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Simple forms of higher-order linear differential systems and their applications in computing regular solutions

Moulay A. Barkatou, Thomas Cluzeau, Carole El Bacha [1]

University of Limoges; CNRS; XLIM UMR 6172; DMI 123 avenue Albert Thomas, 87060 Limoges cedex, France

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ABSTRACT

We propose a direct algorithm for computing regular formal solutions of a given higher-order linear differential system near a singular point. With such a system, we associate a matrix polynomial and we say that the system is simple if the determinant of this matrix polynomial does not identically vanish. In this case, we show that the algorithm developed in Barkatou et al. (2009) can be applied to compute a basis of the regular formal solutions space. Otherwise, we develop an algorithm which, given a non-simple system, computes an auxiliary simple one from which the regular formal solutions space of the original system can be recovered. We also give the arithmetic complexity of our algorithms.

0. Introduction

We consider systems of linear differential equations of arbitrary order $\ell \geq 1$ with meromorphic coefficients and treat the problem of computing their regular formal solutions at a singular point x_0 which can be supposed, without loss of generality, located at the origin. For ease of presentation, we shall use the Euler derivative $\vartheta = x\frac{d}{dx}$ instead of the standard derivative $\frac{d}{dx}$. These derivatives are related by the following formulas:

$$\forall\, i \geq 1, \quad x^i \frac{d^i}{dx^i} = \vartheta(\vartheta-1)\cdots(\vartheta-i+1).$$

E-mail addresses: moulay.barkatou@xlim.fr (M.A. Barkatou), thomas.cluzeau@xlim.fr (T. Cluzeau), carole.el-bacha@xlim.fr (C. El Bacha).

[1] Tel.: +33 5 87 50 67 51; fax: +33 5 55 45 73 22.

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Let K be an extension of $\mathbb{Q}$ and $\overline{K}$ its algebraic closure. We denote by K[x] (resp. K[[x]]) the ring of polynomials (resp. formal power series) over K in the variable x and by K((x)) the field of formal meromorphic power series over K in the variable x.

Consider a system of linear differential equations of order $\ell \geq 1$ of the form

$$\mathcal{L}(x,\vartheta)(y(x)) = A_\ell(x)\vartheta^\ell(y(x)) + A_{\ell-1}(x)\vartheta^{\ell-1}(y(x)) + \cdots + A_0(x)y(x) = 0, \tag{1}$$

where for $i = 0, \ldots, \ell$, the coefficients $A_i(x)$ are square matrices of size n having entries in K[[x]], the *leading coefficient* $A_\ell(x)$ is a nonzero matrix and $y(x)$ is an unknown n-dimensional vector. We are interested in computing the regular formal solutions at the origin of System (1), *i.e.*, solutions of the form $y(x) = x^{\lambda_0} z(x)$ where $\lambda_0 \in \overline{K}$ and $z(x) \in \overline{K}[\log(x)]^n[[x]]$.

In our previous work (Barkatou et al., 2009), we have investigated systems of the form (1) having their leading coefficients $A_\ell(x)$ invertible at the origin, *i.e.*, $A_\ell(0)$ invertible. This hypothesis implies that the matrix polynomial associated with System (1) and defined by

$$\mathcal{L}(0,\lambda) = A_\ell(0)\,\lambda^\ell + A_{\ell-1}(0)\,\lambda^{\ell-1} + \cdots + A_0(0),$$

is regular, *i.e.*, $\det(\mathcal{L}(0,\lambda)) \not\equiv 0$. We have shown that if $y(x) = x^{\lambda_0}\sum_{i\geq 0} U_i x^i$, with $U_i \in \overline{K}[\log(x)]^n$ and $U_0 \neq 0$, is a regular formal solution of (1), then the exponent λ_0 is an eigenvalue of the matrix polynomial $\mathcal{L}(0,\lambda)$, *i.e.*, $\det(\mathcal{L}(0,\lambda_0)) = 0$. We have developed an algorithm (Barkatou et al., 2009, Algorithm 2), based on properties of matrix polynomials (see, *e.g.*, Gohberg et al., 1982) for computing a basis of the regular formal solutions space of such systems. The present paper concerns the more general case where $A_\ell(0)$ is no longer supposed to be invertible. It is divided into two essential parts. In the first part, we show that the method developed in Barkatou et al. (2009) can be extended to handle all systems of the form (1) having a regular associated matrix polynomial $\mathcal{L}(0,\lambda)$ ($A_\ell(0)$ is not necessarily invertible). Such systems are said to be simple. To achieve this, we give new direct proofs of the results showing the correctness of Barkatou et al. (2009, Algorithm 2). The second part is devoted to the computation of regular formal solutions of non-simple linear differential systems. We suppose that the leading coefficient $A_\ell(x)$ is invertible, so the regular formal solutions space of (1) is of finite dimension. Our strategy is to compute a simple linear differential system $\overline{\mathcal{L}}(x,\vartheta)(z(x)) = 0$ from which we can get the regular solutions of the original system. Contrary to the situation in the case of first-order systems (see Barkatou and Pflügel, 1998 and the references therein), the operator $\overline{\mathcal{L}}(x,\vartheta)$ cannot always be obtained from $\mathcal{L}(x,\vartheta)$ via a transformation of the form $\overline{\mathcal{L}}(x,\vartheta) = S(x)\,\mathcal{L}(x,\vartheta)\,T(x)$ where $S(x)$ and $T(x)$ are invertible matrices of size n with coefficients in K((x)) (see Example 3) . For this reason, we are first interested in the existence of a linear substitution $y(x) = T(x)\,z(x)$ with invertible matrix $T(x)$ such that the new linear differential system satisfied by $z(x)$ is simple. We develop an algorithm that either decides on the existence of such a linear substitution or proves that it does not exist. In the latter case, we propose a differential variant of the *EG′-algorithm* proposed by Abramov et al. (2003, Section 4) for matrix recurrence systems. Here we shall suppose that the non-simple system $\mathcal{L}(x,\vartheta)(y(x)) = 0$ has polynomial coefficients. This algorithm consists in applying elementary row operations to $\mathcal{L}(x,\vartheta)(y(x)) = 0$ and always yields a simple system from which the regular solutions of the original system can be recovered. Depending on the elementary row operations performed, the regular solutions spaces of these two systems may or may not be isomorphic; in this case, we explain how regular solutions of the original system can be obtained. Another important contribution of the paper is that we provide a complexity analysis of the algorithms which have been implemented in Maple and will be soon available in the Isolde package (Barkatou and Pflügel, 2006)[2].

The paper is organized as follows. In Section 1, we recall necessary preliminary concepts for matrix polynomials, regular solutions of simple linear differential systems with constant coefficients and minimal bases of singular matrix polynomials. Section 2 deals with simple systems: we prove that the method proposed in Barkatou et al. (2009) can be applied to compute the regular formal solutions space of any simple linear differential system of the form (1). In Section 3, we review the

2 A beta version is available on request from the authors.

algorithm proposed in Barkatou and Pflügel (1998) for computing regular formal solutions of first-order simple linear differential systems. We then compare, from an arithmetic complexity point of view, our method developed in Section 2 to that consisting in transforming System (1) to a first-order system of size $n\,\ell$ and then using the algorithm of Barkatou and Pflügel (1998). Section 4 introduces our approach for handling the case of non-simple systems and points out the main difference from the case of first-order systems. Then, in Section 5, we are interested in computing a linear substitution yielding a simple system. We provide both a necessary condition for the existence of such a linear substitution and an algorithm that computes it when it exists. Furthermore, since we deal with systems having formal power series matrix coefficients, we give a bound on the order at which we have to truncate $\mathcal{L}(x,\vartheta)$ in order to get regular solutions up to a fixed order. Section 6 is concerned with the case when the system cannot be reduced to a simple one by means of a linear substitution: we describe a differential variant of the *EG′-algorithm* proposed in Abramov et al. (2003, Section 4) which always provides a simple system. Then, we explain how to recover the regular formal solutions of the original system from those of the simple one computed by the latter algorithm.

Notation and complexity measures. We denote by $\Re(\lambda_0)$ the real part of a complex number λ_0. Let $\mathbb{A}$ be a ring and $m,\ n \in \mathbb{N}^*$. We denote by $\mathbb{A}^{m\times n}$ the ring of $m \times n$ matrices with entries in $\mathbb{A}$. If $\mathbb{A}$ is a commutative ring with unit element, then $\mathrm{GL}_n(\mathbb{A})$ denotes the group of invertible matrices in $\mathbb{A}^{n\times n}$ and I_n its unit element. Let M denote a matrix of size $m \times n$. In the sequel, we denote by $M(.,i)$ (resp. $M(i,.)$) the ith column (resp. row) of M. If the entries of M depend on a variable λ, then we denote by $M^{(j)}$ the jth derivative of M with respect to λ. For $f \in \mathrm{K}((x))$, we define the *valuation* $v(f)$ as the order of f at 0. For $M = (m_{i,j})_{i,j} \in \mathrm{K}((x))^{m\times n}$, we define $v(M) = \min\left\{v(m_{ij}),\ 1 \le i \le m \text{ and } 1 \le j \le n\right\}$. For a linear differential system of the form (1) and $1 \le i \le n$, we define $\deg_x(\mathcal{L}(x,\vartheta)(i,.)) = \max_{k=0,\dots,\ell} \deg(A_k(x)(i,.))$ and, similarly, $v(\mathcal{L}(x,\vartheta)(i,.)) = \min_{k=0,\dots,\ell} v(A_k(x)(i,.))$. The same definitions hold for the columns $\mathcal{L}(x,\vartheta)(.,i)$. We recall that the product of two operators in $\mathrm{K}((x))[\vartheta]$ where $\vartheta = x\,\frac{d}{dx}$ is obtained by applying the rule

$$\forall\, a \in \mathrm{K}((x)), \quad \vartheta\, a = a\,\vartheta + \vartheta(a) = a\,\vartheta + x\,\frac{da}{dx}.$$

All complexity estimates are given in terms of arithmetic operations in K. We use the notation $f \in \widetilde{\mathcal{O}}(g)$ if f is in $\mathcal{O}(g\,\log^m(g))$ for some $m \ge 1$. We suppose that the fast Fourier transform can be used, so that two univariate polynomials with coefficients in K and degree bounded by d can be multiplied in $\widetilde{\mathcal{O}}(d)$ (see Bürgisser et al., 1997). We further assume that two matrices of size n with entries in K can be multiplied using $\mathcal{O}(n^\omega)$ where $2 \le \omega \le 3$ is the matrix multiplication exponent (von zur Gathen and Gerhard, 1999, Ch. 12). As a consequence, the product of two square matrix polynomials of size n and entries of degree bounded by d can be obtained using $\widetilde{\mathcal{O}}(n^\omega\, d)$ operations in K.

