Symbol Error Rate of Selection Amplify-and-Forward Relay Systems

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SYSTEM MODEL

$$\gamma_r^{AP} = \frac{|h_{s,d}|^2 E_s}{N_{s,d}} + \sum_{i=1}^{m} \frac{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} \frac{|h_{i,d}|^2 E_i}{N_{i,d}}}{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} + \frac{|h_{i,d}|^2 E_i}{N_{i,d}} + 1},$$

$$\gamma_r^{S} = \frac{|h_{s,d}|^2 E_s}{N_{s,d}} + \max_i \frac{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} \frac{|h_{i,d}|^2 E_i}{N_{i,d}}}{\frac{|h_{s,i}|^2 E_s}{N_{s,i}} + \frac{|h_{i,d}|^2 E_i}{N_{i,d}} + 1}$$

$$p_e = E\{Q(\sqrt{c\gamma_r})\} = E\{\int_{\sqrt{c\gamma_r}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt\} = E\{X > \sqrt{c\gamma_r}\}$$

$$Where: X \sim N(0,1)$$

$$= E\{\gamma_r < \frac{X^2}{c}\}, X > 0$$

$$= E\{F_{\gamma_r}(\frac{X^2}{c})\}$$

A. SER of S-AF Systems

$$P_e = \overline{P\left\{X > \sqrt{c\gamma_r}\right\}} = \overline{P\left\{\gamma_r < X^2/c\right\}} = \mathcal{E}_X\left\{F_{\Gamma_R}(\frac{X^2}{c})\right\}$$

$$N_{s,d} = N_{s,i} = N_{i,d} = N_0 = 1/\gamma$$

$$\alpha_0 = |h_{s,d}|^2 E_s,\ \alpha_i = |h_{s,i}|^2 E_s \text{ and } \beta_i = |h_{i,d}|^2 E_i$$

$$\gamma_r^S = \alpha_0\gamma + \max_i \frac{\alpha_i\beta_i\gamma^2}{\alpha_i\gamma + \beta_i\gamma + 1}$$

