Q-Function

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The Q-function is defined as :

$$Q(z) = p(X \geq z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

This function is widely used in BER calculation. There is some bounds that permits us not to calculate the integral. For $x > 0$:

$$\frac{1}{x\sqrt{2\pi}}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2} < Q(x) < \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

One of the good approximations of Q-function is given below (see simulation of communication systems by Michel C. Jeruchim):

$$\begin{aligned} Q(x) &= \int_x^\infty \frac{1}{(2\pi)^{1/2}} e^{-t^2/2} dt \\ &\approx \left[\frac{1}{(1-a)x + a(x^2+b)^{1/2}}\right] \frac{1}{(2\pi)^{1/2}} e^{-x^2/2} \end{aligned}$$

where $a = 1/\pi$ and $b = 2\pi$.

There is also another way to calculate the integral with numerical methods. So, the alternate Q-Function is proposed by Craig as:

$$Q(z) = \frac{1}{\pi} \int_0^{\pi/2} exp\left[\frac{-z^2}{2\sin^2\phi}\right] d\phi, \qquad z > 0$$

Sometimes the Q-function is described as a function of error function *erf*, which is defined over $[0, \infty)$ as:

$$\mathtt{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

with $\mathtt{erf}(0) = 0$ and $\mathtt{erf}(\infty) = 1$. So

$$Q(x) = \frac{1}{2} - \frac{1}{2}\mathtt{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$\texttt{erf}(x) = 1 - 2Q(\sqrt{2}x)$$

There is also the complementary error function $\texttt{erfc}(x)$ which is defined as $\texttt{erfc}(x) = 1 - \texttt{erf}(x)$. With the use of erfc we can write:

$$Q(x) = \frac{1}{2}\texttt{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

The $\texttt{erfc}(x)$ can be approximated by the following series:

$$\texttt{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1.3}{2^2x^4} - \frac{1.3.5}{2^3x^6} + ...\right)$$

The following figure presents the Q-function and its lower and higher bounds. As it can be seen the bounds are quite tight for $x > 2$.

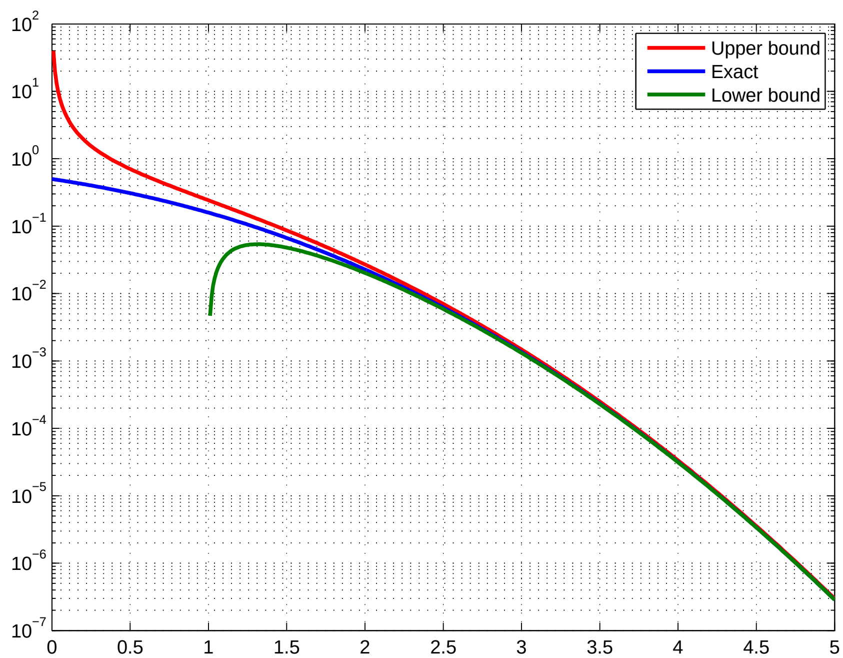

Figure 1: Q-function and its bounds