal{O}$.

(2) Form the generic multiplication matrices $A_1, \ldots, A_\mu$. For $i = 1, \ldots, \mu$, compute the multiplication matrix $M_{t_i} = t_i(A_1, \ldots, A_\mu)$.

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(4) Form the block column matrix $W = \text{Col}(V_1, \ldots, V_\Delta)$ and compute the ideal $J$ generated by the maximal minors of $W$.

(5) Return the ideal $I(\mathbb{B}^\text{df}_\mathcal{O}) + J$. 
The trouble with socialism is that eventually
The trouble with socialism is that eventually you run out of other people’s money.

(Margaret Thatcher)
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Definition 6.1 Let $X$ be a 0-dimensional subscheme of $\mathbb{A}^n$. The scheme $X$ is called a \textbf{strict complete intersection scheme} if the associated graded ring $\text{gr}_F(R_X) \cong P/DF(I)$ is a (local) complete intersection.
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**Definition 6.1** Let $X$ be a 0-dimensional subscheme of $\mathbb{A}^n$. The scheme $X$ is called a **strict complete intersection scheme** if the associated graded ring $\text{gr}_F(R_X) \cong P/DF(I)$ is a (local) complete intersection.

**Idea:** The rings $P/DF(I)$ are parametrized by the **homogeneous border basis scheme**. Apply the characterization of local complete intersections to this family.
Definition 6.2 Let $I^\text{hom}_\mathcal{O}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\text{deg}(t_i) \neq \text{deg}(b_j)$.
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(a) The closed subscheme $\mathbb{B}^\text{hom}_\mathcal{O}$ of $\mathbb{B}_\mathcal{O}$ defined by $I(\mathbb{B}^\text{hom}_\mathcal{O}) = I(\mathbb{B}_\mathcal{O}) + I^\text{hom}_\mathcal{O}$ is called the homogeneous $\mathcal{O}$-border basis scheme. Its affine coordinate ring is $B^\text{hom}_\mathcal{O} = K[c_{ij}]/I(\mathbb{B}^\text{hom}_\mathcal{O})$. 
Definition 6.2 Let $I_{\mathcal{O}}^{\text{hom}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) \neq \deg(b_j)$.

(a) The closed subscheme $\mathbb{B}_{\mathcal{O}}^{\text{hom}}$ of $\mathbb{B}_{\mathcal{O}}$ defined by $I(\mathbb{B}_{\mathcal{O}}^{\text{hom}}) = I(\mathbb{B}_{\mathcal{O}}) + I_{\mathcal{O}}^{\text{hom}}$ is called the homogeneous $\mathcal{O}$-border basis scheme. Its affine coordinate ring is $B_{\mathcal{O}}^{\text{hom}} = K[c_{ij}]/I(\mathbb{B}_{\mathcal{O}}^{\text{hom}})$.

(b) The set of polynomials $G_{\mathcal{O}}^{\text{hom}} = \{g_{1}^{\text{hom}}, \ldots, g_{\nu}^{\text{hom}}\}$ in $K[c_{ij}][x_1, \ldots, x_n]$ given by $g_{j}^{\text{hom}} = b_j - \sum_{\{i | \deg(t_i) = \deg(b_j)\}} c_{ij} t_i$ is called the generic homogeneous $\mathcal{O}$-border prebasis.
Definition 6.2 Let $I^\text{hom}_\mathcal{O}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) \neq \deg(b_j)$.

(a) The closed subscheme $\mathbb{B}_\mathcal{O}^\text{hom}$ of $\mathbb{B}_\mathcal{O}$ defined by $I(\mathbb{B}_\mathcal{O}^\text{hom}) = I(\mathbb{B}_\mathcal{O}) + I^\text{hom}_\mathcal{O}$ is called the homogeneous $\mathcal{O}$-border basis scheme. Its affine coordinate ring is $B^\text{hom}_\mathcal{O} = K[c_{ij}]/I(\mathbb{B}_\mathcal{O}^\text{hom})$.

(b) The set of polynomials $G^\text{hom} = \{g^\text{hom}_1, \ldots, g^\text{hom}_\nu\}$ in $K[c_{ij}][x_1, \ldots, x_n]$ given by $g^\text{hom}_j = b_j - \sum\{i|\deg(t_i) = \deg(b_j)\} c_{ij} t_i$ is called the generic homogeneous $\mathcal{O}$-border prebasis.