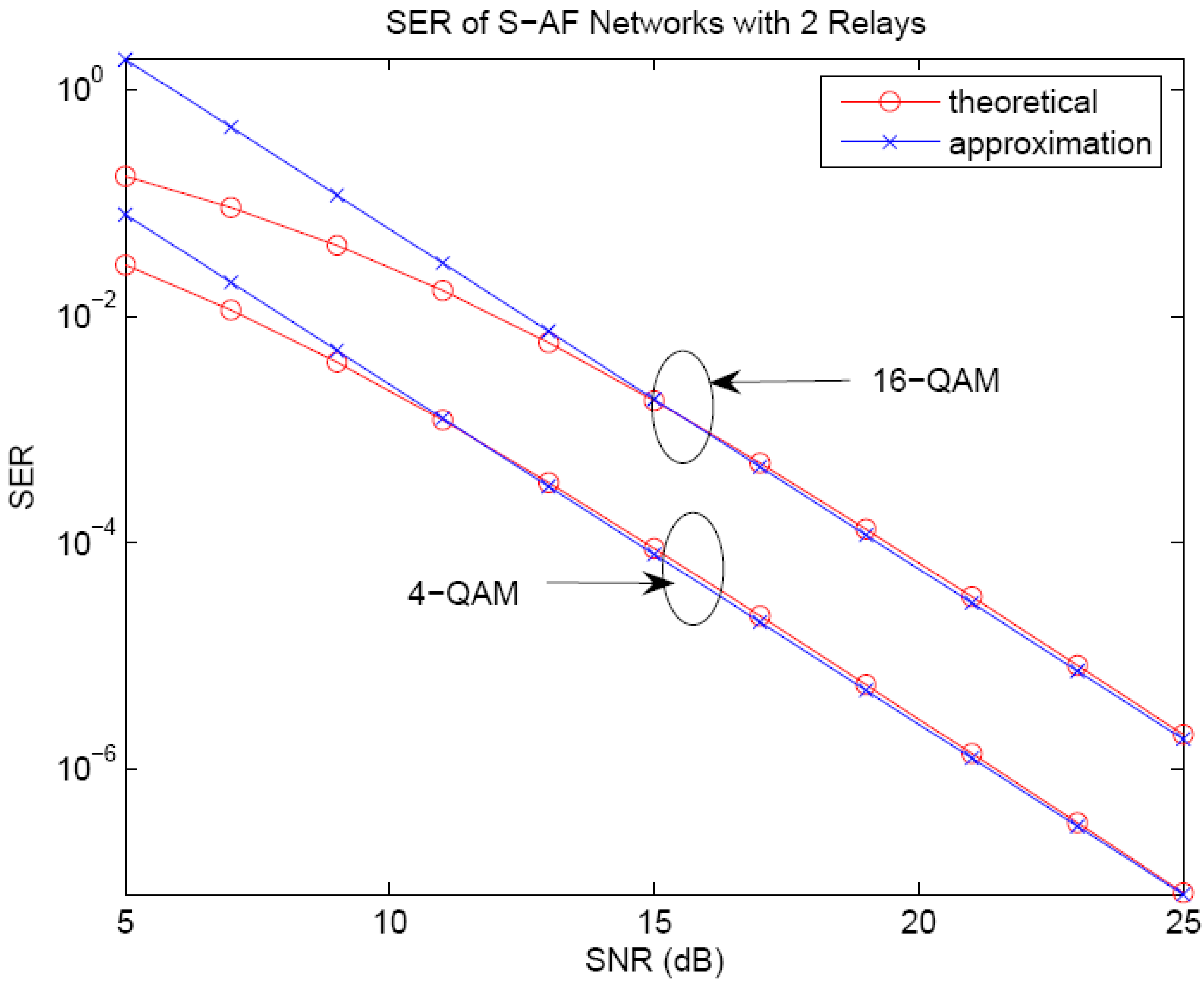
$$F^S_{\Gamma_R}(\gamma_r) \approx \frac{\lambda_0 \prod_{i=1}^{m} (\lambda_i + \xi_i)}{m+1} \left(\frac{\gamma_r}{\gamma}\right)^{m+1}$$

$$f_{\alpha_0}(x) = \lambda_0 \exp(-\lambda_0 x)$$

$$\int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}}$$

$$P^S_e \approx \int_0^\infty \frac{\lambda_0 \prod_{i=1}^{m} (\lambda_i + \xi_i)}{m+1} \left(\frac{x^2}{c\gamma}\right)^{m+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{(2m+1)!\lambda_0 \prod_{i=1}^{m} (\lambda_i + \xi_i)}{(m+1)!(2c\gamma)^{m+1}}.$$


SER of S−AF Networks with 2 Relays
theoretical
approximation
16−QAM
4−QAM
SER
SNR (dB)
10^0
10^{-2}
10^{-4}
10^{-6}
5
10
15
20
25

B. SER Improvement: S-AF vs. AP-AF

- To ensure a fair comparison, unit transmit energy and equal power division among cooperating nodes are assumed for both systems.
- AP-AF: $E_s = E_i = 1/(m + 1)$
- S-AF: $E_s = E_i = 1/2$

$$f_{\gamma_r} = a\gamma_r^t + o(\gamma_r^t)$$

$$P_e \rightarrow \frac{\prod_{i=1}^{t+1}(2i-1)}{2(t+1)c^{t+1}t!} \frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}(0) \quad \frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}(0) = \lim_{\gamma_r \rightarrow 0^+} \frac{\partial^t f_{\gamma_r}}{\partial \gamma_r^t}$$

$$
\begin{aligned}
Y_1 &= \frac{X_0 + \sum_{i=1}^{m} X_i}{m+1}, \\
Y_2 &= \frac{X_0 + \max_i X_i}{2} = \frac{X_0 + V}{2}
\end{aligned}
$$