1. Preliminaries

1.1. Matrix polynomials and regular solutions

In this first subsection, we recall some basic definitions and properties of matrix polynomials that are needed in the sequel. For more details, we refer the reader to Gohberg et al. (1982, Ch. 1 & 7) and Zúñiga (2005, Ch. 3).

Definition 1. A square matrix polynomial $L(\lambda) \in \mathrm{K}[\lambda]^{n\times n}$ is called *regular* if its determinant does not vanish identically, *i.e.*, $\det(L(\lambda)) \not\equiv 0$. Otherwise, it is called *singular*.

Note that if $L(\lambda) \in \mathrm{K}[\lambda]^{n\times n}$ is a regular matrix polynomial of degree ℓ, then the degree in λ of its determinant is at most equal to $n\,\ell$.

Definition 2. Let $L(\lambda) \in \mathrm{K}[\lambda]^{n\times n}$ be a regular matrix polynomial. An element $\lambda_0 \in \overline{\mathrm{K}}$ is called an *eigenvalue of* $L(\lambda)$ if $\det(L(\lambda_0)) = 0$: its multiplicity as a root of $\det(L(\lambda))$ is then called *the algebraic multiplicity of* λ_0 and denoted by $m_a(\lambda_0)$. We denote by $\sigma(L)$ the spectrum of $L(\lambda)$, *i.e.*, the set of all eigenvalues of $L(\lambda)$. For $\lambda_0 \in \sigma(L)$, a nonzero vector $v \in \overline{\mathbb{K}}^n$ satisfying $L(\lambda_0)\, v = 0$ is called an *eigenvector of* $L(\lambda)$ *associated with the eigenvalue* λ_0: the dimension of the right nullspace of $L(\lambda_0)$ is called *the geometric multiplicity of* λ_0 and denoted by $m_g(\lambda_0)$.

In our algorithms, we need to compute a representation of the spectrum $\sigma(L)$ of a square regular matrix polynomial $L(\lambda)$ of size n and degree ℓ. We proceed as follows. We compute and factor over $\mathbb{K}[\lambda]$ the determinant $\det(L(\lambda))$ of $L(\lambda)$. Then, each eigenvalue $\lambda_0 \in \sigma(L)$ is represented by RootOf(π) where π is an irreducible factor of $\det(L(\lambda))$. Note that $\lambda_0 \in \mathbb{K}$ if and only if π is of degree 1. Computing the determinant can be done in $\mathcal{O}(n^{\omega}\,\ell)$ operations in $\mathbb{K}$ (see Jeannerod and Villard, 2006) and factoring it over $\mathbb{K}$ can be done in $\mathcal{O}((n\,\ell)^{12})$ operations in $\mathbb{K}$ (see Mora, 2003, Algorithm 18.7.3). In the sequel, the cost of computing $\sigma(L)$ will not be taken into account in the complexity analysis of our algorithms.

The following lemma introduces the local Smith form of a matrix polynomial at an eigenvalue which is then used to define the useful partial multiplicities associated with the eigenvalue.

Lemma 1 (*Gohberg et al., 1982, Th. S1.10*)**.** *Let $L(\lambda) \in \mathbb{K}[\lambda]^{n\times n}$ be a square regular matrix polynomial, $\lambda_0 \in \sigma(L)$ and $m_g = m_g(\lambda_0)$ its geometric multiplicity. Then, there exist two matrix polynomials $E_{\lambda_0}(\lambda)$ and $F_{\lambda_0}(\lambda)$ invertible at $\lambda = \lambda_0$ such that $L(\lambda) = E_{\lambda_0}(\lambda)\, S_{\lambda_0}(\lambda)\, F_{\lambda_0}(\lambda)$ where $S_{\lambda_0}(\lambda)$ called* the local Smith form of $L(\lambda)$ at λ_0 *is a diagonal matrix polynomial the diagonal entries of which are $1, \ldots, 1, (\lambda-\lambda_0)^{\kappa_1}, \ldots, (\lambda-\lambda_0)^{\kappa_{m_g}}$ for some positive integers κ_i satisfying $\kappa_1 \le \cdots \le \kappa_{m_g}$. These integers are unique and called* the partial multiplicities of $L(\lambda)$ associated with λ_0. *Moreover, we have $m_a(\lambda_0) = \sum_{i=1}^{m_g} \kappa_i$.*

Definition 3 (*Gohberg et al., 1982, Ch. 1*)**.** Let $L(\lambda) \in \mathbb{K}[\lambda]^{n\times n}$ be a square regular matrix polynomial and $\lambda_0 \in \sigma(L)$. For $k \in \mathbb{N}^*$, a sequence of vectors $v_0 \neq 0,\ v_1, \ldots,\ v_{k-1}$ in $\overline{\mathbb{K}}^n$ satisfying

$$\sum_{p=0}^{i} \frac{L^{(p)}(\lambda_0)}{p\,!}\, v_{i-p} = 0, \quad \text{for } i = 0, \ldots, k-1,$$

is called *a Jordan chain of length k associated with* λ_0.

The length k of each Jordan chain associated with λ_0 is always less than or equal to one of the partial multiplicities of $L(\lambda)$ associated with λ_0. In particular, the maximum lengths of Jordan chains associated with an eigenvalue λ_0 of $L(\lambda)$ are equal to its partial multiplicities.

Let $L(\lambda) \in \mathbb{K}[\lambda]^{n\times n}$ be a square regular matrix polynomial of degree ℓ. In Zúñiga (2005, Ch. 3), the author develops several algorithms for computing the partial multiplicities and Jordan chains associated with a given eigenvalue $\lambda_0 \in \sigma(L)$. Since this is an essential tool in our algorithms for computing regular solutions, we have implemented (Zúñiga, 2005, Algorithm 3.3) in Maple and studied its arithmetic complexity: it uses at most $\mathcal{O}(n^5\,\ell^2\,d_{\lambda_0})$ operations in $\mathbb{K}$ where $d_{\lambda_0} \le n\,\ell$ denotes the degree of the extension $\mathbb{K}(\lambda_0)$ over $\mathbb{K}$ (see El Bacha, 2008, Ch. 1).

Consider a linear differential system with constant matrix coefficients of the form

$$L(\vartheta)(y(x)) = A_\ell\, \vartheta^{\ell}(y(x)) + A_{\ell-1}\, \vartheta^{\ell-1}(y(x)) + \cdots + A_0\, y(x) = 0, \tag{2}$$

where for $i = 0, \ldots, \ell$, $A_i \in \mathbb{K}^{n\times n}$, and suppose that its associated matrix polynomial

$$L(\lambda) = A_\ell\, \lambda^{\ell} + A_{\ell-1}\, \lambda^{\ell-1} + \cdots + A_0 \tag{3}$$

is regular. The following lemma is derived from Gohberg et al. (1982, Prop. 1.9 and Th. S1.6) and shows how a basis of the regular formal solutions space of (2) can be computed from the Jordan chains associated with the eigenvalues of (3).

Lemma 2. *Let $L(\vartheta)(y(x)) = 0$ be a linear differential system with constant matrix coefficients of the form (2) and suppose that its associated matrix polynomial $L(\lambda)$ given by (3) is regular. Then, the dimension of the regular formal solutions space of (2) is exactly equal to* $\deg(\det(L(\lambda)))$ *and a basis is given by solutions of the form*

$$y(x) = x^{\lambda_0} \left(v_{k-1} + v_{k-2}\, \frac{\log(x)}{1\,!} + \cdots + v_0\, \frac{\log^{k-1}(x)}{(k-1)\,!} \right),$$

where $\lambda_0 \in \sigma(L)$ and $v_0, \ldots, v_{k-1}$ form a Jordan chain of length k associated with λ_0.

In Barkatou et al. (2009), we naturally deduce an algorithm for computing a basis of the regular formal solutions space of a system of the form (2) having a regular associated matrix polynomial $L(\lambda)$. For each eigenvalue $\lambda_0 \in \sigma(L)$, we only need to compute the partial multiplicities and the Jordan chains of maximal length associated with λ_0. Thus, using Zúñiga (2005, Algorithm 3.3), for each eigenvalue $\lambda_0 \in \sigma(L)$, the algorithm uses at most $\mathcal{O}(n^5\,\ell^2\,d_{\lambda_0})$ operations in $\mathbb{K}$ where $d_{\lambda_0} \le n\,\ell$ denotes the degree of the extension $\mathbb{K}(\lambda_0)$ over $\mathbb{K}$ (see Barkatou et al., 2009, Proposition 1).

1.2. Minimal bases of singular matrix polynomials

Let $L(\lambda)$ be a square singular matrix polynomial of size n, rank $r < n$ and degree ℓ. It is always possible to construct a $\mathbb{K}(\lambda)$-basis of its right (resp. left) nullspace constituted only of vectors of polynomials in the variable λ. Indeed, it suffices to consider an arbitrary basis and to multiply each vector by the common denominator of its entries. Such a basis is called *a right (resp. left) polynomial basis* of $L(\lambda)$.

