

# Matrix-F5 algorithms over finite-precision complete discrete valuation fields

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## ABSTRACT

Let  $(f_1, \dots, f_s) \in \mathbb{Q}_p[X_1, \dots, X_n]^s$  be a sequence of homogeneous polynomials with  $p$ -adic coefficients. Such system may happen, for example, in arithmetic geometry. Yet, since  $\mathbb{Q}_p$  is not an effective field, classical algorithm does not apply.

We provide a definition for an approximate Gröbner basis with respect to a monomial order  $w$ . We design a strategy to compute such a basis, when precision is enough and under the assumption that the input sequence is regular and the ideals  $\langle f_1, \dots, f_i \rangle$  are weakly- $w$ -ideals. The conjecture of Moreno-Socias states that for the grevlex ordering, such sequences are generic.

Two variants of that strategy are available, depending on whether one lean more on precision or time-complexity. For the analysis of these algorithms, we study the loss of precision of the Gauss row-echelon algorithm, and apply it to an adapted Matrix-F5 algorithm. Numerical examples are provided.

## Categories and Subject Descriptors

I.1.2 [Computing Methodologies]: Symbolic and Algebraic Manipulations—*Algebraic Algorithms*

## General Terms

Algorithms, Theory

## Keywords

F5 algorithm, Gröbner bases, Moreno-Socias conjecture,  $p$ -adic algorithm,  $p$ -adic precision

## 1. INTRODUCTION

Ideal study and polynomial system solving are crucial problem in computer algebra, with numerous applications, either theoretical (as in algebraic geometry) or in applied mathematics (as in cryptography). To that intent, Gröbner bases computation is a decisive tool.

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A classical strategy to tackle problems over  $\mathbb{Q}$  consists in reducing it modulo many different primes and then recombine the solutions. In that case, one can choose freely the primes and discard those that lead to inefficient computations. This method applies also to Gröbner bases and leads to the notion of "lucky" primes. Nevertheless, the advent of arithmetic geometry has seen the emergence of questions that are purely local (*i.e.* where the prime  $p$  is fixed at the very beginning and one can not vary it). As an example, one can cite the recent work of Caruso and Lubicz [4] who gave an algorithm to compute lattices in some  $p$ -adic Galois representations. A related question is the study of  $p$ -adic deformation spaces of Galois representations. Since the work of Taylor and Wiles [26], we know that these spaces play a crucial role in many questions in number theory. Being able to compute such spaces appears then as an interesting question of algorithmics and require the use of purely  $p$ -adic Gröbner bases. Yet, no practical survey of Gröbner bases over  $p$ -adic fields are actually available. This motivates our study.

In this document, we present a way to deal with Gröbner bases for ideals of  $\mathbb{Q}_p[X_1, \dots, X_n]$  and  $\mathbb{F}_q((t))[X_1, \dots, X_n]$  with a strong assumption on their structure that assure numerical stability. In that case, we provide a matrix-F5 algorithm to compute an approximate Gröbner bases of such an ideal, while being able to certify the leading monomials of the ideal.

**Related works.** In the last few decades, the need for approximate Gröbner bases for computation over floating-point numbers has risen many studies. Sasaki and Kako provide in [20] [21] a wonderful introduction to this topic, by classifying the cancellation that might happen when handling floating-point number. Shirayanagi & Sweedler [24], Kondratyev, Stetter & Winkler [12], Nagasaka [15], Stetter [25], Traverso & Zanoni [27], Faugère & Liang [8] and many more have contributed to this topic. Yet, their point of view was always that of floating-point, whose behavior is not identical to that of  $\mathbb{Q}_p$  or formal series.

Meanwhile, a  $p$ -adic approach to Gröbner bases over  $\mathbb{Q}$  has been studied by Winkler [28], Pauer [17], Gräbe [11], Arnold [1], and Renault and Yokoyama [18]. Yet, their works all rely on the choice of a specific  $p$ , adapted to the problem over  $\mathbb{Q}$  or  $\mathbb{Z}$  they are interested in.

**Main results.** For  $K = \mathbb{Q}_p$  or  $\mathbb{F}_p((t))$ , and  $R = \mathbb{Z}_p$  or  $\mathbb{F}_p[[t]]$  respectively, polynomials in  $K[X_1, \dots, X_n]$  or  $R[X_1, \dots, X_n]$  can only be handled with finite precision over the coefficients. Let  $w$  be a monomial ordering and let  $f = (f_1, \dots, f_s) \in$

$R^s$  be homogeneous polynomials satisfying the two structure hypotheses:

- **H1:**  $(f_1, \dots, f_s)$  is a regular sequence.
- **H2:** the  $\langle f_1, \dots, f_i \rangle$  are weakly- $w$ -ideals (see Definition 7).

These hypotheses ensure a "continuity" property: in a neighborhood of  $f$  satisfying **H1** and **H2**, the application mapping a sequence to its reduced Gröbner basis is continuous. Hence, around such an  $f$ , one can safely work with approximations. On the opposite, if **H1** or **H2** is relaxed, the continuity is no longer guaranteed (see Subsection 3.5), which means that the computation can not be achieved with approximated inputs. More precisely, under our structure hypotheses, we exhibit an explicit precision

$$\text{prec}_{MF5}(\{f_1, \dots, f_s\}, D, w),$$

essentially given by minors of the Macaulay matrices defined by  $f$ , such that approximations of  $f$  up to  $\text{prec}_{MF5}$  determine well-defined approximation of Gröbner bases, compatible with the precision and with unambiguous leading terms. We provide in Definition 6 a suitable notion of approximate Gröbner bases regarding to finite-precision coefficients. We define an approximate  $D$ -Gröbner basis accordingly. To compute such  $D$ -Gröbner bases, we define in Algorithm 2 the weak Matrix-F5 algorithm, with the following result:

**THEOREM 1.1.** *Let  $(f_1, \dots, f_s) \in K[X_1, \dots, X_n]^s$  be homogeneous polynomials satisfying **H1** and **H2**. Let  $(f'_1, \dots, f'_s)$  be approximations of the  $f_i$ 's with precision  $m$  on the coefficients. Then, if  $m$  is large enough, an approximate  $D$ -Gröbner basis of  $(f'_1, \dots, f'_s)$  regarding to  $w$  is well-defined. It can be computed by the weak Matrix-F5 algorithm.*

Moreover, if the  $f_i$ 's are in  $R[X_1, \dots, X_n]$ , then  $m \geq \text{prec}_{MF5}$  is enough, and the loss in precision is upper-bounded by  $\text{prec}_{MF5}$ .

The complexity is in  $O\left(sD\binom{n+D-1}{D}^3\right)$  operations in  $R$  at precision  $m$ , as  $D \rightarrow +\infty$ .

We remark that the conjecture of Moreno-Socias implies that sequences satisfying **H1** and **H2** for the grevlex ordering are generic. We also remark that operations in  $R = \mathbb{Z}_p$  or  $\mathbb{F}_p[[t]]$  at precision  $m$  can be computed, by usual algorithms, in  $\tilde{O}(m \log p)$  binary operations.

