On the computation of stabilizing controllers of multidimensional systems

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Abstract In this paper, we consider the open problem consisting in the computation of stabilizing controllers of an internally stabilizable MIMO multidimensional system. Based on homological algebra and the so-called *Polydisk Nullstellensatz*, we propose a general method towards the explicit computation of stabilizing controllers. We show how homological algebra methods over the ring of structurally stable SISO multidimensional transfer functions can be made algorithmic based on standard *Gröbner basis* techniques over polynomial rings. The problem of computing stabilizing controllers is then reduced to the problem of obtaining an effective version of the Polydisk Nullstellensatz which, apart from a few cases, stays open and will be studied in forthcoming publications.

Keywords: Fractional representation approach, internal stabilization, stabilizing controllers, multidimensional systems, infinite-dimensional systems, computer algebra

1. THE FRACTIONAL REPRESENTATION APPROACH

Within the *fractional representation approach to analysis and synthesis problems* (Curtain et al. (1991); Desoer et al. (1980); Vidyasagar (1985)), A denotes an *integral domain of SISO stable plants* and

$$K = Q(A) = \{n/d \mid 0 \neq d,\, n \in A\}$$

the quotient field of A. Let us give a few examples.

Example 1. (1) Let $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$ be the *open right-half complex plane*. Then, $H_\infty(\mathbb{C}_+)$ is the integral domain formed by all the holomorphic functions in $\mathbb{C}_+$ which are bounded for the norm $\| f \|_\infty := \sup_{s\in\mathbb{C}_+} |f(s)|$. An element $\widehat{h}$ of $H_\infty(\mathbb{C}_+)$ corresponds to the transfer function of a $L_2(\mathbb{R}_+) - L_2(\mathbb{R}_+)$-stable system (Curtain et al. (1991)).

(2) If $\mathbb{R}(s)$ denotes the field of real rational functions in the complex variable s, then $RH_\infty = H_\infty(\mathbb{C}_+) \cap \mathbb{R}(s)$ is the integral domain of asymptotically stable real rational functions (Vidyasagar (1985)).

(3) Let $\mathbb{R}(z_1, \ldots, z_n)_S$ be the integral domain formed by all the real rational functions in the complex variables $z_1, \ldots, z_n$ whose least common denominators have no poles in the *closed unit polydisc* of $\mathbb{C}^n$, namely:
$\overline{\mathbb{D}}^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1,\ i = 1, \ldots, n\}$.
In other words, we have

$$\mathbb{R}(z_1, \ldots, z_n)_S = \left\{ \frac{n(z)}{d(z)} \mid 0 \neq d,\ n \in \mathbb{R}[z], \ \gcd(d, n) = 1,\ V(\langle d \rangle) \cap \overline{\mathbb{D}}^n = \emptyset \right\},$$

where $V(\langle d \rangle) := \{z \in \mathbb{C}^n \mid d(z) = 0\}$ is the algebraic set defined by the complex zeros of the polynomial d.

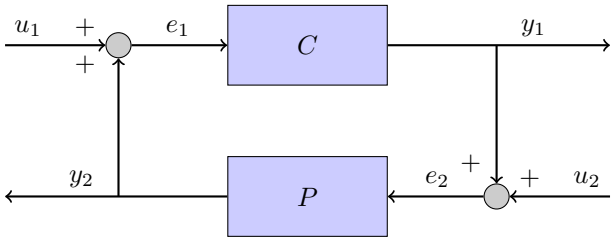

Figure 1. Closed-loop system

Within the fractional representation approach, a SISO plant p is defined by an element of K, i.e., $p = n/d$, where $0 \neq d$, $n \in A$. Then, p is *stable*, more precisely, A*-stable*, if $p \in A$. Hence, $p \in K \setminus A$ is a plant which is not A-stable.

Let $P \in K^{q\times r}$ be a MIMO plant and $C \in K^{r\times q}$ a controller. With the notations of Figure 1, we get $(e_1^T \ \ e_2^T)^T = H(P, C)\,(u_1^T \ \ u_2^T)^T$, where:

$$\begin{aligned} H(P, C) &:= \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \\ &= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}. \end{aligned}$$

If $H(P,C) \in A^{p\times p}$, then one can show that all the entries of any transfer matrix in the closed-loop system shown in Figure 1 are A-stable (see, e.g., Vidyasagar (1985)).

Definition 1. (Vidyasagar (1985)). A plant $P \in K^{q\times r}$ is said to be *internally stabilizable* if there exists $C \in K^{r\times q}$ such that $H(P,C) \in A^{p\times p}$. Then, C is called a *stabilizing*

controller of P. The set of all the stabilizing controllers of P is denoted by $\mathrm{Stab}(P)$.

This paper aims to present an approach towards the computation of stabilizing controllers for a stabilizable plant over an integral domain A of SISO stable plants. Then, based on computer algebra methods (e.g., *Gröbner basis* techniques (Eisenbud (1995))), we detail the main steps of this general approach that must be made effective for the ring $\mathbb{R}(z_1,\ldots,z_n)_S$, i.e., for the explicit computation of stabilizing controllers for multidimensional systems.