Definition 4 (*De Terán et al., 2009*)**.** Let $L(\lambda)$ be a square singular matrix polynomial and $\mathcal{V}$ a right (resp. left) polynomial basis of $L(\lambda)$. Let δ be the sum of the degrees in λ of the elements of $\mathcal{V}$. If δ is minimal among all right (resp. left) polynomial bases of $L(\lambda)$, then $\mathcal{V}$ is called *a right* (resp. *left*) *minimal basis* of $L(\lambda)$.

We illustrate the previous notions with the following example:

Example 1. Let $L(\lambda)$ be the square singular matrix polynomial of size 4, rank 2 and degree 2 given by

$$L(\lambda) = \begin{pmatrix} \lambda^2 & 1 & \lambda & \lambda^2+\lambda \\ 2 & 0 & 0 & 2 \\ 0 & 1 & \lambda & \lambda \\ \lambda & 1 & \lambda & 2\lambda \end{pmatrix}.$$

Then

$$\mathcal{V}_1 = \left(\begin{pmatrix} 0 \\ \lambda \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda \\ 0 \\ -1 \end{pmatrix} \right), \quad \mathcal{V}_2 = \left(\begin{pmatrix} 0 \\ \lambda \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right),$$

are two right polynomial bases of $L(\lambda)$. The sum of the degrees of the elements of $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) is 2 (resp. 1). Thus $\mathcal{V}_1$ is not a right minimal basis of $L(\lambda)$ whereas one can prove that $\mathcal{V}_2$ is a right minimal basis of $L(\lambda)$.

Let $\mathcal{V}_1 = (x_1, \ldots, x_{n-r})$ and $\mathcal{V}_2 = (y_1, \ldots, y_{n-r})$ with $x_i,\, y_i \in \mathbb{K}[\lambda]^n$ be two right (resp. left) minimal bases of $L(\lambda)$. If $\forall\, i,\, j$ such that $1 \le i \le j \le n-r$, one has $\deg(x_i) \le \deg(x_j)$ and $\deg(y_i) \le \deg(y_j)$, then $\forall\, i = 1, \ldots, n-r$, $\deg(x_i) = \deg(y_i)$ (see De Terán et al., 2009 and references therein). In other words, the ordered list of the degrees of the elements of any right (resp. left) minimal basis does not depend on the choice of the right (resp. left) minimal basis. These degrees are called *the right* (resp. *left*) *minimal indices of* $L(\lambda)$ and the sum of the right and left minimal indices is at most equal to $r\,\ell$ (see for example Zúñiga, 2005, Corollary 3.1).

The following lemma will be useful in the proof of Theorem 1.

Lemma 3. *Let $L(\lambda)$ be a square singular matrix polynomial of size n. If $P \in \mathrm{GL}_n(\mathbb{K})$ is an invertible constant matrix, then the right minimal indices of $L(\lambda)$ and $L(\lambda)\,P$ are the same.*

Proof. Since P is invertible, $\widetilde{x}(\lambda) \in \ker(L(\lambda)\,P)$ if and only if $x(\lambda) = P\,\widetilde{x}(\lambda) \in \ker(L(\lambda))$. Moreover, we have $\deg(x(\lambda)) = \deg(\widetilde{x}(\lambda))$. $\square$

Minimal bases of singular matrix polynomials are an important tool in the algorithms that we develop in the sequel. Their calculation has been studied in the literature (see, *e.g.*, Beckermann and Labahn, 2000; Quéré-Stuchlik, 1997; Zhou and Labahn, 2009). In Quéré-Stuchlik (1997, Ch. 4), the author shows that a right minimal basis of a square singular matrix polynomial $L(\lambda)$ of size n, rank

r and degree ℓ can be obtained by computing a σ-basis $\mathcal{B}^\sigma$ with $\sigma = (r+1)\,\ell + 1 \leq n\,\ell + 1$ of the vector Hermite–Padé approximant problem related to $L(\lambda)$. Indeed, the $n - r$ vectors of smallest degrees of $\mathcal{B}^\sigma$ form a right minimal basis of $L(\lambda)$ (see Quéré-Stuchlik, 1997, Th. 4.20). The complexity of computing a σ-basis of a square matrix polynomial of size n is $\widetilde{\mathcal{O}}(n^\omega\,\sigma)$ operations in K (see Zhou and Labahn, 2009). Hence, the cost of computing a minimal basis of a square matrix polynomial of size n and degree ℓ is at most $\widetilde{\mathcal{O}}(n^{\omega+1}\,\ell)$ operations in K. For our implementation in Maple, we use the implementation of the algorithm developed in Beckermann and Labahn (2000) that is available in the MatrixPolynomialAlgebra package.

2. Regular solutions of simple linear differential systems

In Barkatou et al. (2009, Algorithm 2), we give an algorithm for computing a basis of the regular formal solutions space at $x = 0$ of a linear differential system of the form (1) in the particular case when the leading coefficient $A_\ell(x)$ is invertible at the origin, *i.e.*, $A_\ell(0)$ is invertible. In this section, we prove that this algorithm can be applied to any linear differential system of the form (1) where the associated matrix polynomial $\mathcal{L}(0, \lambda)$ is regular. In the sequel, we follow the terminology used in Barkatou and Pflügel (1998, Def. 2.1) for first-order linear differential systems:

Definition 5. A linear differential system $\mathcal{L}(x, \vartheta)(y(x)) = 0$ (or matrix differential operator $\mathcal{L}(x, \vartheta)$) of the form (1) is called *simple* if its associated matrix polynomial $\mathcal{L}(0, \lambda)$ is regular.

A system of the form (1) with invertible matrix $A_\ell(0)$ is simple but the converse is not always true.

Example 2. Consider the matrix differential operator given by

$$\mathcal{L}(x, \vartheta) = \begin{pmatrix} 1+x+x^2 & 0 \\ 3x^2+x^5 & 0 \end{pmatrix} \vartheta^2 + \begin{pmatrix} 2+5x^4 & 3+x^3 \\ 2+x^3+x^4 & 1+2x^2+x^4 \end{pmatrix} \vartheta + \begin{pmatrix} 1+5x^2 & x+x^2+x^3 \\ 0 & 1+2x^2+x^4 \end{pmatrix}.$$

Its associated matrix polynomial

$$\mathcal{L}(0, \lambda) = \begin{pmatrix} \lambda^2+2\lambda+1 & 3\lambda \\ 2\lambda & \lambda+1 \end{pmatrix}$$

is regular. Consequently, $\mathcal{L}(x, \vartheta)$ is simple while $A_2(0)$ is not invertible.

In the sequel, we shall show how Barkatou et al. (2009, Algorithm 2) can be adapted to handle the class of simple systems. To achieve this, we first review the basic ideas of this algorithm and propose direct proofs of Barkatou et al. (2009, Propositions 2 & 4) without converting the corresponding systems into first-order systems.

The approach consists in looking for regular formal solutions $y(x)$ of (1) written as

$$y(x) = \sum_{m\geq 0} U_m(t)\, x^{\lambda_0+m}, \tag{4}$$

where $\lambda_0 \in \overline{\mathrm{K}}$, $t = \log(x)$, for all $m \geq 0$, $U_m(t) \in \overline{\mathrm{K}}[t]^n$ is of degree less than $n\,\ell$ and $U_0(t) \neq 0$. We shall refer to a solution of the form (4) as a *regular solution associated with* λ_0. We write the matrix coefficients $A_i(x)$, for $i = 0, \ldots, \ell$, of System (1) as the formal power series

$$A_i(x) = \sum_{i=0}^{\infty} A_{ij} x^j \tag{5}$$

with $A_{ij} \in \mathrm{K}^{n\times n}$. For $j \geq 1$, we define the matrix differential operator

$$L_j(\vartheta) = \sum_{i=0}^{\ell} A_{ij}\, \vartheta^i, \tag{6}$$

where the A_{ij} are given by (5). System (1) can then be rewritten in the form

$$\sum_{j=1}^{\infty} x^j L_j(\vartheta)(y(x)) + \mathcal{L}(0, \vartheta)(y(x)) = 0. \tag{7}$$

Plugging (4) into (7) and using the equality $L_j(\vartheta)\left(x^{\lambda_0+m}U_m\right) = x^{\lambda_0+m} L_j(\vartheta + \lambda_0 + m)(U_m)$, we find that λ_0 and U_0 must satisfy

$$\mathcal{L}(0, \vartheta)(x^{\lambda_0}U_0) = 0, \tag{8}$$

and for $m \geq 1$, U_m satisfies

$$\mathcal{L}(0, \vartheta + \lambda_0 + m)(U_m) = -\sum_{i=0}^{m-1} L_{m-i}(\vartheta + \lambda_0 + i)(U_i). \tag{9}$$

Eq. (8) shows that $x^{\lambda_0}U_0$ is a regular solution of the homogeneous system with constant matrix coefficients $\mathcal{L}(0, \vartheta)(y(x)) = 0$. Hence, by Lemma 2, λ_0 must be an eigenvalue of $\mathcal{L}(0, \lambda)$ and U_0 of the form

$$U_0 = v_{k-1} + v_{k-2}\frac{\log(x)}{1\,!} + \cdots + v_0\frac{\log^{k-1}(x)}{(k-1)\,!},$$

where $v_0, \ldots, v_{k-1}$ form a Jordan chain of length k associated with λ_0. We recall that k is less than or equal to one of the partial multiplicities of $\mathcal{L}(0, \lambda)$ associated with λ_0.

From Eq. (9), U_m satisfies a non-homogeneous simple linear differential system with constant matrix coefficients and polynomial right-hand side in $t = \log(x)$. The following proposition shows that such a system always admits a polynomial solution in $t = \log(x)$ and gives a bound on its degree.

