BER calculation

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reference: Wireless Communications by Andrea Goldsmith

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1 SER and BER over Gaussian channel

1.1 BER for BPSK modulation

In a BPSK system the received signal can be written as:

$$y = x + n \tag{1}$$

where $x \in \{-A, A\}$, $n \sim C\mathcal{N}(0, \sigma^2)$ and $\sigma^2 = N_0$. The real part of the above equation is $y_{re} = x + n_{re}$ where $n_{re} \sim \mathcal{N}(0, \sigma^2/2) = \mathcal{N}(0, N_0/2)$. In BPSK constellation $d_{min} = 2A$ and γ_b is defined as E_b/N_0 and sometimes it is called SNR per bit. With this definition we have:

$$\gamma_b := \frac{E_b}{N_0} = \frac{A^2}{N_0} = \frac{d_{min}^2}{4N_0} \tag{2}$$

So the bit error probability is:

$$P_b = P\{n > A\} = \int_A^\infty \frac{1}{\sqrt{2\pi\sigma^2/2}} e^{-\frac{x^2}{2\sigma^2/2}} \tag{3}$$

This equation can be simplified using Q-function as:

$$P_b = Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right) \tag{4}$$

where the Q function is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{x^2}{2}} dx \tag{5}$$

1.2 BER for QPSK

QPSK modulation consists of two BPSK modulation on in-phase and quadrature components of the signal. The corresponding constellation is presented on figure 1. The BER of each branch is the same as BPSK:

$$P_b = Q\left(\sqrt{2\gamma_b}\right) \tag{6}$$

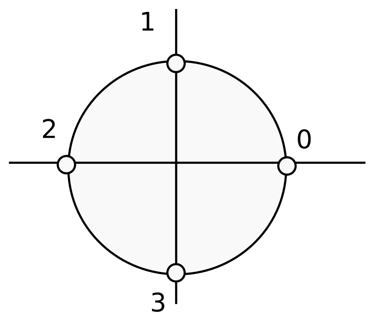

Figure 1: QPSK constellation

The symbol probability of error (SER) is the probability of either branch has a bit error:

$$P_s = 1 - [1 - Q\left(\sqrt{2\gamma_b}\right)]^2 \tag{7}$$

Since the symbol energy is split between the two in-phase and quadrature components, $\gamma_s = 2\gamma_b$ and we have:

$$P_s = 1 - [1 - Q\left(\sqrt{\gamma_s}\right)]^2 \tag{8}$$

We can use the union bound to give an upper bound for SER of QPSK. Regarding figure 1, condition that the symbol zero is sent, the probability of error is bounded by the sum of probabilities of $0 \to 1$, $0 \to 2$ and $0 \to 3$. We can write:

$$P_s \leq Q(d_{01}/\sqrt{2N_0}) + Q(d_{02}/\sqrt{2N_0}) + Q(d_{03}/\sqrt{2N_0}) \tag{9}$$

$$= 2Q(A/\sqrt{N_0}) + Q(\sqrt{2}A/\sqrt{2N_0}) \tag{10}$$

Since $\gamma_s = 2\gamma_b = A^2/N_0$, we can write:

$$P_s \leq 2Q(\sqrt{\gamma_s}) + Q(\sqrt{2\gamma_s}) \leq 3Q(\sqrt{\gamma_s}) \tag{11}$$

Using the tight approximation of Q function for $z \gg 0$:

$$Q(z) \leq \frac{1}{z\sqrt{2\pi}} e^{-z^2/2} \tag{12}$$

we obtain:

$$P_s \leq \frac{3}{\sqrt{2\pi\gamma_s}} e^{-0.5\gamma_s} \tag{13}$$

Using Gray coding and assuming that for high signal to noise ratio the errors occur only for the nearest neighbor, P_b can be approximated from P_s by $P_b \approx P_s/2$.

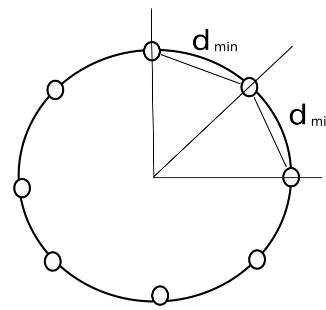

Figure 2: MPSK constellation

1.3 BER for MPSK signaling

For MPSK signaling we can calculate easily an approximation of SER using nearest neighbor approximation. Using figure , the symbol error probability can be approximated by:

$$P_s \approx 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{2A\sin\frac{\pi}{M}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{2\gamma_s}\sin(\pi/M)\right) \tag{14}$$

This approximation is only good for high SNR.