$$
M_{Y_1}(s) = \prod_{i=0}^{m} M_{X_i}\left(\frac{s}{m+1}\right)
$$

$$
\begin{aligned}
\frac{\partial^m f_{Y_1}}{\partial y_1^m}(0) &= \lim_{s\to\infty} s^{m+1} M_{Y_1}(s) = \lim_{s\to\infty} s^{m+1} \prod_{i=0}^{m} M_{X_i}\left(\frac{s}{m+1}\right) \\
&= (m+1)^{m+1} \lim_{s\to\infty} \prod_{i=0}^{m} \frac{s}{m+1} M_{X_i}\left(\frac{s}{m+1}\right) \\
&= (m+1)^{m+1} \prod_{i=0}^{m} f_{X_i}(0).
\end{aligned}
$$

$$Y_2 = \frac{X_0 + \max_i X_i}{2} = \frac{X_0 + V}{2}$$

$$F_V(v) = \prod_{i=1}^{m} F_{X_i}(v)$$

$$\frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) = \frac{\partial^m F_V}{\partial v^m}(0) = m! \prod_{i=1}^{m} f_{X_i}(0) + o(0)$$

$$\frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) = m! \prod_{i=1}^{m} f_{X_i}(0).$$

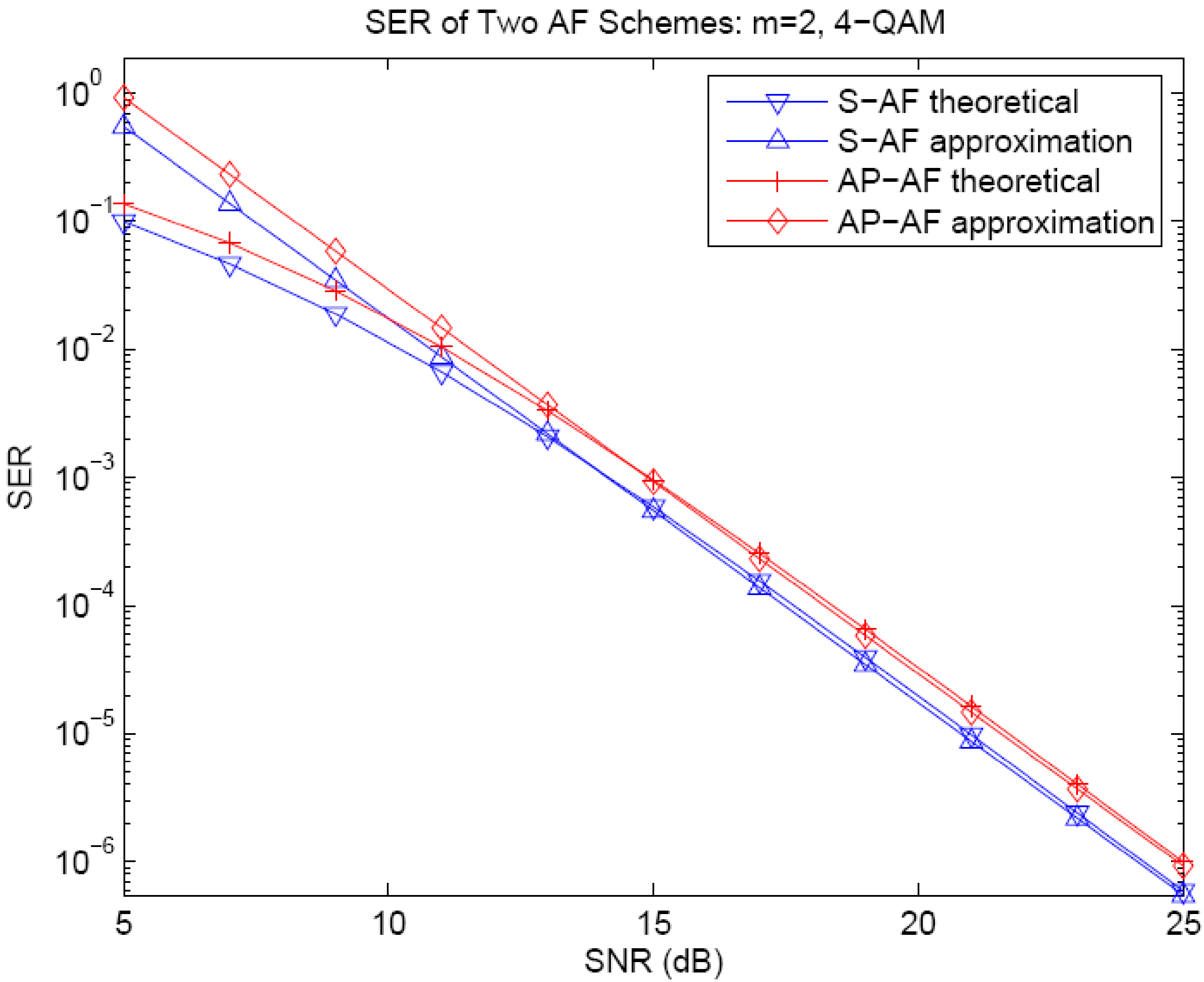
$$Example: F_V(v) = F_{x_1}(v) F_{x_2}(v)$$