Proposition 1. *Consider a non-homogeneous linear differential system with constant coefficients of the form*

$$L(\vartheta)(y) = A_\ell\vartheta^\ell(y) + A_{\ell-1}\vartheta^{\ell-1}(y) + \cdots + A_0 y = \phi(t), \tag{10}$$

where for $i = 0, \ldots, \ell$, $A_i \in \mathrm{K}^{n\times n}$, $t = \log(x)$ *and* $\phi(t) \in \overline{\mathrm{K}}[t]^n$ *is of degree* d. *If the system is simple,* i.e., $\det(L(\lambda)) \not\equiv 0$, *then it admits at least a polynomial solution in* t *of degree* p *such that*

$$\begin{array}{ll} p = d & \text{if } 0 \notin \sigma(L), \\ d \leq p \leq d + \max\{\kappa_i, i = 1, \ldots, m_g(0)\} & \text{if } 0 \in \sigma(L), \end{array}$$

where $\kappa_1, \ldots, \kappa_{m_g(0)}$ *are the partial multiplicities of the eigenvalue* 0 *of* $L(\lambda)$.

The result of the above proposition is similar to that of Barkatou et al. (2009, Proposition 2) except that we only assume here that the system is simple while the leading matrix coefficient A_ℓ is not supposed to be invertible. Therefore, the proof given in Barkatou et al. (2009, Proposition 2) is not valid since it uses explicitly the hypothesis that A_ℓ is invertible in order to convert System (10) into a first-order one. Consequently, we provide a direct proof:

Proof. From the existence of the Smith form $S(\lambda)$ of $L(\lambda)$ (see Gohberg et al., 1982, Th. S1.1), there exist two unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ (*i.e.*, $\det(E(\lambda)), \det(F(\lambda)) \in \mathrm{K}^*$) such that $E(\lambda)\,L(\lambda) = S(\lambda)\,F(\lambda)$. Put $\psi(t) = E(\vartheta)(\phi(t)) \in \overline{\mathrm{K}}[t]^n$ satisfying $\deg(\psi(t)) = \deg(\phi(t)) = d$ and $z = F(\vartheta)(y)$ satisfying $\deg(z) = \deg(y) = p$. Multiplying (10) on the left by $E(\vartheta)$, System (10) is then equivalent to $S(\vartheta)(z) = \psi(t)$, that is, y is a polynomial solution in t of (10) of degree p iff z is a polynomial solution in t of $S(\vartheta)(z) = \psi(t)$ of degree p. Let $S(\lambda) = \mathrm{diag}(a_1(\lambda), \ldots, a_n(\lambda))$ where for $i = 1, \ldots, n$, the $a_i(\lambda)$ are monic polynomials, $z = (z_1, \ldots, z_n)^T$ and $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))^T$. We shall show now that the differential equation $a_i(\vartheta)(z_i) = \psi_i(t)$ has a polynomial solution in $t = \log(x)$ and give a bound on its degree. Write z_i as a polynomial in $t = \log(x)$, plug it into $a_i(\vartheta)(z_i) = \psi_i(t)$ and identify the coefficients of the powers of t. Two cases have to be considered: if $a_i(0) \neq 0$, then the coefficients of z_i are uniquely determined and the degree in t of z_i is equal to that of $\psi_i(t)$. Otherwise, $a_i(\lambda)$ is of the form $\lambda^\kappa b_i(\lambda)$ with $b_i(0) \neq 0$ and $\kappa \in \mathbb{N}^*$ one of the partial multiplicities associated with 0 and $a_i(\vartheta)(z_i) = \psi_i(t)$ admits a polynomial solution z_i in $t = \log(x)$ with $\deg(\psi_i(t)) \leq \deg(z_i) \leq \deg(\psi_i(t)) + \kappa$, which ends the proof. □

The latter proposition shows that for $m \geq 1$, System (9) always admits a polynomial solution $U_m(t)$. Thus, every solution $x^{\lambda_0}\, U_0$ of System (8) can be extended to a regular formal solution of System (1) associated with λ_0. Consequently, from Lemma 2, we can exactly compute $\deg(\det(\mathcal{L}(0,\lambda)))$ linearly independent regular formal solutions of a simple linear differential system of the form (1). In the next proposition, we prove that this is exactly the dimension of its regular formal solutions space.

Proposition 2. *The dimension of the regular formal solutions space of a simple linear differential system of the form* (1) *is equal to* $\deg(\det(\mathcal{L}(0,\lambda)))$.

As for Proposition 1, we shall give a direct proof of this result since that of Barkatou et al. (2009, Proposition 4) uses the hypothesis that the leading coefficient $A_\ell(x)$ of the differential system is invertible which is not an assumption here.

Proof. Let $\mathcal{L}(x,\vartheta)(y(x)) = 0$ be a simple linear differential system of the form (1) and $\mathcal{V}$ the K-vector space spanned by the $\deg(\det(\mathcal{L}(0,\lambda)))$ linearly independent regular formal solutions computed by the method described above. Suppose that the dimension of the regular formal solutions space of $\mathcal{L}(x,\vartheta)(y(x)) = 0$ is greater than $\deg(\det(\mathcal{L}(0,\lambda)))$. Then there exists a regular formal solution $y(x)$ of (1) which is not in $\mathcal{V}$. Let $y(x) = x^{\lambda_0}\sum_{i=0}^{\infty} U_i\, x^i$ be such a solution with $U_i \in \overline{\mathrm{K}}[\log(x)]^n$ and $U_0 \neq 0$. We assume, without loss of generality, that the real part of $\lambda_0 \in \overline{\mathrm{K}}$ is maximal among those of the regular formal solutions of (1) which do not belong to $\mathcal{V}$. Since $y(x)$ is a regular solution of $\mathcal{L}(x,\vartheta)(y(x)) = 0$, we know from the discussion above that necessarily x^{λ_0}, U_0 satisfies $\mathcal{L}(0,\vartheta)(x^{\lambda_0}\, U_0) = 0$. Consequently, there exists a regular solution $z(x) = x^{\lambda_0}\sum_{i=0}^{\infty} V_i\, x^i \in \mathcal{V}$ of (1) with $V_0 = U_0$. Note that $z(x)$ is a linear combination of the regular formal solutions of the system associated with λ_0 and belonging to $\mathcal{V}$. Now, $y(x) - z(x)$ is a nonzero regular formal solution of (1) that does not belong to $\mathcal{V}$. Moreover, since $V_0 = U_0$, we have $y(x) - z(x) = x^{\lambda_0+j}\sum_{i=0}^{\infty} W_i\, x^i$ with $j > 0$ and $W_0 \neq 0$. This is in contradiction with the fact that $\Re(\lambda_0)$ is maximal. □

We have thus proven that Barkatou et al. (2009, Algorithm 2) can be applied to compute a basis of the regular formal solutions space of any linear differential system of the form (1) assuming only that it is simple, *i.e.*, $\det(\mathcal{L}(0,\lambda)) \not\equiv 0$. If the series involved are computed up to a fixed order ν (*i.e.*, we are only interested in the terms x^i with $i \leq \nu$), then for each eigenvalue λ_0 of the matrix polynomial $\mathcal{L}(0,\lambda)$, the algorithm uses at most $\mathcal{O}(n^5\,\ell^2\,\nu^2\,d_{\lambda_0})$ operations in K where d_{λ_0} denotes the degree of the extension $\mathrm{K}(\lambda_0)$ over K. Note that the complexity estimate given in Barkatou et al. (2009, Proposition 3) (namely at most $\mathcal{O}(n^5\,\ell^3\,\nu^2\,d_{\lambda_0})$ operations in K) can be improved, since at the end of the proof, $m_g(\lambda_0)$ can be bounded by n instead of $n\,\ell$, so the dependence on ℓ becomes quadratic instead of cubic.

In the sequel, we give a slightly modified version of Barkatou et al. (2009, Algorithm 2) in order to compute the general regular formal solution of a simple linear differential system of the form (1) in a more efficient way. We proceed as follows. We first gather the eigenvalues of $\mathcal{L}(0,\lambda)$ into sets $\sigma_1,\ldots,\sigma_r$ such that the eigenvalues belonging to the same set σ_i differ by integers. Then for each set σ_i with $i \in \{1,\ldots,r\}$, we shall compute, once and for all, *the general regular solution associated with* σ_i, *i.e.*, the solution $y_i(x) = \sum_{\lambda_0\in\sigma_i}\sum_{j=1}^{m_a(\lambda_0)} c_{\lambda_0,j}\, y_{\lambda_0,j}$ where the $c_{\lambda_0,j}$ are arbitrary constants in K and $y_{\lambda_0,1},\ \ldots, y_{\lambda_0,m_a(\lambda_0)}$ are $m_a(\lambda_0)$ linearly independent regular formal solutions associated with $\lambda_0 \in \sigma_i$. Choosing $\lambda_i \in \sigma_i$ such that its real part is smaller than those of the other elements of σ_i, $y_i(x)$ can then be written as $y_i(x) = \sum_{m\geq 0} U_{i,m}\, x^{\lambda_i+m}$ with $U_{i,m} \in \overline{\mathrm{K}}[\log(x)]^n$ and $U_{i,0} \neq 0$. As we have already seen, λ_i and $U_{i,0}$ must satisfy $\mathcal{L}(0,\vartheta)(x^{\lambda_i} U_{i,0}) = 0$. Hence, we choose $U_{i,0}$ of the form

$$U_{i,0} = \sum_{l=1}^{m_g(\lambda_i)}\ \sum_{j=0}^{\kappa_l(\lambda_i)-1} C_{i,l,j}\left(\sum_{k=0}^{j} v_{l,j-k}\,\frac{\log(x)^k}{k\,!}\right), \tag{11}$$

where the $C_{i,l,j}$ are arbitrary constants in K, the $\kappa_l(\lambda_i)$, for $l = 1,\ldots,m_g(\lambda_i)$, denote the partial multiplicities associated with λ_i and $v_{l,0},\ldots,v_{l,\kappa_l(\lambda_i)-1}$ the Jordan chains of maximal lengths $\kappa_l(\lambda_i)$ associated with λ_i (see Lemma 2). Then, for $m \geq 1$, we choose $U_{i,m}$ as the general polynomial solution in $\log(x)$ of the non-homogeneous system with constant matrix coefficients

$$\mathcal{L}(0, \vartheta + \lambda_i + m)(U_{i,m}) = -\sum_{k=0}^{m-1} L_{m-k}(\vartheta + \lambda_i + k)(U_{i,k}) \tag{12}$$

which is given by the following direct corollary of Proposition 1:

Corollary 1. *Consider a non-homogeneous linear differential system with constant coefficients of the form (10) and let $z(x)$ be a particular polynomial solution in $\log(x)$ of the system. If $0 \notin \sigma(L)$, then $z(x)$ is the unique (general) polynomial solution in $\log(x)$ of System (10). Otherwise, for $i = 1, \ldots, m_g(0)$, let $v_{i,0}, \ldots, v_{i,\kappa_i(0)-1}$ be the Jordan chains of maximal lengths associated with the eigenvalue 0 of $L(\lambda)$. For $i = 1, \ldots, m_g(0)$ and $j = 0, \ldots, \kappa_i(0) - 1$, let*

$$y_{i,j}(x) = v_{i,j} + v_{i,j-1} \frac{\log(x)}{1\,!} + \cdots + v_{i,0} \frac{\log(x)^j}{j\,!}.$$

Then, the general polynomial solution in $\log(x)$ of (10) is given by

$$y(x) = \sum_{\substack{1 \le i \le m_g(0) \\ 0 \le j \le \kappa_i(0)-1}} K_{i,j}\, y_{i,j}(x) + z(x),$$

where the $K_{i,j}$ are arbitrary constants in $\mathbb{K}$.