We explain in Section 4 why  $\prec_{MF5}$  is not sharp, along with numerical examples.

Finally, if one lean more on precision than time-complexity, we show in Theorem 3.5 that, under the assumptions **H1** and **H2** and the  $f_i$ 's in  $R[X_1, \dots, X_n]$ , we can drop the F5 criterion in order to obtain a smaller sufficient precision for an approximate Gröbner basis to be computed:  $\text{prec}_{Mac}$ , see Definition 10. Time-complexity is then in  $O\left(s^2 D \binom{n+D-1}{D}^3\right)$  operations in  $R$  at precision  $m$ , as  $D \rightarrow +\infty$ .

**Structure of the paper.** In Section 2, we explain the setting of our paper: finite-precision complete discrete-valuation fields, and analyze the Gaussian row-echelon algorithm when performed over such fields. Section 3 applies this analysis to the study of the Matrix-F5 algorithm. We then provide and analyze a weak Matrix-F5 algorithm, and a variant for precision-efficiency. Finally, Section 4 provides some experimental examples.

## 2. FINITE-PRECISION CDVF AND ROW-ECHELON FORM COMPUTATION

The objective of this Section is first to introduce finite-precision complete discrete-valuation fields. We study the behavior of the precision when performing elementary operations, and from it, derive an analysis of the loss in precision when performing Gaussian row-echelon form computation

### 2.1 Setting

Throughout this paper,  $K$  is a field with a discrete valuation  $\text{val}$  such that  $K$  is complete with respect to the norm defined by  $\text{val}$ . We denote by  $R = O_K$  its ring of integers,  $m_K$  its maximal ideal and  $k = O_K/m_K$  its fraction field. We denote by CDVF (complete discrete-valuation field) such a field. We refer to Serre's Local Fields [23] for an introduction to such fields. Let  $\pi \in R$  be a uniformizer for  $K$  and let  $S_K \subset R$  be a system of representatives of  $k = O_K/m_K$ . All numbers of  $K$  can be written uniquely under its  $\pi$ -adic power series development form:  $\sum_{k \geq l} a_k \pi^k$  for some  $l \in \mathbb{Z}$ ,  $a_k \in S_K$ .

The case that we are interested in is when  $K$  might not be an effective field, but  $k$  is (*i.e.* there are constructive procedures for performing rational operations in  $k$  and for deciding whether or not two elements in  $k$  are equal). Symbolic computation can then be performed on truncation of  $\pi$ -adic power series development. We will denote by finite-precision CDVF such a field, and finite-precision CDVR for its ring of integers. Classical examples of such CDVF are  $K = \mathbb{Q}_p$ , with  $p$ -adic valuation, and  $\mathbb{Q}[[X]]$  or  $\mathbb{F}_q[[X]]$  with  $X$ -adic valuation. We assume that  $K$  is such a finite-precision CDVF.

Let  $A = K[X_1, \dots, X_n]$ , and  $w$  a monomial order on  $A$ . Let  $B = R[X_1, \dots, X_n]$ . We denote by  $A_d$  the degree- $d$  homogeneous polynomials of  $A$ , and when  $u = (u_1, \dots, u_n) \in \mathbb{Z}_{\geq 0}^n$ , we write  $X^u$  for  $X_1^{u_1} \dots X_n^{u_n}$ . If  $P \in A$  is an homogeneous polynomials, we denote by  $|P|$  its degree.

### 2.2 Precision over $R$ and its behavior

Elements of  $K$  can be symbolically handled only up to a truncation of their  $\pi$ -adic power series development. Therefore, we manipulate quantities of the form  $\sum_{i=k}^{d-1} a_i \pi^i + O(\pi^d)$ , where  $O(\pi^d)$  denotes  $\pi^d R$ .

**Definition 1.** To study the precision on an approximation of a number in  $K$ , we define the **order** (or absolute precision) of  $x = \sum_{i=k}^{d-1} a_i \pi^i + O(\pi^d)$  to be  $d$ .

The number of significant digits of  $x$  would be a much more involved but as least as interesting object to study.

We can track the behavior of the order when performing elementary operations. For this, let  $n_0 < m_0$ ,  $n_1 < m_0$  be integers, and  $\varepsilon = \sum_{j=0}^{m_0-n_0-1} a_j \pi^j$ ,  $\mu = \sum_{j=0}^{m_1-n_1-1} b_j \pi^j$ , with  $a_j, b_j \in S_K$ , and  $a_0, b_0 \neq 0$ . It is then well-known that,

$$(\varepsilon \pi^{n_0} + O(\pi^{m_0})) + (\mu \pi^{n_1} + O(\pi^{m_1})) = \varepsilon \pi^{n_0} + \mu \pi^{n_1} + O(\pi^{\min(m_0, m_1)}),$$

and consequently, the addition of two number know up to order  $n$  is known up to order  $n$ . Similar formulae exist for all elementary operations. We only use the following:

**PROPOSITION 2.1 (DIVISION).**

$$\frac{\varepsilon \pi^{n_1} + O(\pi^{m_1})}{\mu \pi^{n_0} + O(\pi^{m_0})} = \varepsilon \mu^{-1} \pi^{n_1 - n_0} + O(\pi^{\min(m_1 - n_0, m_0 + n_1 - 2n_0)}).$$

As a consequence, we can already see why finite-precision CDVF have a very different behavior than floating-point numbers: if  $a = x + O(\pi^n)$  and  $b = y + O(\pi^n)$  are elements of  $K$  known up to the order  $n$ , then  $a + b = (x + y) + O(\pi^n)$  is known up to the order  $n$ . Because of round-off errors, this does not happen with floating-point numbers.

## 2.3 The Gaussian row-echelon form algorithm

We now apply Lemma 2.1 to the study of Gaussian row-echelon form computation. We first begin with recalling in Algorithm 1 what we mean with row-echelon form and Gaussian elimination.

*Definition 2.* Let  $M$  be an  $n \times m$  matrix. Let  $\text{ind}_M : \{1, r\} \rightarrow \mathbb{Z}_{\leq 0} \cup \{\infty\}$  map  $i$  to the index of the columns of the first non-zero entry on the  $i$ -th row of  $M$ . Then  $M$  is said to be under row-echelon form if the index function is strictly increasing.

$M$  is said to be under row-echelon form up to permutation if there exists  $P$  a permutation matrix such that  $PM$  is under row-echelon form.

**Algorithm 1:** The Gaussian elimination algorithm

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input :  $M$ , an  $n \times m$  matrix.
output:  $\widetilde{M}$ , a row-echelon form of  $M$ , up to
           permutation.

begin
   $\widetilde{M} \leftarrow M;$ 
  if  $n_{\text{col}} = 1$  or  $n_{\text{row}} = 0$  or  $M$  has no non-zero
  entry then
    | Return  $\widetilde{M}$ ;
  else
    Find the coefficient  $M_{i,1}$  on the first column
    with the smallest valuation;
    Swap rows to put it in first row;
    By pivoting with the first row, eliminate the
    coefficients of the other rows on the first
    column;
    Proceed recursively on the submatrix
     $\widetilde{M}_{i \geq 2, j \geq 2};$ 
    Return  $\widetilde{M}$ ;

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We emphasize that when eliminating coefficients with the pivot, we produce real zeroes, and not some  $O(\pi^k)$ . Otherwise, the resulting matrix is not under row-echelon form (up to permutation).