1.4 BER for QAM constellation

The SER for a rectangular M-QAM (16-QAM, 64-QAM, 256-QAM etc) with size $L = M^2$ can be calculated by considering two M-PAM on in-phase and quadrature components (see figure 3 for 16-QAM constellation). The error probability of QAM symbol is obtained by the error probability of each branch (M-PAM) and is given by:

$$P_s = 1 - \left(1 - \frac{2\left(sqrtM - 1\right)}{sqrtM} Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)\right)^2 \tag{15}$$

If we use the nearest neighbor approximation for an M-QAM rectangular constellation, there are 4 nearest neighbors with distance d_{min}. So the SER for high SNR can be approximated by:

$$c \tag{16}$$

In order to calculate the mean energy per transmitted symbol, it can be seen that

$$\overline{E}_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2 \tag{17}$$

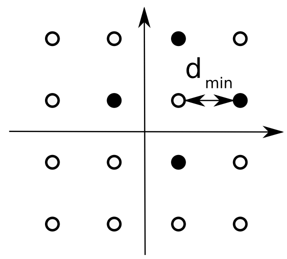

Figure 3: 16-QAM constellation

Modulation	$P_s(\gamma_s)$	$P_b(\gamma_b)$
BPSK		$P_b = Q\left(\sqrt{2\gamma_b}\right)$
QPSK	$P_s \approx 2Q\left(\sqrt{\gamma_s}\right)$	$P_b \approx Q\left(\sqrt{2\gamma_b}\right)$
MPSK	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin\left(\frac{\pi}{M}\right)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin\left(\frac{\pi}{M}\right)\right)$
M-QAM	$P_s \approx 4Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{M-1}}\right)$

Table 1: Approximate symbol and bit error probabilities for coherent modulation

Using the fact that $A_i = (a_i + b_i)$ and a_i and $b_i \in \{2i - 1 - L\}$ for $i = 1, ..., L$. After some simple calculations we obtain:

$$\overline{E}_s = \frac{d_{min}^2}{2L} \sum_{i=1}^{L} (2i - 1 - L)^2 \tag{18}$$

For example for 16-QAM and $d_{min} = 2$ the $\overline{E}_s = 10$. For 64-QAM and $d_{min} = 2$ the $\overline{E}_s = 21$.

1.5 conclusion

The approximations or exact values for SER has the following form:

$$P_s(\gamma_s) \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right) \tag{19}$$

where α_M and β_M depend on the type of approximation and the modulation type. In the table 1 the values for α_M and β_M are semmerized for common modulations.

We can also note that the bit error probability has the same form as for SER. It is:

$$P_b(\gamma_b) \approx \hat{\alpha}_M Q\left(\sqrt{\hat{\beta}_M \gamma_b}\right) \tag{20}$$

where $\hat{\alpha}_M = \alpha_M / \log_2 M$ and $\hat{\beta}_M = \beta_M / \log_2 M$.

Note: $\gamma_s = E_s/N_0$, $\gamma_b = E_b/N_0$, $\gamma_b = \frac{\gamma_s}{\log_2 M}$ and $P_b \approx \frac{P_s}{\log_2 M}$.

1.6 Appendix

In this appendix the reference curve for AWGN channel is presented in figure 4. As we expected , the results for BPQK and QPSK are the same.