We first consider System (12) with $m = 1$. According to Corollary 1, two cases have to be distinguished: if $\lambda_i + 1$ is not an eigenvalue of $\mathcal{L}(0, \lambda)$, then the system has a unique polynomial solution in $\log(x)$: hence, in this case, $U_{i,1}$ only depends on the arbitrary constants appearing in the right-hand side of (12), *i.e.*, on the $C_{i,l,j}$ appearing in the expression of $U_{i,0}$ (see Eq. (11)). Otherwise, *i.e.*, if $\lambda_i + 1$ is an eigenvalue of $\mathcal{L}(0, \lambda)$, then $U_{i,1}$ depends on the $C_{i,l,j}$ and on $m_a(\lambda_i + 1)$ new arbitrary constants appearing in the general polynomial solution in $\log(x)$ of $\mathcal{L}(0, \vartheta + \lambda_i + 1)(y(x)) = 0$. Hence, continuing this process for $m = 2, 3, \ldots$, if we denote by $\nu_i \in \mathbb{N}$ the maximal difference between two elements of σ_i, then U_{i,ν_i} depends on the constants appearing in the right-hand side of (12) with $m = \nu_i$ and on $m_a(\lambda_i + \nu_i)$ new arbitrary constants. For $m > \nu_i$, no new arbitrary constants are introduced. Finally, the general regular solution $y_i(x) = x^{\lambda_i} \sum_{m \ge 0} U_{i,m} x^m$ computed following this method contains $\sum_{\lambda_0 \in \sigma_i} m_a(\lambda_0)$ arbitrary constants; hence it is the general regular solution associated with σ_i.

Let us now explain how to compute the vector polynomials $U_{i,m}$ for $m \ge 0$. According to the discussion following Lemma 2, $U_{i,0}$ can be obtained by computing the partial multiplicities and the Jordan chains associated with the eigenvalue λ_i of $\mathcal{L}(0, \lambda)$ in at most $\mathcal{O}(n^5\, \ell^2\, d_{\lambda_i})$ operations in $\mathbb{K}$. In the sequel, we propose a slightly more efficient method (see Lemma 4) for computing $U_{i,0}$. We then consider the calculation of the $U_{i,m}$ for $m \ge 1$. From (11), we remark that the degree in $t = \log(x)$ of $U_{i,0}$ is equal to $\max\{\kappa_l(\lambda_i), l = 1, \ldots, m_g(\lambda_i)\} - 1 \le m_a(\lambda_i) - 1$. Let α_i denote $m_a(\lambda_i) - 1$ and write $U_{i,0} = \sum_{j=0}^{\alpha_i} U_{(i,0),j} \frac{t^j}{j!}$ where the $U_{(i,0),j} \in \overline{\mathbb{K}}^n$ are to be determined. Plugging $y(x) = x^{\lambda_i} U_{i,0}$ into $\mathcal{L}(0, \vartheta)(y(x)) = 0$ and equating the coefficients of the powers of $t = \log(x)$ to zero, we find that the coefficients $U_{(i,0),j}$, for $j = \alpha_i, \ldots, 0$, satisfy the following linear system:

$$\begin{pmatrix} \mathcal{L}(0,\lambda_i) & & & & \\ \frac{\mathcal{L}'(0,\lambda_i)}{1!} & \mathcal{L}(0,\lambda_i) & & (0) & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \frac{\mathcal{L}^{(\alpha_i)}(0,\lambda_i)}{\alpha_i!} & \cdots & \cdots & \cdots & \mathcal{L}(0,\lambda_i) \end{pmatrix} \begin{pmatrix} U_{(i,0),\alpha_i} \\ U_{(i,0),\alpha_i-1} \\ \vdots \\ \vdots \\ U_{(i,0),0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Consequently, computing $U_{i,0}$ is reduced to solving recursively the $\alpha_i + 1 = m_a(\lambda_i)$ linear systems of size n given by

$$\begin{cases} \mathcal{L}(0,\lambda_i)\, U_{(i,0),\alpha_i} = 0, \\ \mathcal{L}(0,\lambda_i)\, U_{(i,0),j} = -\sum_{k=1}^{\alpha_i - j} \frac{\mathcal{L}^{(k)}(0,\lambda_i)}{k!} U_{(i,0),(j+k)}, \quad \text{for } j = \alpha_i - 1, \ldots, 0. \end{cases} \tag{13}$$

Lemma 4. *With the previous notation, computing $U_{i,0}$ by solving Systems (13) can be done using at most $\mathcal{O}(n^3\, \ell^2\, d_{\lambda_i})$ operations in* K *where d_{λ_i} denotes the degree of the extension* $K(\lambda_i)$ *over* K.

Proof. Computing all right-hand sides of Systems (13) for $j = \alpha_i - 1, \ldots, 0$ costs at most $\mathcal{O}(n^3\, \ell^2\, d_{\lambda_i})$ operations in K (see the proof of Barkatou et al., 2009, Proposition 3). Then, we have to solve $\alpha_i + 1 = m_a(\lambda_i) \leq n\ell$ systems in (13) sharing the same matrix $\mathcal{L}(0,\lambda_i)$. We proceed as follows. First we compute an LU decomposition of $\mathcal{L}(0,\lambda_i)$ which can be done in at most $\mathcal{O}(n^3\, d_{\lambda_i})$ operations in K. Consequently, solving each system of (13) is reduced to solving two systems with triangular matrices in $K(\lambda_i)^{n\times n}$. Now, solving a system with triangular matrix can be done in at most $\mathcal{O}(n^2\, d_{\lambda_i})$ operations in K; hence solving all systems in (13) can be done in at most $\mathcal{O}(n^3\, \ell\, d_{\lambda_i})$ operations in K since the total number of systems with triangular matrices to be solved is $2\,\alpha_i \leq 2\,n\,\ell$. Consequently, computing $U_{i,0}$ can be done in at most $\mathcal{O}(n^3\, \ell^2\, d_{\lambda_i})$ operations in K. □

Now, in order to compute $U_{i,m}$ for $m \geq 1$, we shall proceed in the same way as for $U_{i,0}$. Let $\rho := \lambda_i + m$ and write the right-hand side of (12) in the form $\sum_{j=0}^{d} q_j \frac{t^j}{j!}$ where $q_d \neq 0$. If ρ is an eigenvalue of $\mathcal{L}(0,\lambda)$ then, according to Proposition 1 and Corollary 1, an upper bound on the degree in $t = \log(x)$ of $U_{i,m}$ is given by $d + \max\{\kappa_l(\rho), l = 1, \ldots, m_g(\rho)\} \leq d + m_a(\rho)$. Consequently, write $U_{i,m} = \sum_{j=0}^{p} U_{(i,m),j} \frac{t^j}{j!}$ with

$$\begin{cases} p = d & \text{if } \rho \notin \sigma(\mathcal{L}(0,\lambda)), \\ p = d + m_a(\rho) & \text{if } \rho \in \sigma(\mathcal{L}(0,\lambda)), \end{cases}$$

where the $U_{(i,m),j}$ are constant vectors to be determined. As for $U_{i,0}$, solving a system of the form (12) is equivalent to solving the linear system of size $(p+1)\,n$ given by

$$\begin{pmatrix} \mathcal{L}(0,\rho) & & & & \\ \vdots & \ddots & & (0) & \\ \frac{\mathcal{L}^{(p-d)}(0,\rho)}{(p-d)!} & & \ddots & & \\ \vdots & & & \ddots & \\ \frac{\mathcal{L}^{(p)}(0,\rho)}{p!} & \cdots & \cdots & \cdots & \mathcal{L}(0,\rho) \end{pmatrix} \begin{pmatrix} U_{(i,m),p} \\ \vdots \\ U_{(i,m),d} \\ \vdots \\ U_{(i,m),0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_d \\ \vdots \\ q_0 \end{pmatrix},$$

or, equivalently, solving the $p+1$ linear systems of size n given by

$$\mathcal{L}(0,\rho)\, U_{(i,m),j} = f_j, \quad \text{for } j = p, \ldots, 0, \tag{14}$$

where $f_p = q_p, f_j = q_j - \sum_{k=1}^{p-j} \frac{\mathcal{L}^{(k)}(0,\rho)}{k!} U_{(i,m),(j+k)}$ for $j = p-1, \ldots, 0$ and $q_j = 0$ for $j > d$.

Lemma 5. *With the previous notation, computing $U_{i,m}$ by solving Systems (14) can be done in at most $\mathcal{O}(n^3\, \ell^2\, m\, d_{\lambda_i})$ operations in* K *where d_{λ_i} denotes the degree of the extension* $K(\lambda_i)$ *over* K.