(c) Let $C^\text{hom}$ be the set of all $c_{ij}$ such that $\deg(t_i) \neq \deg(b_j)$. For $k = 1, \ldots, n$, let $A^\text{hom}_k$ be the matrix obtained from $A_k$ by setting all indeterminates in $C^\text{hom}$ equal to zero. Then the matrices $A^\text{hom}_1, \ldots, A^\text{hom}_n$ are called the generic homogeneous multiplication matrices with respect to $\mathcal{O}$. 
Algorithm 6.3 (Computing the Strict CI Locus in $\mathbb{B}^d_\mathcal{O}$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\varrho = \deg(t_\mu)$. Consider the following sequence of instructions.
Algorithm 6.3 (Computing the Strict CI Locus in $\mathbb{B}_{\mathcal{O}}^{df}$)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal with $\deg(t_1) \leq \cdots \leq \deg(t_\mu)$, and let $\varrho = \deg(t_\mu)$. Consider the following sequence of instructions.

(1) For $i = 1, \ldots, \varrho$, determine the number $h_i = \#\{t_j \in \mathcal{O} \mid \deg(t_j) = i\}$. If the tuple $(h_0, \ldots, h_\varrho)$ is not symmetric, then return the ideal $\langle 1 \rangle$ and stop.
Algorithm 6.3 (Computing the Strict CI Locus in $\mathbb{B}^{\text{df}}_{\mathcal{O}}$)

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2. Let $I_{\mathcal{O}}^{\text{df}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$, and let $I(\mathbb{B}^{\text{df}}_{\mathcal{O}}) = I(\mathbb{B}_{\mathcal{O}}) + I_{\mathcal{O}}^{\text{df}}$
Algorithm 6.3 *(Computing the Strict CI Locus in $B^d_O$)*

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$h_i = \# \{t_j \in \mathcal{O} \mid \deg(t_j) = i\}$. If the tuple $(h_0, \ldots, h_\varrho)$ is not symmetric, then return the ideal $\langle 1 \rangle$ and stop.

**(2)** Let $I^d_{\mathcal{O}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$, and let $I(B^d_{\mathcal{O}}) = I(B_{\mathcal{O}}) + I^d_{\mathcal{O}}$

**(3)** Form the generic homogeneous $\mathcal{O}$-border prebasis

$G^\text{hom} = \{g^\text{hom}_1, \ldots, g^\text{hom}_j\}$ and write $g^\text{hom}_j = \sum_{i=1}^n h_{ij} x_i$ with $h_{ij} \in K[c_{ij}][x_1, \ldots, x_n]$ for $j = 1, \ldots, \nu$. 

Algorithm 6.3 (Computing the Strict CI Locus in $\mathcal{B}^{\text{df}}_{\mathcal{O}}$)

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(2) Let $I^{\text{df}}_{\mathcal{O}}$ be the ideal in $K[c_{ij}]$ generated by all indeterminates $c_{ij}$ such that $\deg(t_i) > \deg(b_j)$, and let $I(\mathcal{B}^{\text{df}}_{\mathcal{O}}) = I(\mathcal{B}_{\mathcal{O}}) + I^{\text{df}}_{\mathcal{O}}$.

(3) Form the generic homogeneous $\mathcal{O}$-border prebasis $G_{\text{hom}} = \{g_{1\text{hom}}, \ldots, g_{j\text{hom}}\}$ and write $g_{j\text{hom}} = \sum_{i=1}^{n} h_{ij} x_i$ with $h_{ij} \in K[c_{ij}][x_1, \ldots, x_n]$ for $j = 1, \ldots, \nu$.

(4) Form the matrix $W$ of size $n \times \nu$ whose columns are given by $\sum_{i=1}^{n} h_{ij} e_i$ for $j = 1, \ldots, \nu$. 
(5) Let $k = \binom{\nu}{n}$. Calculate the minors $f_1, \ldots, f_k$ of order $n$ of $W$. 
(5) Let \( k = \binom{\nu}{n} \). Calculate the minors \( f_1, \ldots, f_k \) of order \( n \) of \( W \).

(6) Using border division by \( G^{\text{hom}} \), write the residue classes \( \bar{f}_1, \ldots, \bar{f}_k \in B^{\text{hom}}_\mathcal{O}/\langle G^{\text{hom}} \rangle \) as \( B^{\text{hom}}_\mathcal{O} \)-linear combinations

\[
\bar{f}_j = \sum_{i=1}^{\mu} \bar{a}_{ij}t_i \quad \text{with} \quad \bar{a}_{1j}, \ldots, \bar{a}_{\mu j} \in B^{\text{hom}}_\mathcal{O} \quad \text{for} \quad j = 1, \ldots, k.
\]
(5) Let $k = \binom{\nu}{n}$. Calculate the minors $f_1, \ldots, f_k$ of order $n$ of $W$.

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\[ \bar{f}_1, \ldots, \bar{f}_k \in B^{\text{hom}}_{O}/\langle G^{\text{hom}} \rangle \]
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\[ \bar{f}_j = \sum_{i=1}^\mu \bar{a}_{ij} t_i \]
with $\bar{a}_{1j}, \ldots, \bar{a}_{\mu j} \in B^{\text{hom}}_{O}$ for $j = 1, \ldots, k$.