If $R \in \mathbb{A}^{q\times p}$, where $\mathbb{A}$ is a ring (e.g., A, K), then we can define the *A-homomorphisms* (i.e., A-linear maps)

$$\begin{array}{rcl} .R : \mathbb{A}^{1\times q} & \longrightarrow & \mathbb{A}^{1\times p} \\ \lambda = (\lambda_1 \ \ldots \ \lambda_q) & \longmapsto & \lambda R, \end{array} \qquad \begin{array}{rcl} R. : \mathbb{A}^{p\times 1} & \longrightarrow & \mathbb{A}^{q\times 1} \\ \eta = (\eta_1 \ \ldots \ \eta_p)^T & \longmapsto & R\eta, \end{array}$$

and define the following $\mathbb{A}$-modules:

$$\begin{array}{l} \ker_{\mathbb{A}}(.R) = \{\lambda \in \mathbb{A}^{1\times p} \mid \lambda R = 0\}, \\ \mathrm{im}_{\mathbb{A}}(.R) = \mathbb{A}^{1\times q} R = \{\lambda \in \mathbb{A}^{1\times p} \mid \exists\, \mu \in \mathbb{A}^{1\times q} : \lambda = \mu R\}, \\ \mathrm{im}_{\mathbb{A}}(R.) = R\,\mathbb{A}^{p\times 1} = \{\eta \in A^{p\times 1} \mid \exists\, \xi \in \mathbb{A}^{q\times 1} : \eta = R\xi\}, \\ \ker_{\mathbb{A}}(R.) = \{\eta \in \mathbb{A}^{p\times 1} \mid R\eta = 0\}. \end{array}$$

For information on module theory and homological algebra, we refer the reader to Rotman (2009).

Let us state again a few definitions.

Definition 2. (Quadrat (2003, b); Vidyasagar (1985)). (1) A *fractional representation* of $P \in K^{q\times r}$ is defined by $P = D^{-1}N = \widetilde{N}\,\widetilde{D}^{-1}$, where $R = (D \quad -N) \in A^{q\times p}$ and $\widetilde{R} = \begin{pmatrix} \widetilde{N}^T & \widetilde{D}^T \end{pmatrix}^T \in A^{p\times r}$, with $p = q + r$.

(2) A fractional representation $P = D^{-1}N$ of P is said to be a *weakly left coprime factorization* if:
$$\left(K^{1\times q} R\right) \cap A^{1\times p} = A^{1\times q} R.$$

(3) A fractional representation $P = D^{-1}N$ of P is said to be a *left coprime factorization* if there exist $X \in A^{q\times q}$ and $Y \in A^{r\times q}$ such that $DX - NY = I_q$.

(4) A fractional representation $P = D^{-1}N$ of P is said to be a *weakly right coprime factorization* if we have:
$$\left(K^{1\times r}\,\widetilde{R}^T\right) \cap A^{1\times p} = A^{1\times r}\,\widetilde{R}^T.$$

(5) A fractional representation $P = \widetilde{N}\,\widetilde{D}^{-1}$ of P is said to be a *right coprime factorization* if there exist $\widetilde{X} \in A^{r\times r}$ and $\widetilde{Y} \in A^{r\times q}$ such that $-\widetilde{Y}\,\widetilde{N} + \widetilde{X}\,\widetilde{D} = I_r$.

(6) A fractional representation $P = D^{-1}N = \widetilde{N}\,\widetilde{D}^{-1}$ is a *doubly coprime factorization* of P if $P = D^{-1}N$ is a left coprime factorization and $P = \widetilde{N}\,\widetilde{D}^{-1}$ is a right coprime factorization such as:
$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_p.$$

As explained in Quadrat (2003, b, 2006, a), the existence of a left/right coprime factorization implies internally stablizability. In other words, the existence of a left/right coprime factorization of P is a sufficient condition for internal stabilizability but not a necessary one over a general integral domain A. The equivalence occurs for the so-called *projective-free rings* (Quadrat (2003, b, 2006, a)) such as RH_∞ and $H_\infty(\mathbb{C}_+)$. In Quadrat (2006, a), the equivalence between internal stabilization and the existence of coprime factorization was proved for $\mathbb{R}(z_1,\ldots,z_n)_S$.

In the rest, we assume that P is an internally stabilizable plant which does not necessarily admit left/right coprime factorization.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR INTERNAL STABILIZABILTY

In Quadrat (2003, a), Quadrat (2003, b), Quadrat (2006, a) and Quadrat (2006, b), necessary and sufficient conditions for internal stabilizabilty were obtained within two different mathematical approaches: the *algebraic analysis* approach and the *lattice* one. In this paper, we prove how the condition obtained in the algebraic analysis approach yields the condition obtained in the lattice approach. Due to the length constraint of this paper, the converse implication will be studied in a forthcoming publication.

2.1 The lattice approach

Let $P \in K^{q\times r}$, $P = D^{-1}N$ be a fractional representation of P, and $R = (D \quad -N) \in A^{q\times p}$, where $p = q + r$. We can introduce the so-called *lattice* $\mathcal{P} = R\,A^{p\times 1}$ of $K^{q\times 1}$. For more information on the theory of lattices applied to stabilization problems, see Quadrat (2006, a). To the lattice $\mathcal{P}$, we can associate a *dual lattice* defined by:
$$A : \mathcal{P} := \left\{\mu \in K^{1\times q} \mid \mu R \in A^{1\times p}\right\} \subseteq K^{1\times q}.$$
We can now state the first result on internal stabilizability.