Proof. Computing all right-hand sides f_j of Systems (14) for $j = p, \ldots, 0$ can be done in at most $\mathcal{O}(n^3\, \ell^2\, m\, d_{\lambda_i})$ operations in K (see the proof of Barkatou et al., 2009, Proposition 3). Then, solving the $p + 1 \leq n\,\ell$ systems of (14) can be done in at most $\mathcal{O}(n^3\, \ell\, d_{\lambda_i})$ operations in K following the same arguments as in the proof of Lemma 4. □

Before providing an algorithm derived from the previous discussion and studying its complexity, let us clarify the order of truncation of the power series involved, used in our algorithm.

Remark 1. Let $\nu \in \mathbb{N}$. If the power series in x involved in the general regular solution associated with σ_i are truncated at order ν, then $\mathcal{L}(x,\vartheta)\left(\sum_{m=0}^{\nu} U_{i,m} x^{\lambda_i+m}\right) \equiv 0 \bmod x^{\lambda_i+\nu}$, where $\Re(\lambda_i) = \min_{\lambda\in\sigma_i} \Re(\lambda)$.

Definition 6. Consider a simple linear differential system of the form (1) and $\nu \in \mathbb{N}$. Let $\sigma_1,\ldots,\sigma_r$ be the sets of eigenvalues of $\mathcal{L}(0,\lambda)$ differing by integers. We say that the general regular solution of (1) is *of order* ν if the power series involved in the general regular solution associated with each set σ_i are truncated at order ν.

Note that the computation of the general regular solution of a simple system of the form (1) of order ν requires the knowledge of the operators $\mathcal{L}(0,\vartheta), L_1(\vartheta),\ldots,L_\nu(\vartheta)$ given by (6) (see Eq. (12)). These operators depend only on the first $\nu+1$ coefficients of the expansion (5). Consequently, it suffices to truncate the entries of the matrix coefficients $A_i(x)$ at order ν, *i.e.*, $A_i(x) = \sum_{j=0}^{\nu} A_{ij} x^j + \mathcal{O}(x^{\nu+1})$.

From the method sketched above, we deduce the following algorithm:

Algorithm 1. Input: $\nu \in \mathbb{N}$ and the matrix coefficients $A_i(x)$ of a simple linear differential system of the form (1) truncated at order ν.
Output: The general regular solution of (1) of order ν.

1. Compute $\sigma(\mathcal{L}(0,\lambda))$ and gather the eigenvalues that differ by integers into sets $\sigma_1,\ldots,\sigma_r$;
2. **For** i from 1 to r **do**
   (a) Let $\lambda_i \in \sigma_i$ be such that $\Re(\lambda_i) = \min_{\lambda\in\sigma_i} \Re(\lambda)$;
   (b) Compute $U_{i,0}$ by solving the systems given by (13);
   (c) For $m = 1,\ldots,\nu$, compute $U_{i,m}$ by solving the systems given by (14);
   (d) Let $y_i(x) = x^{\lambda_i} \sum_{m=0}^{\nu} U_{i,m}\, x^m$;
   end do;
3. **Return** $y = \sum_{i=1}^{r} y_i$.

Proposition 3. *The previous algorithm is correct. It computes the general regular solution of order* ν *using at most* $\mathcal{O}(n^4\,\ell^3\,\nu^2)$ *operations in* K.

Proof. The correctness of the algorithm follows from the discussion above. Let us now study its arithmetic complexity. Let σ_i and λ_i be defined as above. According to Lemma 4, the element $U_{i,0}$ can be computed in at most $\mathcal{O}(n^3\,\ell^2\,d_{\lambda_i})$ operations in K, where d_{λ_i} denotes the degree of the extension $\mathrm{K}(\lambda_i)$ over K. Now, from Lemma 5, computing one $U_{i,m}$ can be done in at most $\mathcal{O}(n^3\,\ell^2\,m\,d_{\lambda_i})$ operations in K, and hence computing all $U_{i,m}$ for $m = 1,\ldots,\nu$ costs at most $\mathcal{O}(n^3\,\ell^2\,\nu^2\,d_{\lambda_i})$ operations in K. Consequently, computing the general regular solution associated with σ_i of order ν can be done in at most $\mathcal{O}(n^3\,\ell^2\,\nu^2\,d_{\lambda_i})$ operations in K. Since λ_i is a root of $\det(\mathcal{L}(0,\lambda))$ which is of degree at most $n\,\ell$, then $\sum_{i=1}^{r} d_{\lambda_i} \le n\,\ell$ which ends the proof. □

3. Transformation to a first-order linear differential system

Another approach for computing regular formal solutions of a simple linear differential system of the form (1) consists in converting it into a first-order system of size $n\,\ell$ and then using one of the algorithms dedicated to first-order systems: see Balser (2000), Barkatou and Pflügel (1998) and Coddington and Levinson (1955). If the first-order system obtained is of the first kind, then one can use Balser (2000, Ch. 2) or Coddington and Levinson (1955, Ch. 4). If this is not the case, then we can use the algorithm of Barkatou and Pflügel (1998) which computes regular formal solutions even in the case of an irregular singularity.

In this section, we sketch the algorithm proposed in Barkatou and Pflügel (1998) for computing regular formal solutions of first-order simple linear differential systems; then we provide a complexity analysis. This allows us to compare, from an arithmetic complexity point of view, Algorithm 1 which handles directly simple systems of the form (1) to the approach consisting in converting (1) into a first-order linear differential system of the form $D(x)\vartheta(Y(x)) - N(x)Y(x) = 0$ where

$$D(x) = \begin{pmatrix} I_{n(\ell-1)} & 0 \\ 0 & A_\ell(x) \end{pmatrix}, \quad N(x) = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & I_n \\ -A_0(x) & -A_1(x) & \cdots & \cdots & -A_{\ell-1}(x) \end{pmatrix} \tag{15}$$

and $Y(x) = (y(x)^T, \vartheta(y(x))^T, \ldots, \vartheta^{\ell-1}(y(x))^T)^T$ and applying the algorithm of Barkatou and Pflügel (1998). Note that the first-order differential system obtained is also simple since $\det(D(0)\lambda - N(0)) = \det(\mathcal{L}(0,\lambda))$ (see Gohberg et al., 1982).

We start by giving an overview on the algorithm presented in Barkatou and Pflügel (1998). Consider a first-order linear differential system of the form

$$\mathcal{D}(y(x)) = D\vartheta(y(x)) - Ny(x) = 0, \tag{16}$$

where $D = \sum_{j\geq 0} D_j x^j \in \mathrm{K}[[x]]^{n\times n}$ and $N = \sum_{j\geq 0} N_j x^j \in \mathrm{K}[[x]]^{n\times n}$. In Barkatou and Pflügel (1998), the authors look for regular solutions of (16) written in the form

$$y(x) = x^{\lambda_0}\left(h_s(x) + \log(x)\, h_{s-1}(x) + \cdots + \frac{\log^{s-1}(x)}{(s-1)!}\, h_1(x)\right),$$

where $s \in \mathbb{N}^*$, $\lambda_0 \in \overline{\mathrm{K}}$, $h_k(x) \in \overline{\mathrm{K}}[[x]]^n$ and $h_1(x) \neq 0$.

Suppose that the operator $\mathcal{D}$ is simple, *i.e.*, $\det(D_0\lambda - N_0) \not\equiv 0$; then λ_0 must satisfy $\det(D_0\lambda_0 - N_0) = 0$. Thus, the first step of the algorithm consists in gathering the eigenvalues of the pencil $D_0\lambda - N_0$ into sets $\sigma_1, \ldots, \sigma_r$ such that the eigenvalues belonging to the same set differ by integers. Then for each set σ_i with $i \in \{1, \ldots, r\}$, one computes the general regular solution associated with σ_i truncated at order ν in the following way: let $\lambda_i \in \sigma_i$ be the element having the smallest real part among the elements of σ_i and

$$y_i(x) = x^{\lambda_i} \sum_{k=1}^{s_i} \frac{\log^{s_i-k}(x)}{(s_i-k)!}\, h_{i,k}(x), \tag{17}$$

where $h_{i,k}(x) \in \overline{\mathrm{K}}[[x]]^n$ and $h_{i,1}(x) \neq 0$, be the general regular solution associated with σ_i to be computed. According to Barkatou and Pflügel (1998, Lemma 3.1), $y_i(x)$ is a solution of $\mathcal{D}(y(x)) = 0$ if and only if $\overline{\mathcal{D}_i}(h_{i,1}) = 0$ and $\overline{\mathcal{D}_i}(h_{i,k}) = -D\,h_{i,k-1}$ for $2 \leq k \leq s_i$, where $\overline{\mathcal{D}_i} = D\vartheta - (N - D\lambda_i) \in \mathrm{K}(\lambda_i)[[x]][\vartheta]^{n\times n}$. Note that s_i is bounded by the sum of the algebraic multiplicities of the eigenvalues belonging to σ_i, *i.e.*, $s_i \leq \sum_{\lambda_0\in\sigma_i} m_a(\lambda_0) \leq n$. Consequently, computing a regular solution $y_i(x)$ is reduced to finding formal power series solutions of at most n linear differential systems of first order of the form $\overline{\mathcal{D}_i}(y) = b$ where $b \in \mathrm{K}(\lambda_i)[[x]]^n$.

