p-adic precision and isogeny computation

Applications to cryptography

Xavier Caruso, Pierre Lairez, David Roe **Tristan Vaccon**

Univ.Rennes 1, TU Berlin, Univ. Pittsburgh, 立教大学

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Motivation for isogeny computations

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Cryptosytems

- De Fao, Jao and Plût have proposed cryptosystems based on isogenies between elliptic curves

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p-adic methods

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My personal (long-term) motivation

Computing (some) moduli spaces of p-adic Galois representations.

Background

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- I am **not** an expert in cryptography.

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- I am **not** an expert in cryptography.
- However, one of my goal today is to present **tools** that can be useful for **cryptography** and **computer algebra**: isogenies and p-adic numbers.

Table of contents

What is... an isogeny?

Definition

We can define an isogeny between two elliptic curves E_1 and E_2 to be at the same time:

- a rational map $E_1 \to E_2$;
- a group morphism $E_1 \to E_2$.

Isogeny and quotient

Proposition

*Every isogeny is either **zero** or **surjective**.*

Isogeny and quotient

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Remark

All non-zero isogenies corresponds to taking some quotient:

$$E \twoheadrightarrow E/H.$$

Toward point-counting

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Using isogeny for point-counting

If $\Phi : E_1 \to E_2$ is non-zero then:

$$\sharp E_1 = \sharp E_2 + \sharp Ker(\Phi).$$

Further toward point-counting

Isogeny and kernel, Vélu's formula

For $\Phi : E_1 \to E_2$, Φ can be written in affine coordinates as:

$$\Phi(x, y) = \left(\frac{g(x)}{h(x)}, cy \left(\frac{g(x)}{h(x)} \right)' \right),$$

with g, h polynomials, c scalar.

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For x-coordinates:

$$Ker(\Phi) = \{\infty\} \cup \{ \text{ zeroes of h } \}$$

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Point-counting algorithms

Use isogenies between an elliptic curve E and other curves: twist by Frobenius, quotient by l-torsion.

Recall on Diffie-Hellman key exchange

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Public modulus p and generator g of $\mathbb{Z}/p\mathbb{Z}^\times$.

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Alice

- Choose an integer a.

Bob

- Choose an integer b.

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- Sends $A = g^a \mod p$ to Bob.

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- Sends $B = g^b \mod p$ to Alice.

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- Computes $s = B^a \mod p$.

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Bob

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- Sends $B = g^b \mod p$ to Alice.
- Computes $s = A^b \mod p$.

Shared information

$s = g^{ab} = B^a = A^b \mod p$.

Elliptic-curve based key exchange: De Feo-Jao-Plût (2011)

Preparation

Elliptic curve $E_0/\mathbb{F}_{p^2}$, generators $\{P_A, Q_A\}$, $\{P_B, Q_B\}$ of $E_0[l_A^{e_A}]$, $E_0[l_b^{e_B}]$.

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- $m_A, n_A \in \mathbb{Z}/l_A^{e_A}\mathbb{Z}$, one is inv.

Bob

- $m_B, n_B \in \mathbb{Z}/l_B^{e_B}\mathbb{Z}$, one is inv.

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Bob

- $m_B, n_B \in \mathbb{Z}/l_B^{e_B}\mathbb{Z}$, one is inv.
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- $\Psi_A : E_B \to E_{AB} = E_B/\langle [m_A]\Phi_B(P_A) + [n_A]\Phi_B(Q_A) \rangle$.

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Shared information

$$E_{AB} = \Psi_B(\Phi_A(E_0)) = \Psi_A(\Phi_B(E_0)),$$

and its j-invariant $j(E_{AB})$.

Some remarks

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Not all elliptic curves are safe for this scheme. *e.g.* supersingularity is a requirement in De Feo-Jao-Plût.

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Candidate for **Post-Quantum Cryptography**.

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Isogeny and Differential equations (*cf* Schoof-Elkies-Atkin algorithm, Bostan-Morain-Salvy-Schost 08, Lercier-Sirvent 08, ...)