Theorem 1. (Quadrat (2006, a)). With the above notations, P is internally stabilizable iff there exists a matrix $S = (X^T \quad Y^T)^T$, where $X \in K^{q\times q}$, $\det X \neq 0$, and $Y \in K^{r\times q}$ such that the following two conditions hold:

(1) $RS = DX - NY = I_q$,

(2) We have:
$$SR = \begin{pmatrix} XD & -XN \\ YD & -YN \end{pmatrix} \in A^{p\times p}.$$

Then, $C = YX^{-1}$ is a stabilizing controller of P.

Remark 2. Conditions 1 and 2 of Theorem 1 amount to say that $\mathcal{P}$ is a *projective lattice* of $K^{q\times 1}$, namely, $\mathcal{P}$ is a lattice of $K^{q\times 1}$ which, viewed as an A-module, is a projective A-module of rank q (see Definition 3 below), i.e., there exist an A-module $\mathcal{P}'$ and $r \in \mathbb{Z}_{\geq 0}$ satisfying $\mathcal{P} \oplus \mathcal{P}' \cong A^{r\times 1}$, and $\mathrm{rank}_A(\mathcal{P}) := \dim_K(K \otimes_A \mathcal{P}) = q$, where $K \otimes_A \mathcal{P}$ denotes the K-vector obtained by extending the coefficients of $\mathcal{P}$ from A to $K = Q(A)$ (Rotman (2009)).

Condition 2 of Theorem 1 asserts that $S \in (A : \mathcal{P})^{p\times 1}$. Hence, to get a stabilizing controller, a first approach is to find − if it exists − a certain $S \in (A : \mathcal{P})^{p\times 1}$ satisfying:
$$RS = I_q. \tag{1}$$

To show how the lattice $A : \mathcal{P}$ of $K^{1\times q}$ can explicitly be characterized, we have to introduce the second mathematical setting, i.e., the algebraic analysis approach.

2.2 The algebraic analysis approach

Let $P \in K^{q\times r}$ and $P = D^{-1}N$ a fractional representation of P. Considering $R = (D \quad -N) \in A^{q\times p}$, we can introduce the *finitely presented* A-module (Rotman (2009))
$$M = \mathrm{coker}_A(.R) := A^{1\times p} / \left(A^{1\times q} R\right).$$

The A-module M is formed by the *residue classes* $\pi(\lambda)$ of $\lambda \in A^{1\times p}$ (i.e., $\pi(\lambda') = \pi(\lambda)$ iff $\lambda' - \lambda \in A^{1\times q}\,R$, i.e., iff there exists $\nu \in A^{1\times q}$ such that $\lambda' = \lambda + \nu\,R$) endowed with the following two operations

$$\pi(\lambda) + \pi(\lambda') := \pi(\lambda + \lambda'), \quad a\,\pi(\lambda) := \pi(a\,\lambda),$$

for all $\lambda,\ \lambda' \in A^{1\times p}$ and for all $a \in A$.

The *torsion submodule* $t(M)$ of M is defined by:

$$t(M) = \{m \in M \mid \exists\, 0 \neq a \in A :\ a\,m = 0\}.$$

Let us characterize $t(M)$ in terms of the matrix R. To do that, we first introduce the A-*closure* of the A-submodule $A^{1\times q}\,R$ of $A^{1\times p}$ as the following A-submodule of $A^{1\times p}$:

$$\overline{A^{1\times q}\,R} := \{\lambda \in A^{1\times p} \mid \exists\, 0 \neq a \in A :\ a\,\lambda \in A^{1\times q}\,R\}.$$

Clearly, we have $t(M) = (\overline{A^{1\times q}\,R})/(A^{1\times q}\,R)$, which yields $M/t(M) = A^{1\times p}/(\overline{A^{1\times q}\,R})$ by the *third isomorphism theorem* in module theory (Rotman (2009)). Hence, we have to explicitly characterize $\overline{A^{1\times q}\,R}$.

If $\lambda \in \overline{A^{1\times q}\,R}$, then there exist $a \in A \setminus \{0\}$ and $\nu \in A^{1\times q}$ such that $a\,\lambda = \nu\,R$, which yields $\lambda = (a^{-1}\,\nu)\,R$ and shows that $\lambda \in (K^{1\times q}\,R) \cap A^{1\times p}$, i.e., $\overline{A^{1\times q}\,R} \subseteq (K^{1\times q}\,R) \cap A^{1\times p}$. Conversely, if $\lambda \in (K^{1\times q}\,R) \cap A^{1\times p}$, then there exists $\theta \in K^{1\times q}$ such that $\lambda = \theta\,R$. Cleaning the denominators of the entries of θ, there exist $a \in A \setminus \{0\}$ and $\mu \in A^{1\times q}$ such that $\theta = a^{-1}\,\mu$, which yields $a\,\lambda = \mu\,R$ and shows that $\lambda \in \overline{A^{1\times q}\,R}$, which finally proves the following equality:

$$\overline{A^{1\times q}\,R} = (K^{1\times q}\,R) \cap A^{1\times p}.$$

In Quadrat (2003, b), it was shown that built-in properties of P depend only on the A-closures $\overline{A^{1\times q}\,R}$ and $\overline{A^{1\times r}\,\widetilde{R}^T}$.

Definition 3. (Lang (1993)). Let M be a finitely generated A-module. Then, we say that M is:

- *projective* if there exist an A-module N and $r \in \mathbb{Z}_{\geq 0}$ such that $M \oplus N \cong A^{1\times r}$, where $\oplus$ (resp., $\cong$) stands for direct sum of modules (resp., isomorphic modules).
- *stably free* if there exist $r,\ s \in \mathbb{Z}_{\geq 0}$ such that:

$$M \oplus A^{1\times s} \cong A^{1\times r}.$$

We can now state a second necessary and sufficient condition for internal stabilizability for a plant which does not necessarily admit (weakly) left coprime factorizations.