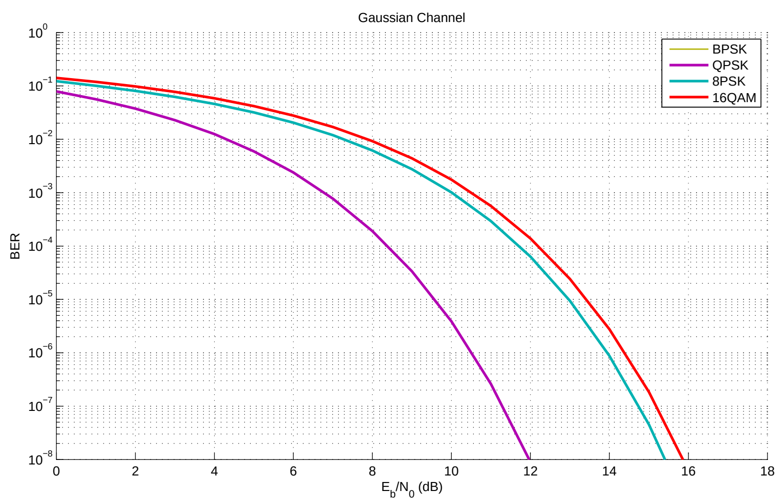

Figure 4: BER over AWGN channel for BPSK, QPSK, 8PSK and 16QAM

The following matlab program illustrates the BER calculations for BPSK over an AWGN channel.

```
%BPSK BER
const=[1 -1];
size=100000;
iter_max=1000;
EbN0_min=0;
EbN0_max=10;
SNR=[];BER=[];
for EbN0 = EbN0_min:EbN0_max
    EbN0_lin=10.^(0.1*EbN0);
    noise_var=0.5/(EbN0_lin); % s^2=N0/2
    iter = 0;
    err = 0;
    while (iter <iter_max && err <100),
        bits=randsrc(1,size,[0 1]);
        s=const(bits+1);
        x = s + sqrt(noise_var)*randn(1,size);
        bit_hat=(-sign(x)+1)/2;
        err = err + sum(bits ~= bit_hat);
        iter = iter + 1;
```

```
    end
    SNR =[SNR EbN0];
    BER = [BER err/(size*iter)];
end
semilogy(SNR,BER);grid;xlabel('E_bN_0');ylabel('BER');
title('BPSK over AWGN channel');
```

The following program uses some advanced functions of matlab to evaluate the symbol error rate for QPSK modulation:

```
M = 4; % Alphabet size
EbN0_min=0;EbN0_max=10;step=2;
SNR=[];SER=[];
for EbN0 = EbN0_min:step:EbN0_max
    SNR_dB=EbN0 + 3; %for QPSK Eb/N0=0.5*Es/N0=0.5*SNR
    x = randint(1000000,1,M);
    y=modulate(modem.qammod(M),x);
    ynoisy = awgn(y,SNR_dB,'measured');
    z=demodulate(modem.qamdemod(M),ynoisy);
    [num,rt]= symerr(x,z);
    SNR=[SNR EbN0];
    SER=[SER rt];
end;
semilogy(SNR,SER);grid;titel('Symbol error rate for QPSK over AWGN');
xlabel('E_b/N_0');ylabel('SER');
```

2 SER and BER over fading channel [1]

2.1 PDF-based approach for binary signal

A fading channel can be considered as an AWGN with a variable gain. The gain itself is considered as a RV with a given *pdf*. So the average BER can be calculated by averaging BER for instantaneous SNR over the distribution of SNR:

$$P_b(E) = \int_0^\infty P_b(E|\gamma) p_\gamma(\gamma) d\gamma$$

The BER is expressed by a Q-function as seen in previous chapter:

$$P_b(E) = \int_0^\infty Q(\sqrt{2g\gamma}) p_\gamma(\gamma) d\gamma \tag{21}$$

where $g = 1$ for the case of coherent BPSK.

Example 1. Rayleigh fading channel with coherent detection:
The received signal in a Rayleigh fading channel is of the form:

$$y = hx + w \tag{22}$$

[1]"Digital Communication over Fading Channel" by Simon and Alouini

where h is the channel attenuation with normal distribution $h \sim \mathcal{CN}(0,1)$ and n is a white additive noise $w \sim \mathcal{CN}(0, N_0)$. The coherent receiver constructs the following metric from the received signal:

$$h^* y = |h|^2 x + h^* w \tag{23}$$

Using BPSK modulation and since the information are real, only the real part of the equation is of interest. So the following sufficient statistic is used for decision at the receiver.

$$\Re \left\{ \frac{h^*}{|h|} y \right\} = |h| x + n \tag{24}$$

The noise n has the same statistics as $\Re w$ because $h^*/|h| = exp(j\theta)$ with θ uniformly distributed in $(0, \pi)$, therefore $n \sim \mathcal{CN}(0, N_0/2)$. This equation shows that we have a normal AWGN channel with the signal scaled by $|h|$. The bit error probability as seen before for this case, given h, will be:

$$P_b = Q\left(\sqrt{2|h|^2 \gamma_b}\right)$$

Now, we compute the SER by averaging this BER over the distribution of h. Since h is complex Gaussian, the distribution of $r = |h|^2$ will be exponential with:

$$\begin{aligned} P_r(r) &= \frac{d}{dr}\left(P(h_r^2 + h_i^2 < r)\right) \\ &= \frac{d}{dr}\left(\int_0^{2\pi}\int_0^{\sqrt{r}} \frac{1}{2\pi 1/2} e^{-x^2} x dx d\theta\right) \\ &= \frac{d}{dr}\left(1 - e^{-r}\right) \\ &= e^{-r} U(r) \end{aligned}$$