Let E and $\tilde{E}$ be two elliptic curves over $\mathbb{Z}/p\mathbb{Z}$:

$$\begin{aligned} E &: y^2 = x^3 + Ax + B, \\ \tilde{E} &: y^2 = x^3 + \tilde{A}x + \tilde{B}. \end{aligned}$$

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Writing $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$, we get :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

Change of variable and the differential equation

The differential equation

Let S be such that $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$.

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- We would like to be in **zero characteristic**: let's go **p-adic**!

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A norm over a field K is a mapping $|\cdot| : K \to \mathbb{R}_+, x \mapsto |x|$ such that :

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$|\cdot|_p$ is an **ultrametric norm** over $\mathbb{Q}$.

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Remark

$\mathbb{Q}$ is complete for none of these norms.

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Definition

We write :

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p,\ v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p,\ |x|_p \leq 1\} = B'_{\mathbb{Q}_p}(0,1).$$

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$\mathbb{Z}_p$ is a sub-ring of $\mathbb{Q}_p$.

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And,
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- *$\mathbb{Z}_p$ is **both** open and closed in $\mathbb{Q}_p$. $\mathbb{Q}_p$ is totally discontinuous.*
- *$\mathbb{Z}$ is a **dense** subset of $\mathbb{Z}_p$.*

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Proposition

- *$\mathbb{Z}_p$ is **both** open and closed in $\mathbb{Q}_p$. $\mathbb{Q}_p$ is totally discontinuous.*
- *$\mathbb{Z}$ is a **dense** subset of $\mathbb{Z}_p$. $\mathbb{Q}$ is a **dense** subset of $\mathbb{Q}_p$.*

Proposition

*If E is an ultrametric vector space, then **any** point in a ball of E is its **center**.*

Summary and algebraic point of view

Proposition

$$\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}.$$

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$$\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}.$$

$$\forall k \in \mathbb{N}, \mathbb{Z}_p/p^k\mathbb{Z}_p = \mathbb{Z}/p^k\mathbb{Z}.$$

Summary and algebraic point of view

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A first idea

- $\mathbb{Q}_p$ is an extension of $\mathbb{Q}$ where one can perform **calculus**, as simply as over $\mathbb{R}$.

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Remark

$$\begin{tikzcd} \mathbb{Q}_p \arrow[dr, dash] & & \mathbb{R} \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d] \\ \mathbb{Z}/p\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \end{tikzcd}$$

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Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=k}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $k \in \mathbb{Z}$ and p a prime number.

While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\boxed{\sum_{i=l}^{d-1} a_i p^i + O(p^d)}$, with $l \in \mathbb{Z}$.

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Definition

The **order**, or the **absolute precision** of $\sum_{i=k}^{d-1} a_i p^i + O(p^d)$ is d.

Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (*p*-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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Remark

It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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Precision formulae

Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1} p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1} p^{-c} + O(p^{d-2c})$$

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A little warm-up on computing determinants : question

An example of determinant computation

$$\begin{bmatrix} p^5 + O(p^{10}) & 1 + O(p^{10}) & 1 + p^3 + O(p^{10}) \\ O(p^{10}) & 1 + O(p^{10}) & 1 + O(p^{10}) \\ 2p^6 + O(p^{10}) & 2p + O(p^{10}) & 2p + p^5 + O(p^{10}) \end{bmatrix}$$

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What is the **precision** on the **determinant**?

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If we expand directly using the expression of the determinant in terms of the coefficients, we get:

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Direct expansion

If we expand directly using the expression of the determinant in terms of the coefficients, we get:

$$-2p^9 + O(p^{10}),$$

because of $1 \times 1 \times O(p^{10})$.

A little warm-up on computing determinants : row-echelon form computation

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If we compute **approximate** SNF, we now get:

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because of $1 \times p^3 \times O(p^{10}) = O(p^{13})$.

Summary: precision and p-adic computations

Direct method for precision

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- No idea on what is **optimal**.