Now we shall explain how the authors of Barkatou and Pflügel (1998) proceed to compute the formal series solutions truncated at order ν of a linear differential system of the form

$$\overline{\mathcal{D}_i}(y) = (D\vartheta - N + D\lambda_i)(y) = b, \tag{18}$$

with $b = \mathrm{lc}(b)x^\delta + \cdots$, where the dots stand for terms of valuation greater than δ. For this purpose, let $y = cx^\mu + z$, where $\mu \in \mathbb{N}$, $\mu \leq \nu$ and c is a nonzero vector to be determined. One searches for μ and c such that $\overline{\mathcal{D}_i}(y) = b$ and $v(z) > \mu$. Plugging y into $\overline{\mathcal{D}_i}(y) = b$, one gets

$$\overline{\mathcal{D}_i}(z) = b - \overline{\mathcal{D}_i}(c\,x^\mu) = \mathrm{lc}(b)\,x^\delta + \cdots - (D_0\mu - N_0 + D_0\lambda_i)\,c\,x^\mu + \cdots. \tag{19}$$

Since $v(\overline{\mathcal{D}_i}(z)) \geq v(z)$ (see Barkatou and Pflügel, 1998, Lemma 2.1), a necessary condition for the existence of μ and c is that the valuation of the right-hand side of (19) must be greater than μ. This holds if one chooses $\mu < \delta$ such that $\det(D_0\mu - N_0 + D_0\lambda_i) = 0$ and c such that $(D_0\mu - N_0 + D_0\lambda_i)c = 0$ or $\mu = \delta$ when δ is less than or equal to ν and c satisfying the linear system $(D_0\mu - N_0 + D_0\lambda_i)c = \mathrm{lc}(b)$ (for more details, see Barkatou and Pflügel, 1998, page 575). After having found possible monomials

cx^{μ}, one performs the substitution $y = cx^{\mu} + z$ in (18). This provides a new linear differential system satisfied by z with $v(z) > \mu$ of the form

$$\overline{\mathcal{D}_i}(z) = b - \overline{\mathcal{D}_i}(cx^{\mu}) \tag{20}$$

and one iterates the process.

Lemma 6. *With the previous notation, computing formal series solutions of System (18) truncated at order ν can be done in at most $\mathcal{O}(n^3\nu^2 d_{\lambda_i})$ operations in* K *where d_{λ_i} denotes the degree of the extension* $\mathrm{K}(\lambda_i)$ *over* K.

Proof. First of all, we compute an LU decomposition of each matrix $D_0\mu - N_0 + D_0\lambda_i$ for $\mu = 0, \ldots, \nu$. This can be done in at most $\mathcal{O}(n^3\,\nu\,d_{\lambda_i})$ operations in K. After this, solving System (18) can be done in at most $\mathcal{O}(n^2\,\nu^2\,d_{\lambda_i})$ operations in K. Indeed, the computation of formal series solutions of System (18) truncated at order ν is a recursive procedure composed of two essential steps: computation of c and that of the right-hand side of (20) truncated at order ν. Moreover, this procedure is repeated at most $\nu + 1$ times since $0 \le \mu \le \nu$. The computation of c is reduced to solving two linear systems with triangular matrices in $\mathrm{K}(\lambda_i)^{n\times n}$. This can be done in at most $\mathcal{O}(n^2\,d_{\lambda_i})$ operations in K. Then, the computation of the right-hand side of (20) truncated at order ν is reduced to that of $\overline{\mathcal{D}_i}(cx^{\mu})$ truncated at order ν, *i.e.*, $\sum_{j=0}^{\nu-\mu}(D_j\mu - N_j + D_j\lambda_i)cx^{\mu+j}$, which costs at most $\mathcal{O}(n^2\,(\nu-\mu)\,d_{\lambda_i})$ operations in K. Consequently, one call of the procedure costs at most $\mathcal{O}(n^2\,(\nu-\mu)\,d_{\lambda_i})$ operations in K. Therefore, computing formal series solutions of System (18) truncated at order ν can be done in at most $\mathcal{O}(n^2\,\nu^2\,d_{\lambda_i})$ operations in K. □

The algorithm of Barkatou and Pflügel (1998) can be sketched as follows:

Algorithm BP. Input: $\nu \in \mathbb{N}$ and the matrix coefficients D and N of a simple first-order linear differential system of the form (16) truncated at order ν.
Output: The general regular solution of (16) of order ν.

1. Compute $\sigma(D_0\lambda - N_0)$ and gather the eigenvalues that differ by integers into sets $\sigma_1, \ldots, \sigma_r$;
2. **For** i from 1 to r **do**
   (a) Let $\lambda_i \in \sigma_i$ be such that $\Re(\lambda_i) = \min_{\lambda\in\sigma_i}\Re(\lambda)$;
   (b) Let $\overline{\mathcal{D}_i} = D\vartheta - (N - D\lambda_i)$;
   (c) Compute the general formal series solution of $\overline{\mathcal{D}_i}(h_{i,1}) = 0$ truncated at order ν. Set $k = 1$;
   (d) Let $k = k + 1$.
   (e) Solve the linear differential system $\overline{\mathcal{D}_i}(h_{i,k}) = -D\,h_{i,k-1}$ (see the discussion above).
      i. **If** the latter system has a parametrized solution for $h_{i,1} \neq 0$ **then** go back to step (d); **else** set $s_i = k - 1$ and $y_i = x^{\lambda_i}\,\sum_{j=1}^{s_i} h_{i,j}(x)\dfrac{\log^{s_i-j}(x)}{(s_i-j)!}$ **end if**;

   end do;
3. **Return** $y = \sum_{i=1}^{r} y_i$.

Proposition 4. *Algorithm* BP *computes the general regular solution of System (16) of order ν using at most $\mathcal{O}(n^4\,\nu^2)$ operations in* K.

Proof. Let us first determine the cost of computing the general regular solution associated with a set σ_i truncated at order ν. Computing the right-hand side $D\,h_{i,k-1}$ of one system $\overline{\mathcal{D}_i}(h_{i,k}) = -D\,h_{i,k-1}$ truncated at order ν can be done in at most $\mathcal{O}(n^2\,\nu^2\,d_{\lambda_i})$ operations in K, where d_{λ_i} denotes the degree of the extension $\mathrm{K}(\lambda_i)$ over K, since the number of matrix–vector products with entries in $\mathrm{K}(\lambda_i)$ is $\frac{1}{2}(\nu+1)(\nu+2)$. Consequently, since $s_i \le n$, the computation of all right-hand sides can be done in at most $\mathcal{O}(n^3\,\nu^2\,d_{\lambda_i})$ operations in K. Now, we shall determine the cost of solving the systems in steps (c) and (e). Note that these systems share the same operator $\overline{\mathcal{D}_i}$, *i.e.*, the same matrices $D_0\mu - N_0 + D_0\lambda_i$ for $\mu = 0, \ldots, \nu$. For this reason, we shall proceed as follows. First, we compute an LU decomposition of each matrix $D_0\mu - N_0 + D_0\lambda_i$ for $\mu = 0, \ldots, \nu$. This can be done in at most $\mathcal{O}(n^3\,\nu\,d_{\lambda_i})$ operations in K. Then, as we have already seen in the proof of Lemma 6, computing a formal series solution of each system in steps (c) and (e) truncated at order ν can be done in at most $\mathcal{O}(n^2\,\nu^2\,d_{\lambda_i})$ operations in K. Since we repeat step (e) at most n times, the total cost of solving systems appearing in this step is at most $\mathcal{O}(n^3\,\nu^2\,d_{\lambda_i})$ operations in K. Consequently, computing the general regular solution associated with σ_i truncated at order ν can be done in at most $\mathcal{O}(n^3\,\nu^2\,d_{\lambda_i})$ operations in K. Now, $\sum_{i=1}^{r} d_{\lambda_i} \le \deg(\det(D_0\lambda - N_0)) \le n$ which ends the proof. □

We have now two different approaches for computing the general regular solution of order ν of a given simple linear differential system (1) of order ℓ and size n:

- Approach 1: apply Algorithm 1 which uses at most $\mathcal{O}(n^4\,\ell^3\,\nu^2)$ operations in K.
- Approach 2: convert (1) into a first-order differential system (see (15)) then apply Algorithm BP. This can be done using at most $\mathcal{O}(n^4\,\ell^4\,\nu^2)$ operations in K since the resulting first-order system is of size $n\,\ell$.

Consequently, when $\ell \ge 2$, the first approach seems to be more efficient than the second one, while for $\ell = 1$, the two approaches are of comparable efficiency.

4. The non-simple case

The linear differential systems that we encounter in applications are not necessarily simple and, consequently, Algorithm 1 cannot be applied directly to them. In order to compute the regular formal solutions space of a non-simple system of the form (1), *i.e.*, for which $\det(\mathcal{L}(0,\lambda)) \equiv 0$, we propose to compute another differential system which is simple and from which one can get the solutions of the non-simple one.

In the remainder of the paper, we consider systems of the form (1) that are non-simple according to Definition 5. We further assume that the leading coefficient $\mathbf{A}_\ell(\mathbf{x})$ **is invertible** in $K((x))^{n\times n}$: this guarantees that the regular formal solutions space of System (1) is of finite dimension since it can be converted into a first-order system of the form $\vartheta(Y(x)) = C(x)\,Y(x)$ where $C(x) = D(x)^{-1}\,N(x) \in K((x))^{n\,\ell\times n\,\ell}$ with $D(x)$ and $N(x)$ given by (15). The invertibility of $A_\ell(x)$ allows us to suppose, without loss of generality, that $\mathbf{A}_\ell(\mathbf{x}) \in K[\mathbf{x}]^{\mathbf{n}\times\mathbf{n}}$: indeed, let $\widetilde{\mathcal{L}}(x,\vartheta) = A_\ell^{-1}(x)\,\mathcal{L}(x,\vartheta) \in K((x))[\vartheta]^{n\times n}$ and let, for $i = 1,\dots,n$, $\alpha_i = \min(0, v(\widetilde{\mathcal{L}}(x,\vartheta)(i,.)))$ and $S = \mathrm{diag}(x^{-\alpha_1},\dots,x^{-\alpha_n}) \in K[x]^{n\times n}$. Multiplying $\widetilde{\mathcal{L}}(x,\vartheta)$ on the left by S, we get a new system with matrix coefficients in $K[[x]]^{n\times n}$ and $S \in K[x]^{n\times n}$ as leading coefficient.