## 2.4 How to pivot

We now make clear how one can pivot and eliminate coefficients.

**PROPOSITION 2.2 (PIVOTING).** Let  $n_0 \leq n_1 < n$  be integers, and  $\varepsilon = \sum_{j=0}^{n-n_1-1} a_j \pi^j$ ,  $\mu = \sum_{j=0}^{n-n_0-1} b_j \pi^j$ , with  $a_j, b_j \in S_K$ , and  $a_0, b_0 \neq 0$ .

To put a “real zero” on the coefficient  $M_{i,j} = \varepsilon \pi^{n_1} + O(\pi^n)$ , we eliminate it with a pivot  $\text{piv} = \mu \pi^{n_0} + O(\pi^n)$  on row  $L$ . This can be performed by the following operation on the  $i$ -th row  $L_i$ :

$$L_i \leftarrow L_i - \frac{M_{i,j}}{\text{piv}} L = L_i + (\varepsilon \mu^{-1} \pi^{n_1-n_0} + O(\pi^{n-n_0})) L,$$

along with the symbolic operation  $M_{i,j} \leftarrow 0$ .

**PROOF.** The symbolic operations  $M_{i,j} \leftarrow 0$  is just a part of the symbolic operation  $L_i \leftarrow L_i - \frac{M_{i,j}}{\text{piv}} L$ . Yet, for any other coefficient of  $L_i$ , symbolic computation is not relevant and what is performed is  $L_i + (\varepsilon \mu^{-1} \pi^{n_1-n_0} + O(\pi^{n-n_0})) L$ .

Indeed, we prove that  $\frac{M_{i,j}}{\text{piv}} = \varepsilon \mu^{-1} \pi^{n_1-n_0} + O(\pi^{n-n_0})$ .

This is a direct consequence of Proposition 2.1:  $\frac{M_{i,j}}{\text{piv}} = \frac{\varepsilon \pi^{n_1} + O(\pi^n)}{\mu \pi^{n_0} + O(\pi^n)}$ , and therefore  $\frac{M_{i,j}}{\text{piv}} = \varepsilon \mu^{-1} \pi^{n_1-n_0} + O(\pi^{n-n_0})$ , since  $\min(n + n_1 - 2n_0, n - n_0, n + n - 2 * n_0) = n - n_0$ .

□

## 2.5 Gaussian row-echelon form computation

We are now able to track the loss of precision when performing Algorithm 1 to compute a row echelon form of a matrix. The result is the following:

**THEOREM 2.3.** Let  $M$  be a matrix  $n \times m$  ( $0 \leq n \leq m$ ) with coefficients in  $R$  all known with absolute precision  $k \geq 0$  and such that its principal minor  $\Delta = \det((M_{i,j})_{1 \leq i \leq n, 1 \leq j \leq n})$  satisfies  $\text{val}(\Delta) < k$ .

Then, the maximum loss of absolute precision while performing Gaussian row-reduction on  $M$  can be upper-bounded by  $\text{val}(\Delta)$ .

**PROOF.** To prove this result, we first study the pivoting process, and then conclude by induction on the number of rows.

When performing the Gauss row-echelon form computation, we first look for the coefficient  $M_{i,1}$  on the first column with the smallest valuation. Then, we put it (via permutation of rows) to the first row,  $L_1$ . We denote it by  $\text{piv}$ , and let  $n_1$  be its valuation.

As in Proposition 2.2, we then pivot all the rows  $L_i$  below the first one:  $L_i \leftarrow L_i - \frac{M_{i,1}}{\text{piv}} L_1$ , and if we denote by  $M^{(1)}$  the resulting matrix, then the coefficients of the sub-matrix  $M_{2 \leq i \leq n, 2 \leq j \leq m}^{(1)}$  are known up to  $O(\pi^{n-n_1})$ .

We then proceed recursively with the pivoting process on this sub-matrix.

We first remark that the result is clear for matrices with only  $n = 1$  rows. We also remark that in the previous pivoting process, the operations performed on the rows change the principal minor only up to a sign, and we have  $\Delta = \pm \text{piv} \times \det M_{2 \leq i \leq n, 2 \leq j \leq n}^{(1)}$ .

Then the result is clear by induction on  $n$  the number of columns. □

## 2.6 A more refined result

In the following section, we will apply this result on row-echelon form computation to study the computation of Gröbner bases, but beforehand, a more sophisticated result is available if one consider matrices with possibly more rows than columns:

**PROPOSITION 2.4.** Let  $M$  be a matrix  $n \times m$  ( $0 \leq n, m$ ) with coefficients in  $R$  all known with absolute precision  $k \geq 0$ . Let  $l \leq m$  be such that there is an  $l$ -minor on the  $l$  first columns  $C_1, \dots, C_l$ , with valuation strictly less than  $k$ .

Let  $\Delta$  be the product of the pivots of the Gaussian row-reduction of  $M$  up to column  $l$ .

Then, the maximum loss of absolute precision while performing Gaussian row-reduction on  $M$ , up to the  $l$ -th column, can be upper-bounded by  $\text{val}(\Delta)$  and moreover,  $\text{val}(\Delta)$  attains the smallest valuation of an  $l$ -minor on  $(C_1, \dots, C_l)$ .

PROOF. This comes from the following fact: in the ring of integers of a complete discrete valuation ring, an ideal  $I$  is generated by any of its element whose valuation attains  $\min(\text{val}(I))$ .

Here, if we define  $I_{\text{minor}}$  to be the ideal in  $R$  generated by the  $l$ -minors on  $(C_1, \dots, C_l)$ , then  $I_{\text{minor}}$  remains unchanged by any of the operations on the row of  $M$  performed during the Gaussian row-echelon form computation (the matrices of these operations all are invertible over  $R$ ).

Once the row-echelon form computation is completed, there is only one non-zero  $l$ -minor on  $(\tilde{C}_1, \dots, \tilde{C}_l)$  (the  $l$  first columns of  $\tilde{M}$ ), and its value is the product  $\Delta$  of the pivots chosen during the computation.

Therefore,  $\Delta$  generates  $I_{\text{minor}}$ , and attains  $\min(\text{val}(I))$ .  $\square$

As a consequence, Gaussian reduction on such a matrix  $M$  up to column  $l$  provide the choice of pivots which yields the smallest upper-bound of the loss in precision.

### 3. MATRIX F5 ALGORITHM AND PRECISION ISSUES

In this section, we show that our analysis of the loss in precision during Gauss reduction can be applied to understand how we can compute Gröbner bases of some ideals in  $K[X_1, \dots, X_n]$ . Our main tool will be Faugère's Matrix F5 algorithm in a slightly modified version, and to that intent, we first describe the idea of the Matrix F5 algorithm, in a general setting.