$$\Rightarrow \frac{\partial F_V(v)}{\partial v} = f_{x_1}(v) F_{x_2}(v) + F_{x_1}(v) f_{x_2}(v)$$

$$\Rightarrow \frac{\partial^2 F_V(v)}{\partial v^2} = \frac{\partial f_{x_1}(v)}{\partial v} F_{x_2}(v) + 2 f_{x_1}(v) f_{x_2}(v) + F_{x_1}(v) \frac{\partial f_{x_2}(v)}{\partial v}$$

$$
\begin{aligned}
\frac{\partial^m f_{Y_2}}{\partial y_2^m}(0) &= \lim_{s \to \infty} s^{m+1} M_{Y_2}(s) \\
&= \lim_{s \to \infty} s^{m+1} M_{X_0}\left(\frac{s}{2}\right) M_V\left(\frac{s}{2}\right) \\
&= 2^{m+1} f_{X_0}(0) \frac{\partial^{m-1} f_V}{\partial v^{m-1}}(0) \\
&= m! 2^{m+1} \prod_{i=0}^{m} f_{X_i}(0).
\end{aligned}
$$

$$
\boxed{\frac{P_e^S}{P_e^{AP}} = m! \left(\frac{2}{m+1}\right)^{m+1}}
$$


SER of Two AF Schemes: m=2, 4−QAM
S−AF theoretical
S−AF approximation
AP−AF theoretical
AP−AF approximation
SER
SNR (dB)
10^0
10^−1
10^−2
10^−3
10^−4
10^−5
10^−6
5
10
15
20
25

All-Participate Amplify-and-Forward

- Basically, a relay amplifies the signal received in source transmission slot e.g. relay i receives

$$y_{s,i} = \sqrt{E_s} h_{s,i} x + n_{s,i}$$

where

$$\begin{aligned} E_s &= \text{Transmitted symbol energy;} \\ h_{s,i} &= \text{Complex channel gain from source to relay } i\text{;} \\ x &= \text{Unit-energy transmitted symbol;} \\ n_{s,i} &= \text{Thermal noise at relay } i \end{aligned}$$

All-Participate Amplify-and-Forward

- If it needs to transmit a unit-energy signal, that would be

$$x_i = \frac{y_{s,i}}{\sqrt{E|y_{s,i}|^2}} = \frac{\sqrt{E_s}h_{s,i}x + n_{s,i}}{\sqrt{E_s|h_{s,i}|^2 + N_{s,i}}}$$

  where $N_{s,i}$ is the noise variance.