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The Main lemma of p-adic differential precision

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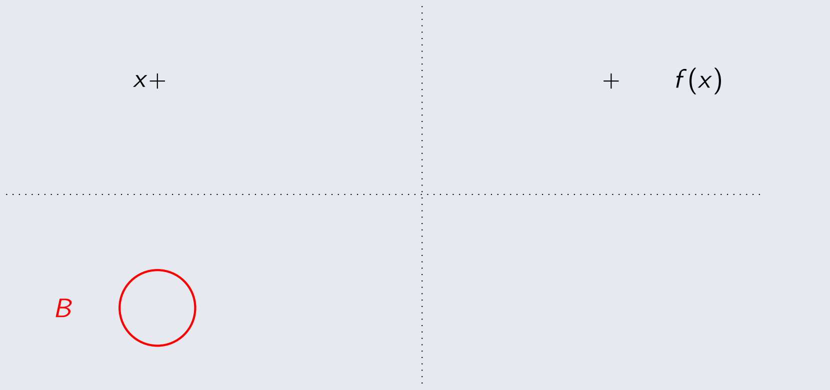
Geometrical meaning

Interpretation



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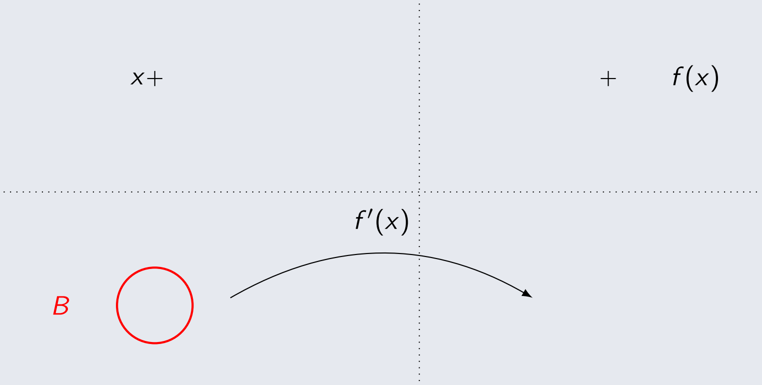

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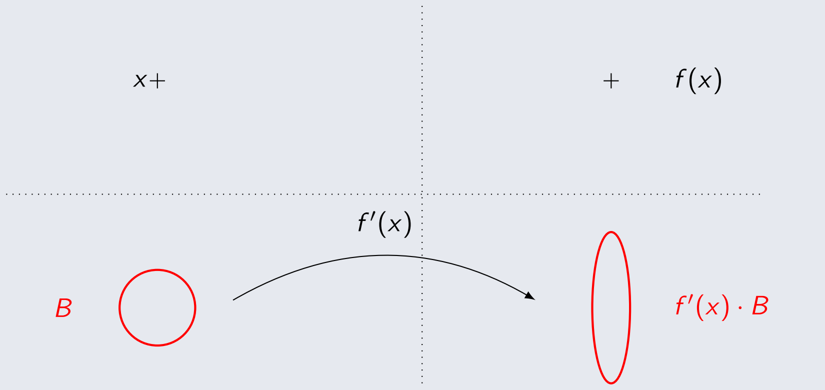

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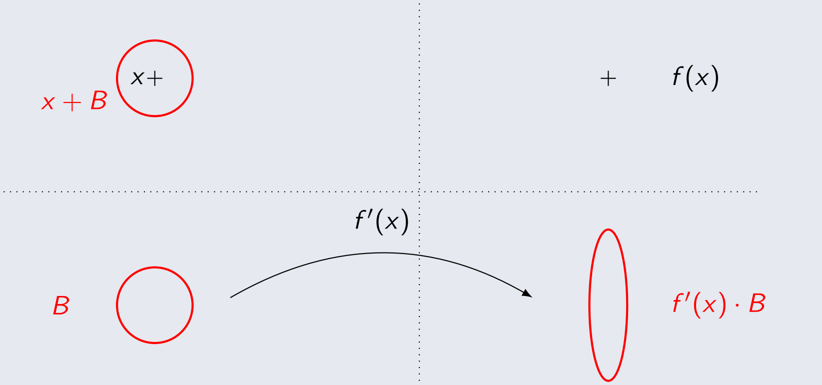