#### 3.1 Matrix-F5

The main reference concerning Matrix-F5 is Bardet's PhD thesis [2].

We first recall some basic facts about matrix-algorithm to compute Gröbner bases, and present the Matrix-F5 algorithm.

**Definition 3.** Let  $B_{n,d} = (x^{d_i})_{1 \leq i \leq \binom{n-1}{n+d-1}}$  be the monomials of  $A_d$ , ordered decreasingly regarding to  $\omega$ . Then for  $f_1, \dots, f_s \in A$  homogeneous polynomials, with  $|f_i| = d_i$ , and  $d \in \mathbb{N}$ , we define  $\text{Mac}_d(f_1, \dots, f_s)$  to be the following matrix:

$$\begin{array}{ccccccccc} x^{d_1} & > & \dots & > & \dots & > & x^{\binom{n+d-1}{n-1}} \\ & & & & & & \\ \begin{matrix} x^{\alpha_{1,1}} f_1 \\ \vdots \\ x^{\alpha_{1,\binom{n+d-d_1-1}{n-1}}} f_1 \\ x^{\alpha_{2,1}} f_2 \\ \vdots \\ x^{\alpha_{s,\binom{n+d-d_s-1}{n-1}}} f_s \end{matrix} & & & & & & \left[ \begin{array}{c} * \\ \vdots \\ * \end{array} \right] \end{array}$$

with  $x^{\alpha_{i,j}} \in B_{n,d-d_i}$ . The rows of the matrix  $\text{Mac}_d(f_1, \dots, f_s)$  are the polynomials  $x^{\alpha_{i,j}} f_i$  written in the basis  $B_{n,d}$  of  $A_d$ .

We note that  $\text{Im}(\text{Mac}_d(f_1, \dots, f_s)) = I \cap A_d$ , and the first non-zero coefficient of a row of  $\text{Mac}_d(f_1, \dots, f_s)$  is the leading coefficient of the corresponding polynomial.

**THEOREM 3.1** (LAZARD [13]). *For an homogeneous ideal  $I = (f_1, \dots, f_s)$ ,  $f_1, \dots, f_s$  is a Gröbner basis of  $I$  if and only if: for all  $d \in \mathbb{N}$ ,  $\text{Mac}_d(f_1, \dots, f_s)$  contains an echelon basis of  $\text{Im}(\text{Mac}_d(f_1, \dots, f_s))$ .*

By *echelon basis*, we mean the following

**Definition 4.** Let  $g_1, \dots, g_r$  be homogeneous polynomials of degree  $d$ . Let  $M$  be the matrix whose  $i$ -th row is the row vector corresponding to  $g_i$  written in  $B_{n,d}$ . Then we say that  $g_1, \dots, g_r$  is an *echelon basis* of  $\text{Im}(M)$  if there is a permutation matrix  $P$  such that  $PM$  is under row-echelon form.

From this theorem, it is easy to derive an algorithm to compute Gröbner bases: compute the row-echelon form of all the  $\text{Mac}_d(f_1, \dots, f_s)$ , for varying  $d$ . Faugère's F5 criterion provides a decisive improvement with a way to remove most of the unnecessary computation. One can look at Faugère's original article [7] or Bardet's PhD thesis [2] for an introduction to the F5 criterion, but it can be summed up in the following theorem.

**THEOREM 3.2** (F5 CRITERION). *If  $i \in \llbracket 2, s \rrbracket$ , and if we discard all the rows  $x^\alpha f_j$  of  $\text{Mac}_d(f_1, \dots, f_i)$  such that  $x^\alpha \in \text{LM}(I_{j-1})$ , for all  $j \in \llbracket 2, i \rrbracket$ , and if we denote by  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$  this matrix,  $\text{Im}(\overline{\text{Mac}_d(f_1, \dots, f_i)}) = \text{Im}(\text{Mac}_d(f_1, \dots, f_i))$ . A reduction to zero of a row by elementary operations over the rows leads to a syzygy that is not principal.*

*If  $(f_1, \dots, f_s)$  is a regular sequence, then  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$  is injective, and no reduction to zero can be performed.*

Theorem 3.2 yields the *matrix-F5 algorithm*: compute the row-echelon form of the  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$  sequentially in  $d$  and  $i$ , with the computation of  $\overline{\text{Mac}_{d-d_i}(f_1, \dots, f_{i-1})}$  being enough to apply the F5-criterion on  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$  to provide  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$ . One can build a Gröbner basis of  $I$  by adding to the  $f_i$ 's the polynomials corresponding to the rows of the row-echelon form of  $\overline{\text{Mac}_d(f_1, \dots, f_i)}$  which provide new leading monomial.

Nevertheless, there is no criterion on up to what  $d$  Macaulay matrices should be echelonized. This is why we define  $D$ -Gröbner basis (see for example [9]):

**Definition 5.** Let  $I$  be an ideal of  $A$ ,  $w$  a monomial ordering on  $A$  and  $D$  an integer.

Then  $(g_1, \dots, g_l)$  is a  $D$ -Gröbner basis of  $I$  if for any  $f \in I$ , homogeneous of degree less than  $D$ , there exists  $1 \leq i \leq l$  such that, regarding to  $w$ ,  $\text{LM}(g_i)$  divides  $\text{LM}(f)$ .

Thus, if we perform the Matrix-F5 algorithm up to the degree  $D$  Macaulay matrix  $\text{Mac}_D(f_1, \dots, f_s)$ , we obtain a  $D$ -Gröbner basis.

#### 3.2 Precision issues

We now try to understand what happens when the entries are known only up to finite precision. To that intent, we give a definition of what we expect an approximate Gröbner basis to be.

**Definition 6.** Let  $f_i + \sum_{|u|=d_i} O(\pi^{n_{u,i}})X^u$ ,  $1 \leq i \leq s$ , be approximations of homogeneous polynomials in  $A$ . The  $n_{u,i}$  belong to  $\mathbb{Z}_{\geq 0} \cup \{+\infty\}$ . Then an approximate Gröbner basis, regarding to the monomial ordering  $w$ , of the ideal generated by these polynomials is a finite sequence  $(g_i + \sum_{|u|=|g_i|} O(\pi^{m_{u,i}}))$ ,  $m_{u,i} \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ , of approximation of polynomials such that: for any  $a_{u,i} \in \pi^{n_{u,i}} R$ , there exists some  $b_{u,i} \in \pi^{m_{u,i}} R$  such that the  $g_i + \sum_{|u|=|g_i|} b_{u,i}$ 's form a Gröbner basis, regarding to  $w$ , of the ideal generated by

the  $f_i + \sum_{|u|=d_i} a_{u,i} X^u$ 's. Moreover, we require that if  $X^u$  is a monomial of degree  $|g_i|$  such that  $X^u >_w LM(g_i)$ , then  $a_{u,i} = +\infty$  (and the coefficient of  $X^u$  in  $g_i$  is zero). In other words, we require that the leading monomials of the  $g_i$  does not depend on the approximation.