Therefore the signal-to-noise-ratio distribution $\gamma = |h|^2 \gamma_b$ will be:

$$p_\gamma(\gamma) = \frac{1}{\gamma_b} e^{-\gamma/\gamma_b}$$

The error probability can be calculated by:

$$P_b = \int_0^\infty Q(\sqrt{2\gamma}) p_\gamma(\gamma) d\gamma = \int_0^\infty Q(\sqrt{2\gamma}) \frac{1}{\gamma_b} e^{-\gamma/\gamma_b} d\gamma$$

Using the following form of Q-function and MGF function, the integral can eb calculated.

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp(-\frac{x^2}{2\sin^2\theta}) d\theta$$

$$p_b = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_b}{1+\gamma_b}}\right)$$

Example 2. Consider a SIMO system with L receive antennas. Each branch has a SNR per bit of γ_l and therefore the SNR at the output of MRC combiner is $\gamma_t = \sum_{l=1}^{L} \gamma_l$. Suppose a Rayleigh channel, the *pdf* of SNR for each channel will be (supposing i.i.d. channels):

$$p_{\gamma_l}(\gamma_l) = \frac{1}{\bar{\gamma}} e^{-\gamma_l/\bar{\gamma}}$$

At the output of combiner, the SNR follows the distribution of chi-square (or gamma) with L degrees of freedom:

$$p_{\gamma_t}(\gamma_t) = \frac{1}{(L-1)!\bar{\gamma}^L} \gamma_t^{L-1} e^{-\gamma_t/\bar{\gamma}}$$

The average probability can be calculated using the integration by part and resulting in the following formula:

$$P_b(E) = \left(\frac{1-\mu}{2}\right)^L \sum_{l=0}^{L-1} \left(\begin{array}{c} L-1+l \\ l \end{array} \right) \left(\frac{1+\mu}{2}\right)^l$$

2.2 MGF-based approach

2.2.1 Binary PSK

We can use the other representation of Q-function to simplify the calculations.

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{y^2}{2}\right) dy = \frac{1}{\pi} \int_0^{\pi/2} exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta$$

Therefore the equation (5) can be written as:

$$P_b(E|\{\gamma_l\}_{l=1}^L) = \frac{1}{\pi} \int_0^{\pi/2} exp(-\frac{g\gamma_t}{\sin^2\phi}) d\phi = \frac{1}{\pi} \int_0^{\pi/2} \prod_{l=1}^{L} exp(-\frac{g\gamma_l}{\sin^2\phi}) d\phi \quad (25)$$

This form of Q-function is more convenient because it allows us to average first over the individual distributions of γ_l and then perform the integral over ϕ.

$$P_b(E) = \int_0^{\infty} \int_0^{\infty} ... \int_0^{\infty} P_b(\{\gamma_l\}_{l=1}^L) \prod_{l=1}^{L} p_{\gamma_l}(\gamma_l) d\gamma_1 d\gamma_2 ... d\gamma_L \quad (26)$$

Using (25) in (26) and changing the order of integration gives:

$$P_b(E) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{l=1}^{L} M_{\gamma_l} \left(-\frac{g}{\sin^2\phi}\right) d\phi \quad (27)$$

2.2.2 MPSK

For MPSK signaling the SER given all the SNRs is:

$$P_s(E|\{\gamma_l\}_{l=1}^L) = \frac{1}{\pi} \int_0^{(M-1)\pi/M} exp\left(-\frac{g\gamma_t}{\sin^2\phi}\right) d\phi \tag{28}$$

$$P_s(E|\{\gamma_l\}_{l=1}^L) = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \prod_{l=1}^{L} exp\left(-\frac{g\gamma_l}{\sin^2\phi}\right) d\phi \tag{29}$$

where $g = \sin^2(\pi/m)$.