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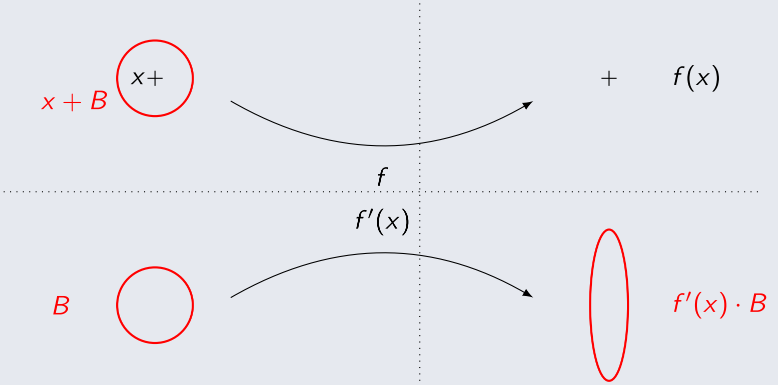

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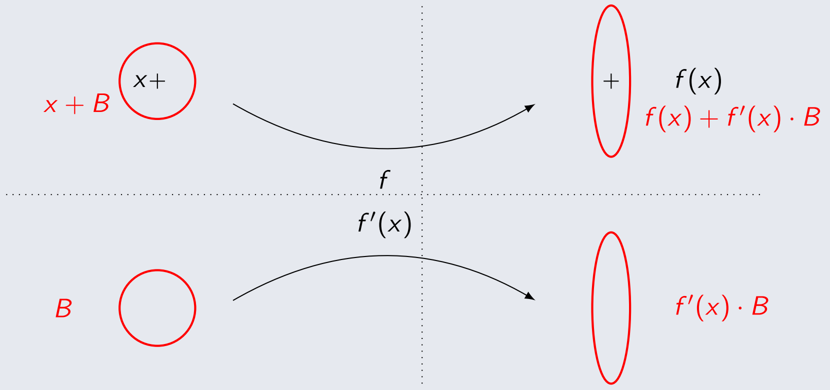

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Remark

Our framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* being $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$, $\mathbb{R}((\varepsilon))^s$).

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This can be determined with **Newton-polygon** techniques.

Looking back to the case of the determinant

Differential of the determinant

It is well known:

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- **Approximate SNF is optimal.**

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An example of p-adic algorithm

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Let E and $\tilde{E}$ be two elliptic curves over $\mathbb{Z}/p\mathbb{Z}$:

$$\begin{aligned} E &: y^2 = x^3 + Ax + B, \\ \tilde{E} &: y^2 = x^3 + \tilde{A}x + \tilde{B}. \end{aligned}$$

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Let us assume that there exists some normalized isogeny I between E and $\tilde{E}$. Then, for some rational fraction U,

$$I(x, y) = (U(x), yU'(x)),$$

Writing $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$, we get :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

Change of variable and the differential equation

The differential equation

Let S be such that

$$U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}.$$

Then if $A, B, \tilde{A}, \tilde{B}$ are in $\mathbb{Z}_p$,

$$S \in \mathbb{Z}_p[[t]]$$

We have the following differential equation for S :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

A p-adic computation of a solution

Computing the isogeny

Given E and $\widetilde{E}$, the goal is to compute the isogeny I *via* the differential equation:

$$\begin{cases} S(0) = 0, \\ (Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6. \end{cases}$$

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$$\begin{cases} S(0) = 0, \\ (Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6. \end{cases}$$

Going through $\mathbb{Z}_p$

Not easy to solve a differential equation in $\mathbb{Z}/p\mathbb{Z}$. Consequently:

1. Lift (consistently) from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}_p$.
2. Solve the differential equation in $\mathbb{Z}_p$.
3. Reduce mod p to get the solution in $\mathbb{Z}/p\mathbb{Z}$.