As seen in the previous section, if the input polynomials form a regular sequence, then all matrices considered in the F5 algorithm are injective. Yet, this is not enough in order to be able to certify that we get an approximate Gröbner bases.

For example, the injective matrix,

$$\begin{bmatrix} 1 + O(\pi^{10}) & 1 + O(\pi^{10}) & 1 + O(\pi^{10}) & 0 \\ 1 + O(\pi^{10}) & 1 + O(\pi^{10}) & 1 + O(\pi^{10}) & 1 + O(\pi^{10}) \end{bmatrix},$$

become, after the first step in the computation of its row-echelon form.

$$\begin{bmatrix} 1 + O(\pi^{10}) & 0 & 1 + O(\pi^{10}) & 0 \\ 0 & O(\pi^{10}) & O(\pi^{10}) & 1 + O(\pi^{10}) \end{bmatrix},$$

Yet, there is no way, with only operations on the rows, to decide whether the coefficient of index  $(2, 2)$  is the first non-zero coefficient of the second row or if it is the one of index  $(2, 3)$  or  $(2, 4)$ . Thus, we can not know what is the row-echelon form of the matrix nor the leading monomials of polynomials corresponding to its rows.

Hence, the  $(f_1, \dots, f_s)$  such that the Matrix-F5 algorithm could give a satisfactory answer must have a special shape: when performing the row-echelon computation on the Macaulay matrices, no column without pivot is encountered. With  $w$  being our monomial ordering, an ideal  $\langle f_1, \dots, f_s \rangle$  such that every  $Mac_d(f_1, \dots, f_j)$  satisfies this property is called a  $w$ -ideal.

$w$ -ideal have been heavily studied in the field of generic initial ideal: for example, in [6], Conca and Sidman proved that the generic initial ideal (*i.e.* with a generic change of variable) of a generic set of points in  $\mathbb{P}^r$  is a  $w$ -ideal.

Yet, this is not the generic case as, for example, Pardue showed in [16] that the ideal generated by 6 quadrics in 6 variables is not a grevlex-ideal.

Fortunately, we can study a somehow weaker condition: weakly- $w$ -ideal.

**Definition 7.** Let  $I$  be an ideal in  $A$ , and  $w$  be a monomial order on  $A$ . Then  $I$  is said to be a weakly- $w$ -ideal if, for all leading monomial  $x^\alpha$  of the reduced Gröbner basis of  $I$ , regarding to  $w$ , for all  $x^\beta$  such that  $|\alpha| = |\beta|$  and  $x^\beta > x^\alpha$ ,  $x^\beta$  belongs to  $LM(I)$  (according to  $w$ ).

Moreno-Socias has conjectured that this is generic in the following sense:

**CONJECTURE 1 (MORENO-SOCIAS).** If  $k$  is an infinite field,  $s \in \mathbb{N}$ ,  $d_1, \dots, d_s \in \mathbb{N}$ , then there is a non-empty Zariski-open subset  $U$  in  $R_{d_1} \times \dots \times R_{d_s}$  such that for all  $(f_1, \dots, f_s) \in U$ ,  $I = (f_1, \dots, f_s)$  is a weakly-grevlex ideal.

As a consequence, if Moreno-Socias conjecture holds, sequences satisfying **H1** and **H2** are generic. We refer to Moreno-Socias's PhD Thesis [14] or Pardue's article [16] for an introduction to this conjecture.

**Remark 1.** The choice of grevlex is important: as seen in [16], if we take 3 quadrics  $(f_1, f_2, f_3)$  in  $\mathbb{Q}[X_1, \dots, X_6]$ ,

then generically, the ideal  $I$  they span is neither lex nor weakly-lex ! Indeed, for lex  $X_1 > \dots > X_6$ , the leading monomials of  $I$  in degree 2 are generically  $X_1^2$ ,  $X_1 X_2$  and  $X_1 X_3$ . Yet, in degree 3, we generically have  $X_2^3 \in LT(I)$  and  $X_1 X_6^2 \notin LT(I)$  while  $X_1 X_6^2 > X_2^3$ , and  $X_2^3$  is not a multiple of any of the leading monomial of  $I$  in degree 2,  $X_1^2$ ,  $X_1 X_2$  and  $X_1 X_3$ . Therefore, the ideal generated by 3 generic quadrics in 6 variables is neither lex nor weakly-lex.

### 3.3 The weak matrix-F5 algorithm

We provide in Algorithm 2 the algorithm weak-MF5. We will see in the following subsections that if  $(f_1, \dots, f_s)$  is a sequence of homogeneous polynomials in  $A = K[X_1, \dots, X_n]$  satisfying **H1** and **H2**, and if the  $f_i$ 's are known up to a large enough precision  $O(\pi^k)$  on their coefficients, then weak-MF5 can compute approximate  $D$ -Gröbner bases of  $\langle f_1, \dots, f_s \rangle$ .

---

#### Algorithm 2: The weak-MF5 algorithm

---

```

input :  $F = (f_1, \dots, f_s) \in R[X_1, \dots, X_n]^s$ ,  

         homogeneous polynomials with respective  

         degrees  $d_1 \leq \dots \leq d_s$ , and  $D \in \mathbb{N}$ ,  

         a term order  $w$ .  

output:  $(g_1, \dots, g_k) \in A^k$ , a  $D$ -Gröbner basis of  

          $Id(F)$ , or error if  $(f_1, \dots, f_k)$  does not  

         satisfy H1, H2 or the precision is not  

         enough

begin
   $G \leftarrow F$ 
  for  $d \in [0, D]$  do
     $\widetilde{\mathcal{M}}_{d,0} := \emptyset$ 
    for  $i \in [1, s]$  do
       $\mathcal{M}_{d,i} := \widetilde{\mathcal{M}}_{d,i-1}$ 
      for  $\alpha$  such that  $|\alpha| + d_i = d$  do
        if  $x^\alpha$  is not the leading term of a row  

          of  $\widetilde{\mathcal{M}}_{d-d_i, i-1}$  then
          Add  $x^\alpha f_i$  to  $\mathcal{M}_{d,i}$ 
    Compute  $\widetilde{\mathcal{M}}_{d,i}$ , the row-echelon form of  

     $\mathcal{M}_{d,i}$ , up to the first column with no  

    non-zero pivot
    Replace the remaining rows of  $\widetilde{\mathcal{M}}_{d,i}$  by  

    multiple of rows of  $\widetilde{\mathcal{M}}_{d-1,i}$ , so as to  

    obtain an injective matrix in row-echelon  

    form  $\widetilde{\mathcal{M}}_{d,i}$ .
    if  $\widetilde{\mathcal{M}}_{d,i}$  could not be completed then
      Return "Error", the ideals are not  

      weakly- $w$ , the sequence is not regular,  

      or the precision is not enough".
    else
      Add to  $G$  all the rows of  $\widetilde{\mathcal{M}}_{d,i}$  with a  

      new leading monomial.
  Return  $G$ 

```

---

**Remark 2.** At the beginning of the second **for** loop, the classical Matrix-F5 uses  $\mathcal{M}_{d,i} := \widetilde{\mathcal{M}}_{d,i-1}$  instead of  $\mathcal{M}_{d,i} := \widetilde{\mathcal{M}}_{d,i-1}$ . The former is faster ( $\widetilde{\mathcal{M}}_{d,i-1}$  is already under row-echelon form), but we have chosen the latter since the analysis of the precision is simpler, and more efficient.