Theorem 3. (Quadrat (2003, b)). $P \in K^{q\times r}$ is internally stabilizable iff the A-module $M = A^{1\times p}/(A^{1\times q}\,R)$ is such that $M/t(M) = A^{1\times p}/\left(\overline{A^{1\times q}\,R}\right)$ is projective of rank r.

There is a second characterization of $t(M)$. We first recall that an integral domain A is said to be *coherent* if for any finitely generated ideal I of A, i.e., an ideal of the form of $I = \sum_{i=1}^{s} A\,a_i$ for a certain $s \in \mathbb{Z}_{\geq 0}$ and a_i's in A, then the A-module of relations of I, namely,

$$\mathrm{Syz}(I) := \left\{\rho = (\rho_1\ \ldots\ \rho_s) \in A^{1\times s} \mid \sum_{i=1}^{s} \rho_i\,a_i = 0\right\}$$

is finitely generated as an A-module, i.e., $\mathrm{Syz}(I) = A^{1\times t}\,L$ for a certain $L \in A^{t\times s}$. If A is an integral domain, then a result of homological algebra asserts that

$$t(M) \cong \mathrm{ext}^1_A(T(M), A),$$

where $T(M) = A^{q\times 1}/(R\,A^{p\times 1}) \cong A^{1\times q}/\left(A^{1\times p}\,R^T\right)$ is the so-called *Auslander transpose* of M and

$$\mathrm{ext}^1_A(T(M), A) = \ker_A(.Q)/(A^{1\times q}\,R),$$

where $Q \in A^{p\times m}$ is a matrix satisfying:
$\ker_A(R.) := \{\eta \in A^{p\times 1} \mid R\,\eta = 0\} = \mathrm{im}_A(Q.) := Q\,A^{m\times 1}$, and $\ker_A(.Q) := \{\lambda \in A^{1\times p} \mid \lambda\,Q = 0\}$. Since $\ker_A(.Q)$ is a finitely generated A-module, there exists $R' \in A^{q'\times p}$ such that $\ker_A(.Q) = \mathrm{im}_A(.R') = A^{1\times q'}\,R'$. For more details, see Quadrat (2003, a). Since a *noetherian ring* (namely, a ring whose ideals are finitely generated as A-modules) is a coherent ring, such a result holds for any noetherian integral domain. For more details, see Quadrat (2003, a).

Finally, since $A^{1\times q}\,R \subseteq A^{1\times q'}\,R'$, the rows of R belong to the A-module $A^{1\times q'}\,R'$, which shows that there exists $R'' \in A^{q'\times q}$ such that $R = R''\,R'$. Finally, note that R'' has full row rank since $\lambda\,R'' = 0$ yields $\lambda\,R = 0$, and thus $\lambda = 0$ since R has full row rank.

Lemma 4. (Quadrat (2003, b)). With the above notations, we have the following results

$$\begin{aligned}
\overline{A^{1\times q}\,R} &:= \{\lambda \in A^{1\times p} \mid \exists\, 0 \neq a \in A :\ a\,\lambda \in A^{1\times q}\,R\}\\
&= (K^{1\times q}\,R) \cap A^{1\times p},\\
t(M) &= (\overline{A^{1\times q}\,R})/(A^{1\times q}\,R)\\
&= ((K^{1\times q}\,R) \cap A^{1\times p})/(A^{1\times q}\,R)\\
&= (A^{1\times q'}\,R')/(A^{1\times q}\,R),\\
M/t(M) &= A^{1\times p}/(\overline{A^{1\times q}\,R}) = A^{1\times p}/(A^{1\times q'}\,R'),
\end{aligned}$$

where the matrix $R' \in A^{q'\times p'}$ is defined by

$$\ker_A(.Q) = \mathrm{im}_A(.R'),$$

and the matrix $Q \in A^{p\times m}$ is such that:

$$\ker_A(R.) = \mathrm{im}_A(Q.).$$

Finally, there exists a full row rank matrix $R'' \in A^{q\times q'}$ such that:

$$R = R''\,R'. \tag{2}$$

Remark 5. Using Lemma 4, $P = D^{-1}\,N$ is a weakly left coprime factorization (see 2 of Definition 2) iff we have $t(M) = 0$ or $M/t(M) = M$ (Quadrat (2003, a)).

Finally, let us give a third characterization of $t(M)$. Since R has full row rank, we have the following *short exact sequence* of A-modules (see, e.g., Rotman (2009)):

$$0 \longrightarrow A^{1\times q} \xrightarrow{.R} A^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{3}$$

Applying the *tensor functor* $K \otimes_A \cdot$ to (3) and using the fact that $K = Q(A)$ is a *flat* A-module (Rotman (2009)), we obtain the following exact sequence of A-modules:

$$0 \longrightarrow K^{1\times q} \xrightarrow{.R} K^{1\times p} \xrightarrow{\mathrm{id}_K \otimes \pi} K \otimes_A M \longrightarrow 0.$$