- Note that in AF (or DF), each relay is assumed to know the channel from the source $h_{s,i}$, the transmitted symbol energy E_s, and $N_{s,i}$.

AP AF Signal Structure

- If tx'ed symbol energy at relay i is E_i, signal received at destination is

$$y_{i,d} = \sqrt{E_i} h_{i,d} x_i + n_{i,d} = \sqrt{\frac{E_s E_i}{E_s |h_{s,i}|^2 + N_{s,i}}} h_{i,d} h_{s,i} x + \tilde{n}_{i,d}$$

  where $\tilde{n}_{i,d}$ is a linear combination of $n_{s,i}$ and $n_{i,d}$.

- Noise term $\tilde{n}_{i,d}$ has a known variance, if variances of $n_{s,i}$ and $n_{i,d}$ are known, so we can scale $y_{i,d}$ to have unit-variance noise (for convenience).

- We can write all $m + 1$ normalized signals received at destination in a vector:

$$\mathbf{y}_d = \mathbf{h} x + \mathbf{n}.$$

AP AF Capacity

- MRC gives a sufficient statistic for detecting x since $\mathbf{n}$ assumed to have independent identically distributed components.

- The capacity of the AP-AF scheme is thus (in bits per frame)

$$\mathcal{I}_{AP} = \frac{1}{m+1} \log_2(1 + \|\mathbf{h}\|^2).$$

- After some algebra, we can simplify the capacity expression to

$$\mathcal{I}_{AP} = \frac{1}{m+1} \log_2 \left[1 + E_s a_0 + \sum_{i=1}^{m} \frac{E_s E_i a_i b_i}{E_s a_i + E_i b_i + 1} \right]$$

where $a_0 = |h_{s,d}|^2/N_{s,d}$, $a_i = |h_{s,i}|^2/N_{s,i}$, $b_i = |h_{i,d}|^2/N_{i,d} \ldots$

AP AF Optimal Power Allocation

- ...and then to

$$\mathcal{I}_{AP} = \frac{1}{m+1}\log_2\left[1 + E_s\sum_{i=1}^{m} a_i - \sum_{i=1}^{m}\frac{E_s^2 a_i^2 + E_s a_i}{E_s a_i + E_i b_i + 1}\right].$$

- We want to maximize $\mathcal{I}_{AP}$ over the power allocations E_s and $\{E_i\}$, subject to $E_s + \sum_i E_i \leq E_T$, and $E_i \leq E_i^{max}$.

- Problem cannot be solved, but if we fix E_s and search for the best power allocation among relays, the problem becomes

$$\text{minimize} \sum_{i=1}^{m}\frac{E_s^2 a_i^2 + E_s a_i}{E_s a_i + E_i b_i + 1} \text{ s.t. } \sum_{i=1}^{m} E_i \leq E_T - E_s, 0 \leq E_i \leq E_i^{max}.$$

$$
\begin{aligned}
P_{out}^{AP} &= P\left[\mathcal{I}_{AP} < R\right] \\
&= P\left[\alpha_0 + \sum_{i=1}^{m} \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1} < \right. \\
&\qquad\qquad\qquad\qquad \left. \frac{2^{(m+1)R} - 1}{\gamma}\right]
\end{aligned}
$$

$$
\sum_{i=1}^{m} \frac{\gamma^2\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1} > \max_i \frac{\gamma^2\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1}
$$

So an upper bound can be introduced as

$$
\begin{aligned}
\overline{P_{out}^{AP}} &= P\left[\frac{1}{m+1}\log_2\left(1+\gamma\alpha_0+\right.\right.\\
&\qquad\qquad\left.\left.\max_i \frac{\gamma^2\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1}\right)<R\right],\\
&= P\left[\alpha_0+\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1}<\frac{2^{(m+1)R}-1}{\gamma}\right],\\
&= P\left[\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1}<\delta-\alpha_0\right], \qquad (36)
\end{aligned}
$$

where $\delta = \frac{2^{(m+1)R}-1}{\gamma}$

Since α_0 is exponentially distributed variable with parameter λ_0, we have:

$$\overline{P_{out}^{AP}} = \int_0^{\delta} P\left[\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \delta - x\right] \lambda_0 e^{-\lambda_0 x} dx$$

$$= \int_0^1 P\left[\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \delta x'\right] \delta\lambda_0 e^{-\lambda_0\delta(1-x')} dx'$$