*Remark 3.* Instead of adding to  $G$  all the rows of  $\widetilde{\mathcal{M}}_{d,i}$  with a new leading monomial, it is enough to add the rows whose leading monomial is not a multiple of the leading monomial of a polynomial in  $G$ , therefore, we will directly obtain at the end a minimal Gröbner base.

### 3.3.1 Correctness

We prove here that, regarding to symbolic computation (*i.e.* disregarding precision issues), the weak-MF5 algorithm indeed compute  $D$ -Gröbner bases.

**PROPOSITION 3.3.** *Let  $(f_1, \dots, f_s) \in B^s$  be a sequence of homogeneous polynomials satisfying **H1** and **H2**. Then for any  $D \in \mathbb{Z}_{\geq 0}$ , the result of weak-MF5( $(f_1, \dots, f_s), D$ ) is a  $D$ -Gröbner basis of the ideal  $I$  generated by  $(f_1, \dots, f_s)$ . If  $(f_1, \dots, f_s)$  does not satisfy **H1** or **H2**, an error is raised.*

**PROOF.** Let  $(f_1, \dots, f_s) \in B^s$ , homogeneous of degree  $d_1 \leq \dots \leq d_s$  and satisfying **H1** and **H2**. Let  $\mathcal{M}_{d,i}$  be the matrix built with the F5-criterion at the beginning of the second **for** loop in Algorithm 2, and  $\widetilde{\mathcal{M}}_{d,i}$  be the result at the end of this very same loop of  $\mathcal{M}_{d,i}$  after row-echelon computation and completion with  $\widetilde{\mathcal{M}}_{d-1,i}$ .

Let  $\mathcal{P}(d, i)$  be the proposition:  $\mathcal{M}_{d,i} = \overline{Mac_d(f_1, \dots, f_i)}$ ,  $\widetilde{\mathcal{M}}_{d,i}$  is put under row-echelon form (up to permutation) by Algorithm 2 without raising an error, and  $Im(\widetilde{\mathcal{M}}_{d,i}) = Im(Mac_d(f_1, \dots, f_i))$ . We prove by induction on  $d$  and  $i$  that for any  $d \in \llbracket 0, D \rrbracket$  and  $i \in \llbracket 1, s \rrbracket$ ,  $\mathcal{P}(d, i)$ , holds.

First of all,  $\mathcal{P}(d, i)$  is clear if  $d < d_1$  since the corresponding matrices are empty.

Now, let  $d \in \llbracket d_1, D \rrbracket$  be such that for any  $0 \leq \delta \leq d$  and  $i \in \llbracket 1, s \rrbracket$ ,  $\mathcal{P}(\delta, i)$  is true. We prove  $\mathcal{P}(d, i)$  for all  $i \in \llbracket 1, s \rrbracket$ .

It is clear for  $i = 1$  since the ideal generated by  $f_1$  is monogeneous. Let  $i \in \llbracket 1, s \rrbracket$  be such that for all  $j \in \llbracket 1, i-1 \rrbracket$ ,  $\mathcal{P}(\delta, i)$  is true.

Then, by the induction hypothesis,  $\mathcal{M}_{d,i-1} = \overline{Mac_d(f_1, \dots, f_{i-1})}$ ,  $\widetilde{\mathcal{M}}_{d-d_i, i-1}$  is under row-echelon form (up to permutation) and  $Im(\widetilde{\mathcal{M}}_{d-d_i, i-1}) = Im(Mac_d(f_1, \dots, f_{i-1}))$ . By the F5 criterion (Proposition 3.2), we then indeed have  $\mathcal{M}_{d,i} = Mac_d(f_1, \dots, f_i)$ ,  $\widetilde{\mathcal{M}}_{d,i}$ .

Now, we prove that the completion process can be performed without error. Let us denote by  $x^{\alpha_u}$ , for  $u$  from 1 to  $\binom{n+d-1}{n-1}$ , the monomials of degree  $d$ , ordered decreasingly according to  $w$ , and let  $l$  be the index of the first column without pivot found during the computation of the Gaussian row-echelon form of  $\mathcal{M}_{d,i}$ . Let us denote by  $r_i$ , with  $i$  from 1 to  $l-1$ , the  $l$  polynomials corresponding to rows of  $Mac_d(f_1, \dots, f_s)$  with leading monomial  $x^{\alpha_i}$ . Their leading monomials belong to the  $x^{\alpha_u}$ , with  $u \geq l$ .

Let  $(g_1, \dots, g_r)$  be the reduced Gröbner basis of  $I$  according to  $w$ . Then, since there is no pivot on the column of index  $l$ ,  $x^{\alpha_l}$  is not a monomial of  $LM(I)$ . By definition of a weakly- $w$ -ideal (hypothesis **H2**), this implies that if  $x^{\alpha_u} \in LM(I)$  for some  $u \geq l$ , then  $x^{\alpha_u}$  is not one of the  $LM(g_i)$ . This means that any  $x^{\alpha_u} \in LM(I)$  with  $u \geq l$  is a non-trivial multiple of one of the  $LM(g_i)$ . Such  $x^{\alpha_u}$  is therefore a multiple of a monomial in  $LM(I \cap A_{d-1})$ . As a consequence, since  $Im(\widetilde{\mathcal{M}}_{d-1,i}) = I \cap A_{d-1}$ , and  $\widetilde{\mathcal{M}}_{d-1,i}$  is under row echelon form (up to permutation), then for any  $u \geq l$  such that  $x^{\alpha_u} \in LM(I)$ , there exists a polynomial  $P_u$  corresponding to a row of  $\widetilde{\mathcal{M}}_{d-1,i}$  and  $k_u \in \llbracket 1, n \rrbracket$  such

that  $LM(X_{k_u} P_u) = x^{\alpha_u}$ . With the F5 criterion and **H1**,  $\mathcal{M}_{d,i} = \overline{Mac_d(f_1, \dots, f_i)}$ ,  $\mathcal{M}_{d,i}$  is injective and its number of rows,  $m$ , is exactly the number of monomials in  $LM(I \cap A_d)$ . This implies that the completion process can be executed without error.