Applying the tensor functor $(K/A) \otimes_A \cdot$ to (3) and using the fact that K is a flat A-module, which yields $\mathrm{tor}^A_1(K, M) = 0$ − where $\mathrm{tor}^A_1(L, M)$ is the *first torsion* A-module (Rotman (2009)) − we get the exact sequence

$$0 \longrightarrow \ker\beta \longrightarrow (K/A)^{1\times q} \xrightarrow{\beta} (K/A)^{1\times p} \xrightarrow{\alpha} (K/A) \otimes_A M \longrightarrow 0,$$

where $\ker\beta \cong \mathrm{tor}^A_1(K/A, M)$, and $\alpha = \mathrm{id}_{K/A} \otimes \pi$ and $\beta = \mathrm{id}_{K/A} \otimes .R$ are defined by

$$\begin{aligned}
&\forall\, \nu \in K^{1\times p}, \quad \alpha(\kappa_p(\nu)) = \sigma(1 \otimes \nu),\\
&\forall\, \theta \in K^{1\times q}, \quad \beta(\kappa_q(\theta)) = \kappa_p(\theta\,R),
\end{aligned}$$

with the following notations for the canonical projections:
$\sigma : K \otimes_A M \longrightarrow (K/A) \otimes_A M, \quad \kappa_l : K^{1\times l} \longrightarrow (K/A)^{1\times l}.$

Combining the above three exact sequences of A-modules, we obtain the commutative exact diagram given in Fig. 2. A chase in this diagram (see Rotman (2009)) shows that:

$$\begin{aligned} t(M) &\cong \mathrm{tor}_1^A(K/A, M) \\ &= \ker\beta = \left\{\kappa_q(\theta) \mid \theta \in K^{1\times q} :\ \theta\, R \in A^{1\times p}\right\} \\ &= \kappa_q\left(A : \mathcal{P}\right). \end{aligned}$$

The above isomorphism is defined by

$$\begin{aligned} \phi : \ker\beta &\longrightarrow t(M) \quad & \phi^{-1} : t(M) &\longrightarrow \ker\beta \\ \kappa_q(\theta) &\longmapsto \pi(\theta\, R), & \pi(\lambda) &\longmapsto \kappa_q(\theta), \end{aligned}$$

where $\theta \in K^{1\times q}$ such that $\theta\, R = \lambda$.

If we denote by $R'_{i\bullet} \in A^{1\times p}$ the i^{th} row of R', i.e.,

$$R' = \begin{pmatrix} R'_{1\bullet} \\ \vdots \\ R'_{q'\bullet} \end{pmatrix},$$

using $t(M) = (A^{1\times q'}\, R')/(A^{1\times q}\, R)$ (see Lemma 4), then $\{\pi(R'_{i\bullet})\}_{i=1,\ldots,q'}$ define a generating family of the A-module $t(M)$. Thus, there exists $\Theta \in K^{q'\times q}$ such that:

$$R' = \Theta\, R.$$

Now, writing $R' = (D' \quad -N')$, where $D' \in A^{q'\times q}$ and $N' \in A^{q'\times r}$, we then obtain $D' = \Theta\, D$, which yields:

$$\Theta = D'\, D^{-1}.$$

Since we proved that $t(M) \cong \kappa_q\left(A : \mathcal{P}\right)$, we obtain:

$$A : \mathcal{P} = A^{1\times q'}\, \Theta = A^{1\times q'}\, (D'\, D^{-1}). \tag{4}$$

In particular, we have

$$R' = \Theta\, R = D'\, D^{-1}\, (D \quad -N) = (D' \quad -D'\, P) \in A^{q'\times p},$$

which shows that $N' = D'\, P \in A^{q'\times r}$.

Lemma 6. With the above notations, we have:

$$A : \mathcal{P} = A^{1\times q'}\, \left(D'\, D^{-1}\right). \tag{5}$$

In particular, we have $D'\, P \in A^{q'\times r}$.

Example 2. Since $t(M) = (A^{1\times q'}\, R')/(A^{1\times q}\, R)$ is a torsion A-module, its rank is equal to 0, and thus we get $\mathrm{rank}_A(A^{1\times q'}\, R') = \mathrm{rank}_A(A^{1\times q}\, R) = q$ since the rows of R are A-linearly independent. The short exact sequence

$$0 \longrightarrow \ker_A(.R') \xrightarrow{\ i\ } A^{1\times q'} \xrightarrow{\ .R'\ } \mathrm{im}_A(.R') \longrightarrow 0$$

yields $\mathrm{rank}_A(A^{1\times q'}\, R') = q' - \mathrm{rank}_A(\ker_A(.R'))$. Hence, if $q' = q$, then $\mathrm{rank}_A(\ker_A(.R')) = 0$, which yields $\ker_A(.R') = 0$ since $\ker_A(.R')$ cannot be a non-trivial torsion A-module because it is a A-submodule of the free A-module $A^{1\times q'}$. Hence, R' has full row rank. Finally, using the identity $R = R''\, R'$ (see (2)), where $R'' \in A^{q\times q}$ has full row rank, we get $R' = (R'')^{-1}\, R$, i.e., $\Theta = R''^{-1}$.

Using Lemma 6 and Section 2.1, we obtain the corollary.

Corollary 7. With the above notations, $P = D^{-1}\, N$ is internally stabilizable iff there exists a matrix $Z \in A^{p\times q'}$ such that $S := Z\, D'\, D^{-1}$ satisfies (1), i.e., $R\, S = I_q$.

3. MAIN RESULTS

Let us state again a standard result due to Serre.