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Change of equation

When $p \neq 2$, we can replace $y'^2 \times G = H(y)$ by $y' = g \times h(y)$ with $g, h \in \mathbb{Z}_p^\times$.

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To compute $y \mod x^{2^N+1}$, we need an initial precision of $O(N^2)$ digits.

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Differential and differential equation

Theorem

Let $\Phi : (g, h) \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$*. Then,*

$$\Phi'(g, h) \cdot (\delta g, \delta h) = h(y) \int \delta g + \frac{g \delta h(y)}{h(y)}.$$

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$$\Phi'(y) \cdot (\delta g, \delta h) \mod x^{2^N+1} \in \frac{O(p^k)}{p^N} \mathbb{Z}_p[\![x]\!].$$

First conclusion on the application of the lemma

Proposition

$\Phi(g,h) \mod (p, t^{2^n})$ *is determined by* $g, h \mod (p^{1+\log_p 2^n}, t^{2^n})$. *In other words, we have a logarithmic loss in precision.*

What happens in practice ?

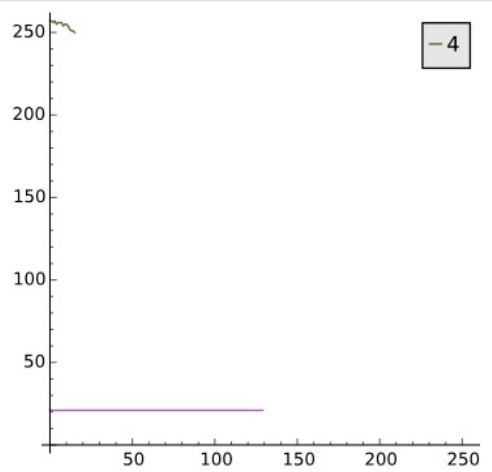

Figure: Precision over the output

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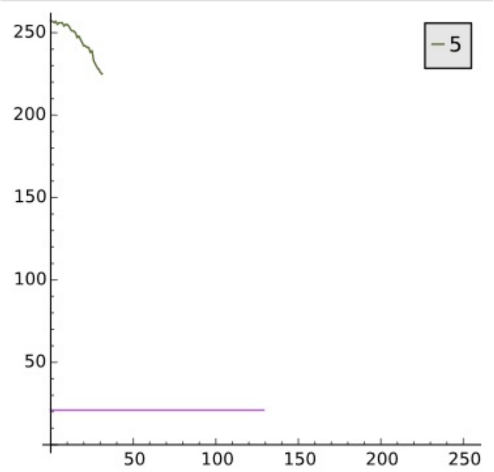

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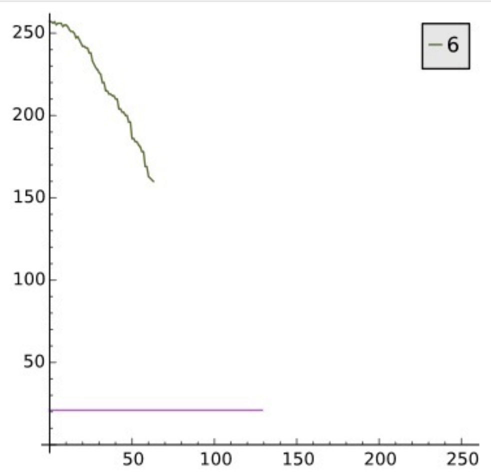

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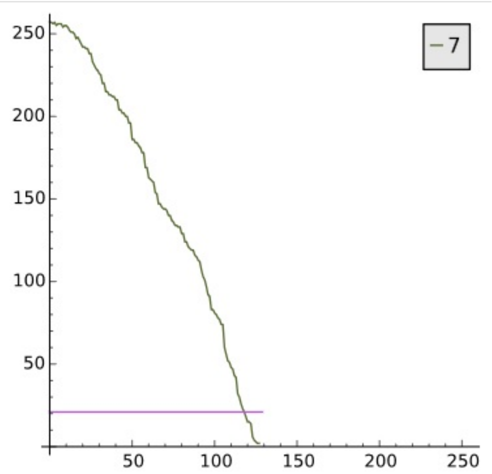