Let  $(t_l, \dots, t_m)$  be the rows of  $\widetilde{\mathcal{M}}_{d,i}$  obtained as multiples of rows of  $\widetilde{\mathcal{M}}_{d-1,i}$ . Then the polynomials corresponding to  $(r_1, \dots, r_{l-1}, t_l, \dots, t_m)$  have respectively distinct leading monomials, and therefore,  $\widetilde{\mathcal{M}}_{d,i}$  is under row-echelon form (up to permutation). Finally,  $Im(\widetilde{\mathcal{M}}_{d,i}) \subset I \cap R_d = Im(Mac_d(f_1, \dots, f_i))$  and both have dimension  $m$  over  $K$ . Hence,  $Im(\widetilde{\mathcal{M}}_{d,i}) = Im(Mac_d(f_1, \dots, f_i))$ .

$\mathcal{P}(d, i)$  is then proven. By induction, it is now true for all  $d \in \llbracket 0, D \rrbracket$  and  $i \in \llbracket 1, s \rrbracket$ . As a consequence, the output of Algorithm 2 is indeed a  $D$ -Gröbner basis of  $(f_1, \dots, f_s)$ .

Now, if  $(f_1, \dots, f_s)$  is either not regular or their exist some  $i$  such that  $(f_1, \dots, f_i)$  is not weakly- $w$ , then in the first case, it means some rows of one of  $Mac_d(f_1, \dots, f_i)$  reduce to zero (see [2]). Therefore, the row-echelon computation will encounter a column without pivot and the completion of  $\mathcal{M}_{d,i}$  in an echelon basis will not be possible, raising an error. The second case is similar.  $\square$

*Remark 4.* As seen in the proof, the completion process is only here to ensure that no leading monomial is missing and therefore, we indeed have produced a  $D$ -Gröbner basis. It does not produce new polynomials for the Gröbner basis in development. Had  $K$  been an effective field, then under the hypothesis **H2**, to stop the row-echelon form computation after the first columns without pivot is enough to get the polynomial in a minimal Gröbner basis. One could then ensure that the output is a Gröbner basis by the Buchberger criterion.

### 3.3.2 Termination

Since we restrict to computation of row-echelon form of the Macaulay matrices up to degree  $D$ , there is no termination issue.

Yet, if we want a Gröbner basis instead of a  $D$ -Gröbner basis, one can use the following result: (see [3], [10], [13])

**PROPOSITION 3.4.** *If  $(f_1, \dots, f_s)$  is a regular sequence of homogeneous polynomials in  $A$ . Then, after a generic linear change of variables, the highest degree of elements of a Gröbner basis of  $\langle f_1, \dots, f_s \rangle$  for the grevlex ordering is upper-bounded by the Macaulay bound:  $\sum_{i=1}^s (|f_i| - 1) + 1$ .*

### 3.3.3 Precision

We can now prove Theorem 1.1. Let  $(f_1, \dots, f_s)$  be a sequence of homogeneous polynomials in  $B$  satisfying **H1** and **H2**. To that intent, we first define the  $\Delta_{d,i}$ , which corresponds to the precision sufficient, by Proposition 2.4, to compute the  $\widetilde{\mathcal{M}}_{d,i}$  from  $\mathcal{M}_{d,i}$ .

**Definition 8.** Let  $l_{d,i}$  be the maximum of the  $l \in \mathbb{Z}_{\geq 0}$  such that the  $l$ -first columns of  $\overline{Mac_d(f_1, \dots, f_i)} = \mathcal{M}_{d,i}$  are linearly free. We define

$$\Delta_{d,i} = \min(val(\{\text{minor over the } l_{d,i}\text{-first columns of } \mathcal{M}_{d,i}\})).$$

We can now define  $prec_{MF5}$ .

*Definition 9.* We define the Matrix-F5 precision of  $(f_1, \dots, f_s)$  regarding to  $w$  and  $D$  as:

$$prec_{MF5}(\{f_1, \dots, f_s\}, D, w) = \max_{d \leq D, 1 \leq i \leq s} val(\Delta_{d,i}).$$

With Proposition 2.4 and the Proposition 3.3, this upper-bound is enough to compute the  $\widetilde{\mathcal{M}}_{d,i}$ . Indeed, it is enough to compute the Gaussian row-echelon form of  $\mathcal{M}_{d,i}$  up to column  $l_{d,i}$  and then to complete this matrix with multiples of  $\widetilde{\mathcal{M}}_{d-1,i}$ . This way, either they come from row-reduction or are multiple of rows of  $\widetilde{\mathcal{M}}_{d-1,i}$ , the leading monomials of the rows of  $\widetilde{\mathcal{M}}_{d,i}$  are unambiguous. The fact that the completion process can be successfully completed implies that we are ensured we have obtained an echelon basis for  $Im(\mathcal{M}_{d,i})$ .

Therefore,  $prec_{MF5}(\{f_1, \dots, f_s\}, D, w)$  is enough to compute approximate  $D$ -Gröbner bases by the weak-MF5 algorithm.

To conclude the proof, we remark that, in order order to facilitate the precision analysis, we have assumed that the input polynomials in the algorithm,  $(f_1, \dots, f_i)$ , are in  $B$ . Yet, if the  $(f_1, \dots, f_i)$ , are in  $A$ , one can still replace the  $f_i$ 's by the  $\pi^{l_i} f_i \in A$  (for some  $l_i$ ) and still generate the same ideal. This does not affect **H1** and **H2**, and one can still compute an approximate Gröbner basis if we know the  $f_i$ 's up to a large enough precision. Only our precision bound  $prec_{MF5}$  is no longer available.

### 3.3.4 Complexity

Asymptotically, the complexity to compute a  $D$ -Gröbner basis of  $(f_1, \dots, f_s)$  is the same as the classical MatrixF5 algorithm, that is to say,  $O(sD(n+D-1)^3)$  operations in  $R$  at precision  $m$ , as  $D \rightarrow +\infty$ . One can see [2] and [3] for more about the complexity of the Matrix-F5 algorithm.

## 3.4 Precision versus time-complexity

In order to achieve a better loss in precision for the Gaussian row-echelon form computation, we suggest the following weak-Matrix algorithm:

- Compute the  $\mathcal{M}_{d,i}$  as before, with the F5 criterion;
- Instead of computing the row-echelon form of  $\mathcal{M}_{d,i}$ , one can perform the row-echelon form of the whole  $Mac_d(f_1, \dots, f_i)$ , up to the first column without non-zero pivot;
- Finally, build  $\widetilde{\mathcal{M}}_{d,i}$  by filling  $\mathcal{M}_{d,i}$  with the linearly independent rows found by the previous computation over  $Mac_d(f_1, \dots, f_i)$  and multiples of rows of  $Mac_{d-1,i}$ , into a matrix under row-echelon form.

The following quantity defines a sufficient precision to compute  $D$ -Gröbner bases through this algorithm.