Theorem 8. (Lang (1993)). A finitely generated projective A-module L is stably free iff L admits a *finite free resolution*, i.e., there exists an exact sequence of the form:

$$\begin{aligned} 0 \longrightarrow A^{1\times p_n} &\xrightarrow{\ .R_n\ } A^{1\times p_{n-1}} \xrightarrow{\ .R_{n-1}\ } \\ \ldots &\xrightarrow{\ .R_2\ } A^{1\times p_1} \xrightarrow{\ .R_1\ } A^{1\times p_0} \xrightarrow{\ \pi\ } L \longrightarrow 0. \end{aligned}$$

Let us give a standard characterization of stably free modules obtained by *splitting* the above finite free resolution.

Corollary 9. Let $U \in A^{t\times s}$ and a projective A-module $L = A^{1\times s}/(A^{1\times t}\, U)$. Then, L is stably free iff there exists a generalized inverse $V \in A^{s\times t}$ of U, i.e., such that:

$$U\, V\, U = U.$$

Let us now suppose that P is an internally stabilizable plant. By Theorem 3, $M/t(M) = A^{1\times p}/(A^{1\times q'}\, R')$ is a projective A-module. If the ring A is such that every finitely presented A-module admits a finite free resolution, then by Theorem 8, $M/t(M)$ is a stably free A-module and using Corollary 9, there exists $S' \in A^{p\times q'}$ such that:

$$R'\, S'\, R' = R'. \tag{6}$$

Pre-multiplying this identity by R'' and using (2), we get

$$R\, S'\, R' = R,$$

and using $R = (D \quad -N)$ and $R' = (D' \quad -N')$, we then obtain $R\, S'\, D' = D$, which yields

$$R\, (S'\, D'\, D^{-1}) = I_q,$$

and shows that Corollary 7 holds where $Z = S' \in A^{p\times q'}$ is a generalized inverse of R', i.e., such that (6) holds. If we note $S = S'\, D'\, D^{-1} = (X^T \quad Y^T)^T$, where $X \in K^{q\times q}$ and $Y \in K^{r\times q}$, then Theorem 1 shows that $C = Y\, X^{-1}$ internally stabilizes P.

Corollary 10. With the above notations, if A is an integral domain satisfying that every finitely presented A-module admits a finite free resolution, and if P is an internally stabilizable plant, then the matrix Z defined in Corollary 7 can be taken as a generalized inverse S' of R'. Writing

$$S := S'\, D'\, D^{-1} = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad X \in K^{q\times q}, \quad Y \in K^{r\times q},$$

if $\det(X) \neq 0$, then $C = Y\, X^{-1} \in \mathrm{Stab}(P)$.

4. APPLICATION TO MULTIDIMENSIONAL SYSTEMS

In the rest of the paper, we apply Corollary 10 to multidimensional systems, i.e., to $\mathbb{R}(z_1, \ldots, z_n)_S$, and more precisely, since we want develop a computer algebra approach, to the integral domain of stable rational functions with rational coefficients, i.e., $A = \mathbb{Q}(z_1, \ldots, z_n)_S$.

Let $B := \mathbb{Q}[z_1, \ldots, z_n]$ be the commutative polynomial ring in $z_1, \ldots, z_n$ with coefficients in $\mathbb{Q}$ and the following *multiplicatively closed subset* of B (Eisenbud (1995)):

$$S = \{d \in B \mid V(\langle d\rangle) \cap \overline{\mathbb{D}}^n = \emptyset\}.$$

Then, we can consider the *localization* of B with respect to S, namely the following integral domain:

$$B_S := \left\{n\, d^{-1} \mid d \in S\right\} = \mathbb{Q}(z_1, \ldots, z_n)_S = A.$$

For more details, see Eisenbud (1995). A standard result of module theory asserts that $A = B_S$ is a flat A-module, i.e., the functor $A \otimes_B \cdot$ is *exact* (Rotman (2009)).

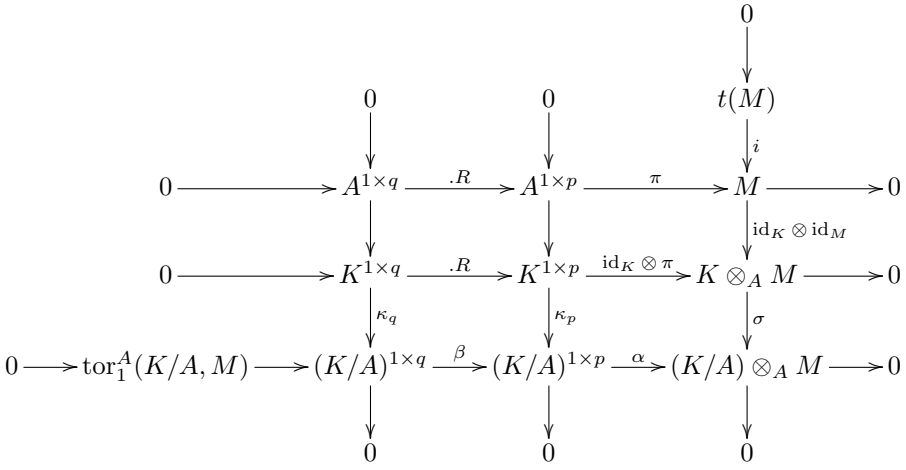

Figure 2. Commutative exact diagram of A-modules

Note that $K = Q(A) = Q(B) = \mathbb{Q}(z_1, \ldots, z_n)$ is the ring of rational fractions in the z_i's with rational number coefficients. Hence, if $P \in K^{q\times r}$, then there exist two polynomial matrices $D \in B^{q\times q}$ and $N \in B^{q\times r}$ such that $P = D^{-1}N$. Let $R = (D \quad -N) \in B^{q\times p}$ and $L = B^{1\times p}/(B^{1\times q}R)$. Since $A = B_S$ is a flat B-module, we have:

$$A \otimes_B L \cong M := A^{1\times p}/(A^{1\times q}R).$$

For more details, see, e.g., Rotman (2009). Introducing the Auslander transpose $T(L) = B^{q\times 1}/(R\,B^{p\times 1})$, since B is a noetherian ring, there exists $Q \in B^{p\times m}$ such that $\ker_B(R.) = \mathrm{im}_B(Q.)$. Using *Gröbner basis techniques* (see, e.g., Eisenbud (1995)), such a matrix Q can be computed. See, e.g., Chyzak et al. (2005); Quadrat (2010). Then, we have the following exact sequence of B-modules:

$$0 \longleftarrow T(L) \longleftarrow B^{q\times 1} \xleftarrow{R.} B^{p\times 1} \xleftarrow{Q.} B^{m\times 1}.$$

Applying the exact functor $A\otimes_B \cdot$ to it, we get the following exact sequence of A-modules:

$$0 \longleftarrow A \otimes_B T(L) \longleftarrow A^{q\times 1} \xleftarrow{R.} A^{p\times 1} \xleftarrow{Q.} A^{m\times 1}.$$

Using again the fact that A is a flat B-module, we obtain:

$$\begin{aligned} A \otimes_B T(L) &= A \otimes (B^{q\times 1}/(R\,B^{p\times 1})) \\ &\cong A^{q\times 1}/(R\,A^{p\times 1}) = T(M). \end{aligned}$$

Applying the functor $\hom_A(\cdot, A)$ to the above exact sequence, we get the complex of A-modules:

$$A^{1\times q} \xrightarrow{.R} A^{1\times p} \xrightarrow{.Q} A^{1\times m}.$$

Hence, according to Section 2.2, we have:

$$t(M) \cong \mathrm{ext}^1_A(T(M), A) = \ker_A(.Q)/\mathrm{im}_A(.R).$$

Let $R' \in B^{q'\times p}$ be such that $\ker_B(.Q) = \mathrm{im}_B(.R')$, i.e., such that we have the exact sequence of B-modules:

$$B^{1\times q'} \xrightarrow{.R'} B^{1\times p} \xrightarrow{.Q} B^{1\times m}.$$

Such a matrix R' can also be obtained by means of Gröbner basis techniques. Applying the exact functor $A \otimes_B \cdot$ to the above exact sequence, we get the following exact sequence of A-modules:

$$A^{1\times q'} \xrightarrow{.R'} A^{1\times p} \xrightarrow{.Q} A^{1\times m}.$$

Hence, we obtain $\ker_A(.Q) = \mathrm{im}_A(.R')$, which yields:

$$t(M) \cong \mathrm{im}_A(.R')/\mathrm{im}_A(.R) = (A^{1\times q'}\,R')/(A^{1\times q}\,R).$$

Since the entries of R and R' belong to B, based on Gröbner basis techniques, we can obtain $R'' \in B^{q'\times q}$ such that $R = R''\,R'$ (Chyzak et al. (2005)). Writing $R' = (D' \quad -N')$, where $D' \in B^{q'\times q}$ and $N' \in B^{q'\times r}$, then Lemma 6 shows that $A : \mathcal{P} = A^{1\times q'}\left(D'\,D^{-1}\right)$, where $\mathcal{P} = R\,A^{p\times 1}$ is the lattice of $K^{q\times 1}$ associated with P.

By Theorem 3, P is internally stabilizable iff the A-module $M/t(M) = A^{1\times p}/(A^{1\times q'}\,R')$ is projective of rank r.

Let us now characterize when a finitely presented module over a commutative ring is projective.

Theorem 11. (Eisenbud (1995)). Let $\mathbb{A}$ be a commutative ring, $U \in \mathbb{A}^{t\times s}$, and $L = \mathbb{A}^{1\times s}/(\mathbb{A}^{1\times t}\,U)$ the $\mathbb{A}$-module finitely presented by U. Then, L is projective of rank r iff

$$\mathrm{Fitt}_i(L) = 0, \quad i = 0, \ldots, r-1, \quad \mathrm{Fitt}_r(L) = \mathbb{A},$$

where $\mathrm{Fitt}_i(L)$ denotes the i^{th} *Fitting ideal* of L defined as the ideal of $\mathbb{A}$ formed by all the $(s-i)\times(s-i)$ minors of U, with the following conventions:

$$\mathrm{Fitt}_i(L) = 0, \quad i < s-t, \quad \mathrm{Fitt}_i(L) = \mathbb{A}, \quad i \geq s.$$

Since the entries of R' belong to B, the Fitting ideals of $M/t(M) = A^{1\times p}/(A^{1\times q'}\,R')$ belong to B and not to A. Hence, $\mathrm{Fitt}_r(M/t(M)) = A$ iff $\mathrm{Fitt}_r(M/t(M)) \cap S \neq \emptyset$, i.e., iff there exists $b \in \mathrm{Fitt}_r(M/t(M))$ such that:

$$V(\langle b\rangle) \cap \overline{\mathbb{D}}^n = \emptyset.$$

A more efficient way to check projectivity of $M/t(M) = A^{1\times p}/(A^{1\times q'}\,R')$ is to consider the ideal of A defined by

$$I = \bigcap_{i=2}^{n} \mathrm{ann}_A(\mathrm{ext}^i_A(T(M/t(M)), A)),$$

where $T(M/t(M)) = A^{q'\times 1}/(R'\,A^{p\times 1})$ is the Auslander transpose of $M/t(M)$, which defines the obstructions for $M/t(M)$ to be projective (Chyzak et al. (2005)). Hence, $M/t(M)$ is projective iff $A/I = 0$, i.e., iff we have $I = A$.