(7) Let $C^{\text{hom}} = \{ c_{ij} \mid \deg(t_i) = \deg(b_j) \}$. For $i = 1, \ldots, \mu$ and
\[ j = 1, \ldots, k, \]
let $a_{ij} \in K[C^{\text{hom}}]$ be a polynomial which represents the
\[ \bar{a}_{ij} \]
with respect to $B^{\text{hom}}_{O} \cong K[C^{\text{hom}}]/I(B^{\text{hom}}_{O})$. Return the ideal
\[ J = I(B^{\text{df}}_{O}) + \langle a_{ij} \mid i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, k\} \rangle \]
and stop.
(5) Let $k = \binom{n}{\nu}$. Calculate the minors $f_1, \ldots, f_k$ of order $n$ of $W$.

(6) Using border division by $G^\text{hom}$, write the residue classes $ar{f}_1, \ldots, \bar{f}_k \in B^\text{hom}_O / \langle G^\text{hom} \rangle$ as $B^\text{hom}_O$-linear combinations

$$
\bar{f}_j = \sum_{i=1}^{\mu} \bar{a}_{ij} t_i \text{ with } \bar{a}_{1j}, \ldots, \bar{a}_{\mu j} \in B^\text{hom}_O \text{ for } j = 1, \ldots, k.
$$

(7) Let $C^\text{hom} = \{ c_{ij} \mid \deg(t_i) = \deg(b_j) \}$. For $i = 1, \ldots, \mu$ and $j = 1, \ldots, k$, let $a_{ij} \in K[C^\text{hom}]$ be a polynomial which represents the $\bar{a}_{ij}$ with respect to $B^\text{hom}_O \cong K[C^\text{hom}] / \bar{I}(B^\text{hom}_O)$. Return the ideal

$$
J = I(B^\text{df}_O) + \langle a_{ij} \mid i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, k\} \rangle
$$

and stop.

This is an algorithm which computes an ideal $J$ in the ring $K[c_{ij}]$ which defines a closed subscheme $\text{NonSCI}(O) \cap B^\text{df}_O$. The $K$-rational points of this subscheme represent the 0-dimensional subschemes of $\mathbb{A}^n$ which have a degree filtered $O$-border basis, but are not strict complete intersection schemes.
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- strict Cayley-Bacharach schemes
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Outlook

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- The closed subscheme $B_\mathcal{O}(\overline{H})$ of $B_\mathcal{O}$ corresponds to all schemes whose Hilbert function is dominated by a fixed Hilbert function $\mathcal{H}$. 
Outlook

(1) There are many other loci in the border bases scheme which we can describe explicitly, e.g.

• strict Cayley-Bacharach schemes
• strict Gorenstein schemes
• locally Gorenstein Cayley-Bacharach schemes

(2) Many properties require us to fix the (affine) Hilbert function.

• The closed subscheme $B_\mathcal{O}(\overline{H})$ of $B_\mathcal{O}$ corresponds to all schemes whose Hilbert function is dominated by a fixed Hilbert function $H$.
• Its open subset $B_\mathcal{O}(H)$ corresponds to all schemes whose Hilbert function is $H$. 
• The defining equations of $\mathcal{B}_\mathcal{O}(\mathcal{H})$ can be computed.
• The defining equations of $\mathcal{B}_\mathcal{O}(\overline{\mathcal{H}})$ can be computed.

(3) The various Hilbert function subschemes of $\mathcal{B}_\mathcal{O}$ form a tree at whose root lies $\mathcal{B}_\mathcal{O}^{\text{df}}$ and whose unique leaf is the subscheme corresponding to $\mathcal{H} : 1 2 \cdots \mu \mu \cdots$. 
• The defining equations of $\mathbb{B}_\mathcal{O}(\overline{\mathcal{H}})$ can be computed.

(3) The various Hilbert function subschemes of $\mathbb{B}_\mathcal{O}$ form a tree at whose root lies $\mathbb{B}_\mathcal{O}^{df}$ and whose unique leaf is the subscheme corresponding to $\mathcal{H} : 1 2 \cdots \mu \mu \cdots$.

(4) Inside the parts of this stratification we can calculate the equations defining the loci of the subschemes which are locally Gorenstein, Cayley-Bacharach, strict complete intersections, etc. In general, these loci are constructible and can be described by a pair of ideals.
The trouble with Spanish food is that 5-6 days later you are hungry again.

Thank you for your attention!
The trouble with Spanish food is that 5-6 days later you are hungry again.

Thank you for your attention!

Humor is if you laugh anyway.