*Definition 10.* Let

$$\square_{d,i} = \min \left( val \left( \left\{ \begin{array}{c} \text{minor over the } l_{d,i}\text{-first} \\ \text{columns of } Mac_d(f_1, \dots, f_i) \end{array} \right\} \right) \right).$$

We define the Macaulay precision of  $(f_1, \dots, f_s)$  regarding to  $w$  and  $D$  as:

$$prec_{Mac}(\{f_1, \dots, f_s\}, D, w) = \max_{d \leq D, 1 \leq i \leq s} \square_{d,i}.$$

Indeed, with Proposition 2.4,  $prec_{Mac}(\{f_1, \dots, f_s\}, D, w)$  is enough to compute approximate  $D$ -Gröbner bases of sequences of homogeneous polynomials satisfying **H1** and **H2**, and it would achieve the best loss in precision that Gaussian row-reduction of Macaulay matrices can attain. We have  $prec_{Mac} \leq prec_{MF5}$ .

Yet, this precision would come with a cost in time-complexity: row-reducing of the full Macaulay matrix  $Mac_d(f_1, \dots, f_i)$  is  $O\left(\binom{n+d-1}{d}^2 \times i \binom{n+d-1}{d}\right)$  (see [3]). This leads to a total time-complexity in  $O\left(s^2 D \binom{n+D-1}{D}^3\right)$  operations in  $R$  at precision  $m$ , as  $D \rightarrow +\infty$ , while when using the F5-criterion, we only need  $O\left(sD \binom{n+D-1}{D}^3\right)$ . To sum up:

**THEOREM 3.5.** *Let  $(f_1, \dots, f_s) \in A^s$  be homogeneous polynomials satisfying **H1** and **H2**. Let  $(f'_1, \dots, f'_s)$  be approximations of the  $f_i$ 's with precision  $m$  on the coefficients. Then, if  $m$  is large enough, an approximate  $D$ -Gröbner basis of  $(f'_1, \dots, f'_s)$  regarding to  $w$  is well-defined. It can be computed by the weak Matrix algorithm.*

*Let  $prec_{Mac} = prec_{Mac}(\{f_1, \dots, f_s\}, D, w)$ . Then, if the  $f_i$ 's are in  $B$ , a precision  $m \geq prec_{Mac}$  is enough, and the loss in precision is upper-bounded by  $prec_{Mac}$ . The complexity is in  $O\left(s^2 D \binom{n+D-1}{D}^3\right)$  operations in  $R$  at precision  $m$ , as  $D \rightarrow +\infty$ .*

## 3.5 Continuity and optimality

We can reinterpret Theorem 1.1 in the following way: the application  $\Phi : A_{d_1} \times \dots \times A_{d_s} \rightarrow \mathcal{P}(A)$  that sends  $f = (f_1, \dots, f_s)$  to the set  $LM(\langle f_1, \dots, f_s \rangle)$  (its initial ideal) is locally constant at any sequence satisfying **H1** and **H2**. Similarly, sending  $f$  to a Gröbner basis by Algorithm 2 is "continuous" in some reasonable sense. It is these properties that allow numerical stability at  $f$ . One could show similarly they still holds for  $K = \mathbb{R}$ . Yet in that case, finding an explicit neighborhood of  $f$  would be much more involved since we could not apply Proposition 2.1 and Theorem 2.3.

Concerning the optimality of the structure hypotheses, we remark that without **H1** or **H2**, the locally-constant property of  $LM$  is not necessarily satisfied. For example, in  $K[X, Y, Z]$ ,  $f = (X + Y, XY + Y^2 + Z^2)$  satisfy **H1** and not **H2**, and one can consider the approximations  $(X + (1 + \pi^n)Y, XY + (1 - \pi^n)Y^2 + Z^2)$ , intersecting any neighborhood of  $f$  but yielding a different  $LM$  than  $LM(\langle f \rangle)$ . Likewise,  $f = (X + Y, X^2 + XY)$  satisfy **H2** and not **H1**, with the same issue.

## 4. IMPLEMENTATION

A toy implementation in Sage [19] of the previous algorithm is available at [http://perso.univ-rennes1.fr/tristan.vaccon/toy\\_F5.py](http://perso.univ-rennes1.fr/tristan.vaccon/toy_F5.py).

The purpose of this implementation was the study of the precision. It is therefore not optimized regarding to time-complexity.

We have experimented the weak-Matrix-F5 algorithm up to degree  $D$  on homogeneous polynomials  $f_1, \dots, f_s$ , of degree  $d_1, \dots, d_s$  in  $\mathbb{Z}_p[X_1, \dots, X_n]$ , with coefficients taken randomly in  $\mathbb{Z}_p$  up to initial precision 30. This experiment is repeated  $n_{exp}$  times, and the monomial ordering was grevlex. **max** denotes the maximal loss in precision noticed on a polynomial in all the  $n_{exp}$  output bases,  $\bar{m}$  the mean loss in precision over all coefficients of the output bases, **gap** is the

maximum of the differences for one experiment between the effective maximal loss in precision and the theoretical bound  $prec_{MF5}$ , and  $\mathbf{f}$  is the number of failures. We present the results in the following array:

$d =$	D	$p$	$n_{exp}$	$\max$	$\bar{m}$	$\text{gap}$	$\mathbf{f}$
[3,4,7]	12	2	30	11	.5	141	0
[3,4,7]	12	7	30	2	0	42	0
[2,3,4,5]	11	2	20	25	2.2	349	3
[2,3,4,5]	11	7	20	5	.3	84	0
[2,4,5,6]	14	2	20	28	3.1	581	3
[2,4,5,6]	14	7	20	14	.4	73	0

These results suggest that the loss in precision is less when working with bigger primes. It seems reasonable since the loss in precision comes from pivots with positive valuation, while, with the Haar measure on  $\mathbb{Z}_p$ , the probability that  $val(x) = 0$  for  $x \in \mathbb{Z}_p$  is  $\frac{p-1}{p}$ . Similarly, it increases when the size of the Macaulay matrices increases.

Concerning the gap between  $prec_{MF5}$  and the effective loss in precision, we remark that  $prec_{MF5}$  derive from Theorem 2.3 which was about dense matrices. A pivot with a big valuation echos into  $prec_{MF5}$ , even though it might generate no loss in precision if there is no non-zero coefficient on its column to eliminate. Hence, Theorem 2.3 does not take into account the sparsity of the Macaulay matrices, which explains why **gap** might be so big compared to **max**.

## 5. FUTURE WORKS

We suggest that if the initial polynomials  $(f_1, \dots, f_s)$  are known up to some precision  $n_0 \gg prec_{MF5}$  and one want to achieve high precision on the result, then Hensel liftings could be used to speed-up the computation, as in [1], [18] or [28]. One would first truncate the  $(f_1, \dots, f_s)$  up to some sufficient precision  $n_1 > prec_{MF5}$ , and compute an approximate reduced  $D$ -Gröbner basis  $(g_1, \dots, g_t)$  up to precision  $m_1$  along with the decomposition  $g_i = \sum_{j=1}^s a_{i,j} + O(\pi^{m_1})$ . Then, one could lift this approximate Gröbner bases up to the desired precision with Hensel lifting applied on the decompositions.

Besides, in order to produce an optimal control on the loss in precision, the differential tracking of precision in [5] seems promising since, more than continuity, our study prove the differentiability (in their sense) of the computation of Gröbner bases at point satisfying **H1** and **H2**.

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