Figure: Precision over the output

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Different way of representing the p-adics

Another take on the computation

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- In the previous computation, we start with some given approximations of g, h, u_0 and try **to follow** the algorithm for the exact counterparts of g, h, u_0. This is somehow **much stronger** than our desire: computing a good approximate solution.
- Another way is then to modify the current g, h, u_0 **at each step**, in a consistent way, so as to keep on getting better approximate solutions.
- A third way here will be to work entirely in $\mathbb{Z}/p^\kappa\mathbb{Z}$.

New framework

In this new computation, we consider h as given, and not varying for the lemma.

Lemma

Let $Y : g \mapsto y$ such that $y(0) = 0$ and $y' = gh(y)$. Then,

$$Y'(g) \cdot (\delta g) = h(y) \int \delta g.$$

In concrete terms

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In the end

- $u_l' = g h(u_l) \mod (p^k, t^{2^l})$
- $g_l = g \mod p^k$

Final take on the Newton scheme

As a consequence, we can prove that it is harmless to work in $\mathbb{Z}/p^k\mathbb{Z}$ for our computation.

Proposition

We can obtain the solution $\Phi(g,h) \mod (p, t^{n+1})$ *knowing* $g, h \mod (p^{\lfloor \log_p n \rfloor + 1}, t^{n+1})$ *and applying the following iteration:*

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modulo $p^{\lfloor \log_p n \rfloor + 1}$ *and growing order of truncation.*

Timings

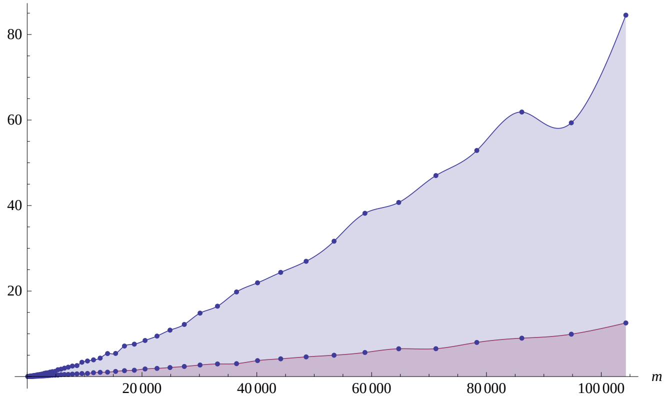

Figure: Timings in seconds, measured on a laptop, of our Algorithm run at precision λ_{old} (upper curve) and λ_{new} (lower curve) in order to compute an approximation modulo $(5, t^{4m+1})$ of some given m-isogenies.

Speedup

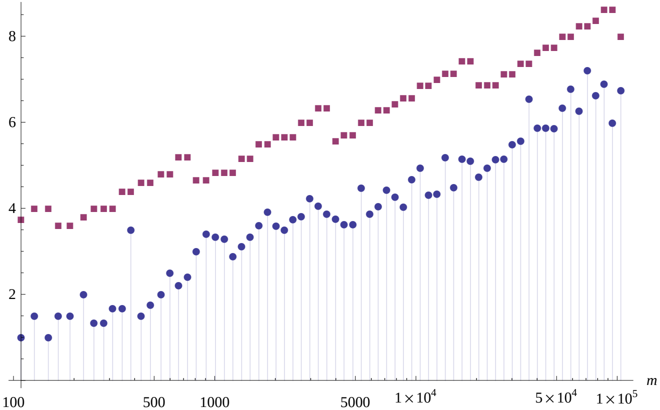

Figure: Practical speedup obtained with the new precision analysis compared with the theoretical improvement (m-axis in logarithmic scale). (■) is the ratio on precisions, (●) is the actual speedup.

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- Can attain **optimal loss** in precision for differential equations with separation of variables.
- Future works: higher order and $p = 2$.

References

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Linear Algebra

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Differential equations

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Thank you for your attention

Thanks