Using the fact that A is a flat B-module, if we define

$$J = \bigcap_{i=2}^{n} \mathrm{ann}_B(\mathrm{ext}^i_B(T(\mathrm{coker}_B(R'.)), B)),$$

where $T(\mathrm{coker}_B(R'.)) = B^{q'\times 1}/(R'\,B^{p\times 1})$, then we have:

$$I \cong A \otimes_B J.$$

See Rotman (2009). Hence, we get that $I = A$ iff $J \cap S \neq \emptyset$, i.e., iff there exists $b \in J$ such that $V(\langle b \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$.

If we either consider $\mathrm{Fitt}_r(M/t(M))$ or the ideal J, we are led to the problem to decide whether or not a finitely generated ideal $\mathcal{I}$ of B has a non-empty intersection with S, i.e., contains an element which does not vanish in $\overline{\mathbb{D}}^n$.

This problem is known as the *Polydisc Nullstellensatz*.

Theorem 12. (*Polydisc Nullstellensatz*, Bridges et al. (2004)) Let $\mathcal{I} = \langle f_1, \ldots, f_m \rangle$ be a finitely generated ideal of $B = \mathbb{Q}[z_1, \ldots, z_n]$. Then, the two assertions are equivalent:

(1) $V(\mathcal{I}) = \{z \in \mathbb{C}^n \mid f_1(z) = \ldots = f_m(z) = 0\} \cap \overline{\mathbb{D}}^n = \emptyset$.
(2) There exists $b \in \mathcal{I}$ such that $V(\langle b \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$.

According to Theorem 12, we first have to check that $V(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$, where $\mathcal{I} = \mathrm{Fitt}_r(M/t(M))$ or I, and thus develop an effective version of the Polydisc Nullstellensatz to explicitly obtain such a b. This problem was effectively solved in Bouzidi et al. (2015) for principal ideals of B and in Bouzidi et al. (2017) for *zero-dimensional ideals*, i.e., ideals whose sets of complex zeros are finite. To our knowledge, an effective version of the general case is still lacking despite the fact that the proof of Theorem 12 is constructive in the sense that it does not use choice axioms.

Example 3. If $\mathcal{I} = \langle f_1, \ldots, f_m \rangle$, then using Bouzidi et al. (2015), we can effectively check whether or not there exists $i = 1, \ldots, m$ such that $V(\langle f_i \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$. If such an i exists, then we can take $b = f_i$.

Let us suppose that we can effectively solve the problem to testing $V(\mathrm{Fitt}_r(M/t(M))) \cap \overline{\mathbb{D}}^n = \emptyset$ or $V(J) \cap \overline{\mathbb{D}}^n = \emptyset$ (e.g., when $n = 2$), then we can effectively prove whether or not the multidimensional plant P is internally stabilizable.

If P is internally stabilizable and if $b \in \mathrm{Fitt}_r(M/t(M))$ or $b \in J$ can be determined such that $V(\langle b \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$, then we can consider the multiplicatively closed subset $S' = \{1, b, b^2, \ldots\}$ of B and the localization $B_b := S'^{-1} B$ of B (see, e.g., Eisenbud (1995)). We have $B_b \subset A$. By construction, we have $B_b \otimes_B \mathrm{Fitt}_r(M/t(M)) = B_b$ or $B_b \otimes_B J = B_b$, which yields $A \otimes_B \mathrm{Fitt}_r(M/t(M)) = A$ or $A \otimes_B J = A$. Since every finitely presented B-module admits a finite free resolution (see, e.g., Eisenbud (1995)), so has the B-module $L/t(L) = B^{1\times p}/(B^{1\times q'} R')$, and thus $M/t(M) = A^{1\times p}/(A^{1\times q'} R')$ since A is a flat B-module. Since B_b is a localization of B, B_b is a flat B-module. Using Gröbner basis techniques (see Chyzak et al. (2005); Eisenbud (1995); Quadrat (2010)), we can effectively compute a generalized inverse $S' \in B_b^{p\times q'} \subset A^{p\times q'}$ of $R' \in B^{q'\times p}$, i.e., $R'\,S'\,R' = R'$. Finally, Corollary 10 gives an explicit stabilizing controller C of P.

5. CONCLUSION

In this paper, we have shown how the algebraic analysis condition for internal stabilizability implies the one obtained in the lattice approach. This result is general for any class of plants. In particular, it holds for infinite-dimensional systems (Curtain et al. (1991)). For multidimensional systems, we have explained how testing internal stabilizability is reduced to checking whether or not an algebraic set, defined by a polynomial ideal, intersects the closed unit polydisc. To your knowledge, the problem of effectively checking this last condition is still open apart from a few cases. We have also shown that this condition is a necessary and sufficient condition for the existence of an element in the defining ideal which does not intersect the closed unit polydisc (Polydisc Nullstellensatz). Finally, if such a "stable polynomial" can be computed, a stabilizing controller can then be effectively obtained by means of standard computer algebra methods and implementations.

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