$x' = 1 - x/\delta$

$$= \int_0^1 \left(\prod_{i=1}^m P\left[\frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \delta x'\right]\right) \delta\lambda_0 e^{-\lambda_0\delta(1-x')} dx'$$

$$= \delta^{m+1}\lambda_0 \int_0^1 \left(\prod_{i=1}^m \frac{P\left[\frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \delta x'\right]}{\delta x'}\right) (x')^m e^{-\lambda_0\delta(1-x')} dx', \quad (37)$$

- We know: $\lim_{\gamma \to \infty} e^{-\lambda_0 \delta (1 - x')} = 1.$
- λ_0, λ_i and ξ_i the exponential distribution parameters of the Rayleigh channel amplitude squares α, α and β

By Lemma 1 in Appendix 1 of [2], we have

$$\lim_{\gamma \to \infty} \frac{P\left[\frac{\gamma \alpha_i \beta_i}{\gamma \alpha_i + \gamma \beta_i + 1} < \delta x'\right]}{\delta x'} = \lambda_i + \xi_i$$

$$\lim_{\gamma\to\infty} \frac{P\left[\frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \delta x'\right]}{\delta x'} = \lambda_i + \xi_i$$

$$\lim_{\gamma\to\infty} \frac{\overline{P_{out}^{AP}}}{\delta^{m+1}} = \lambda_0 \int_0^1 \left(\prod_{i=1}^{m}(\lambda_i+\xi_i)\right)(x')^m dx'$$

$$= \frac{\lambda_0 \prod_{i=1}^{m}(\lambda_i+\xi_i)}{m+1}.$$

$$P_{out}^{AP} < \frac{\lambda_0 \prod_{i=1}^{m}(\lambda_i+\xi_i)}{m+1}\delta^{m+1}.$$

$$\sum_{i=1}^{m} \frac{\gamma^2\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < m \max_i \frac{\gamma^2\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1}$$

$$
\begin{aligned}
P_{out}^{AP} &> \underline{P_{out}^{AP}} \\
&= P\left[\alpha_0 + m \max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1} < \frac{2^{(m+1)R} - 1}{\gamma}\right] \\
&= P\left[\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1} < \frac{\frac{2^{(m+1)R}-1}{\gamma} - \alpha_0}{m}\right] \\
&= \int_0^{\delta} P\left[\max_i \frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i + \gamma\beta_i + 1} < \frac{\frac{2^{(m+1)R}-1}{\gamma} - x}{m}\right] \lambda_0 e^{-\lambda_0 x} dx
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{2^{(m+1)R}-1}{\gamma}\right)^{m+1} \frac{\lambda_0}{m^m}\int_0^1 \\
&\qquad \left(\prod_{i=1}^{m} \frac{P\left[\frac{\gamma\alpha_i\beta_i}{\gamma\alpha_i+\gamma\beta_i+1} < \frac{2^{(m+1)R}-1}{m\gamma}x'\right]}{\frac{2^{(m+1)R}-1}{m\gamma}x'}\right) \\
&\qquad\qquad (x')^m e^{-\lambda_0\frac{2^{(m+1)R}-1}{\gamma}(1-x')}dx' \\
&\overset{highSNR}{\approx} \frac{\lambda_0\prod_{i=1}^{m}(\lambda_i+\xi_i)}{(m+1)m^m}\left(\frac{2^{(m+1)R}-1}{\gamma}\right)^{m+1}, \qquad (43)
\end{aligned}
$$