The problem of computing a simple system, from which we can recover the solutions of the non-simple one, has been already treated in Barkatou and Pflügel (1998) for the case $\ell = 1$: the authors show that one can always find $S(x)$ and $T(x)$ in $\mathrm{GL}_n(K((x)))$ such that the operator $\overline{\mathcal{L}}(x,\vartheta) = S(x)\mathcal{L}(x,\vartheta)T(x) \in K[[x]][\vartheta]^{n\times n}$ is simple. However, in the case $\ell \ge 2$, it is not always possible to reduce a non-simple system of the form (1) to a simple one using only algebraic transformations $S(x)$ and $T(x)$ in $\mathrm{GL}_n(K((x)))$:

Example 3. Consider the second-order linear differential system $\mathcal{L}(x,\vartheta)(y(x)) = 0$ where

$$\mathcal{L}(x,\vartheta) = A_2(x)\,\vartheta^2 + A_1(x)\,\vartheta + A_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\vartheta + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (21)$$

Its associated matrix polynomial

$$\mathcal{L}(0,\lambda) = \begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix}$$

is singular and the leading coefficient $A_2(x)$ is invertible in $\mathbb{Q}((x))^{2\times 2}$. We shall show that for any matrices $S(x)$ and $T(x)$ in $\mathbb{Q}((x))^{2\times 2}$ such that $\widetilde{\mathcal{L}}(x,\vartheta) = S(x)\,\mathcal{L}(x,\vartheta)\,T(x) \in \mathbb{Q}[[x]][\vartheta]^{2\times 2}$, the matrix differential operator $\widetilde{\mathcal{L}}(x,\vartheta)$ is always non-simple. Indeed, write

$$T(x) = T_\alpha\,x^\alpha + \cdots = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x^\alpha + \cdots \quad (\text{resp. } S(x) = S_\beta\,x^\beta + \cdots),$$

where $\alpha \in \mathbb{Z}$ (resp. $\beta \in \mathbb{Z}$), $T_\alpha \in \mathbb{Q}^{2\times 2}$ (resp. $S_\beta \in \mathbb{Q}^{2\times 2}$) is a nonzero matrix and the dots stand for terms of valuation greater than α (resp. β). The matrix coefficients of the new operator

$\widetilde{\mathcal{L}}(x,\vartheta) = \widetilde{A}_2(x)\vartheta^2 + \widetilde{A}_1(x)\vartheta + \widetilde{A}_0(x)$ defined by $\widetilde{\mathcal{L}}(x,\vartheta) = S(x)\,\mathcal{L}(x,\vartheta)\,T(x)$ can then be written as follows:

$$\begin{aligned}\widetilde{A}_2(x) &= S(x)\,A_2(x)\,T(x) = S_\beta\begin{pmatrix} a & b \\ 0 & 0\end{pmatrix} x^{\alpha+\beta} + \cdots,\\ \widetilde{A}_1(x) &= S(x)\,(2\,A_2(x)\,\vartheta(T(x)) + A_1(x)\,T(x)) = S_\beta\begin{pmatrix} c+2\,\alpha\,a & d+2\,\alpha\,b \\ a & b\end{pmatrix} x^{\alpha+\beta} + \cdots,\\ \widetilde{A}_0(x) &= S(x)(A_2(x)\vartheta^2(T(x)) + A_1(x)\vartheta(T(x)) + A_0(x)T(x))\\ &= S_\beta\begin{pmatrix} \alpha^2\,a+\alpha\,c & \alpha^2\,b+\alpha\,d \\ c+\alpha\,a & d+\alpha\,b\end{pmatrix} x^{\alpha+\beta} + \cdots,\end{aligned}$$

where the dots stand for terms of valuation greater than $\alpha+\beta$. Now consider the matrix polynomial

$$M(\lambda) = \begin{pmatrix} a & b \\ 0 & 0\end{pmatrix}\lambda^2 + \begin{pmatrix} c+2\,\alpha\,a & d+2\,\alpha\,b \\ a & b\end{pmatrix}\lambda + \begin{pmatrix} \alpha^2\,a+\alpha\,c & \alpha^2\,b+\alpha\,d \\ c+\alpha\,a & d+\alpha\,b\end{pmatrix}.$$

One can check that it is a singular matrix polynomial and none of its rows is zero since T_α is assumed to be a nonzero constant matrix. Moreover, its left nullspace is spanned by the row vector $(-1 \quad \lambda+\alpha)$. If $\alpha+\beta<0$, then the matrix coefficients of $x^{\alpha+\beta}$ in $\widetilde{A}_2(x)$, $\widetilde{A}_1(x)$ and $\widetilde{A}_0(x)$ are necessarily equal to zero since $\widetilde{\mathcal{L}}(x,\vartheta) \in \mathbb{Q}[[x]][\vartheta]^{2\times 2}$. This implies that $S_\beta\,M(\lambda) = 0$ with S_β a nonzero constant matrix which is impossible since there is no nonzero constant vector in the left nullspace of $M(\lambda)$. Consequently, $\alpha+\beta \geq 0$ and $\widetilde{\mathcal{L}}(0,\lambda)$ is equal to either the zero matrix or $S_\beta\,M(\lambda)$ and hence it is always a singular matrix polynomial.

In the next section, we provide a necessary condition for the existence of a linear substitution $y(x) = T(x)\,z(x)$ with $T(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ such that the new system $(\mathcal{L}(x,\vartheta)\,T(x))(z(x)) = 0$ is simple. An algorithm deciding the existence of such a linear substitution and computing it explicitly is developed. Note that in this case, the regular solutions of the original system are easily obtained by multiplying those of the simple one on the left by $T(x)$. In Section 6, the case where such a linear substitution does not exist is investigated. We propose a differential variant of the *EG′-method* developed in Abramov et al. (2003): the latter algorithm can only be applied to systems with polynomial coefficients. It consists in performing elementary operations on the rows of the input system and always yields a simple linear differential system $\overline{\mathcal{L}}(x,\vartheta)(y(x)) = 0$ having among its regular solutions the ones of the input one. Note that $\overline{\mathcal{L}}(x,\vartheta)(y(x)) = 0$ may have order greater than ℓ and it is not necessarily equivalent to the original system in the sense that the regular formal solutions spaces of these two systems are not necessarily isomorphic. However, we shall explain at the end of Section 6 how to obtain the regular solutions of the original system from those of the simple one.

5. Reduction to the simple case by linear substitutions

Example 3 shows that for non-simple linear differential systems of the form (1) and order $\ell \geq 2$, there do not always exist matrices $S(x)$ and $T(x)$ in $\mathrm{GL}_n(\mathrm{K}((x)))$ such that the new linear differential system $(S(x)\,\mathcal{L}(x,\vartheta)\,T(x))(z(x)) = 0$ is simple. In this section, we are merely interested in the existence of a linear substitution $y(x) = T(x)\,z(x)$ with $T(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ such that the system $\overline{\mathcal{L}}(x,\vartheta)(z(x)) = 0$, where $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{L}(x,\vartheta)\,T(x)$, is simple.

Lemma 7 (*Moser, 1960, Lemma 1*). *Every invertible matrix $T(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ can be written as*

$$T(x) = P(x)\,x^\alpha\,Q(x),$$

where $P(x) \in \mathrm{K}[x]^{n\times n}$ with $\det(P(x)) = 1$, $Q(x) \in \mathrm{K}[[x]]^{n\times n}$ with $\det(Q(0)) \neq 0$ and $\alpha = \mathrm{diag}(\alpha_1 I_{n_1},\ldots,\alpha_s I_{n_s})$ where $\alpha_1 < \cdots < \alpha_s$ are integers, $\forall\, i = 1,\ldots,s,\ n_i \in \mathbb{N}^*$ and $\sum_{i=1}^{s} n_i = n$.

Following Moser (1960) and using the notation of Lemma 7, we shall refer to $\alpha_s - \alpha_1$ as the *span* of $T(x)$ and denote it by $\mathrm{span}(T)$. This quantity is also called the *lag* of $T(x)$ in Babbitt and Varadarajan (1983).

Lemma 8 (Babbitt and Varadarajan, 1983, *Prop. 1*)**.** $\forall\ T_1,\ T_2 \in \mathrm{GL}_n(\mathrm{K}((x))),\ \ \mathrm{span}(T_1\,T_2) \leq \mathrm{span}(T_1) + \mathrm{span}(T_2)$.

The following theorem gives a necessary condition for the existence of a linear substitution leading to a simple system.

Theorem 1. *Let $\mathcal{L}(x,\vartheta)(y(x)) = 0$ be a non-simple linear differential system of the form (1). If there exists a linear substitution $y(x) = T(x)\,z(x)$ with $T(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ such that the new system $\overline{\mathcal{L}}(x,\vartheta)(z(x)) = 0$, where $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{L}(x,\vartheta)\,T(x) \in \mathrm{K}[[x]][\vartheta]^{n\times n}$, is simple, then the elements of a right minimal basis of $\mathcal{L}(0,\lambda)$ are contained in K^n,* i.e.*, the right minimal indices of $\mathcal{L}(0,\lambda)$ are all equal to zero.*

Proof. Write $T(x) = P(x)\,x^{\alpha}\,Q(x)$ where $P(x)$, α and $Q(x)$ are as in Lemma 7. If $T(x) \in \mathrm{K}[[x]]^{n\times n}$, then $\overline{\mathcal{L}}(0,\lambda) = \mathcal{L}(0,\lambda)\,T(0)$ is singular which is in contradiction with the hypotheses of the theorem. Therefore, there exists $k \in \{1,\ldots,s\}$ such that $\alpha_1 < 0,\ldots,\alpha_k < 0$ and $\alpha_{k+1} \geq 0,\ldots,\alpha_s \geq 0$. Let $\mathcal{L}_1(x,\vartheta) = \mathcal{L}(x,\vartheta)\,P(x) \in \mathrm{K}[[x]][\vartheta]^{n\times n}$. The matrix polynomial $\mathcal{L}_1(0,\lambda) = \mathcal{L}(0,\lambda)\,P(0)$ is singular and $P(0)$ is invertible, so from Lemma 3, the task reduces to proving that all the right minimal indices of $\mathcal{L}_1(0,\lambda)$ are zero. To achieve this, let $\mathcal{L}_2(x,\vartheta) = \overline{\mathcal{L}}(x,\vartheta)\,Q^{-1}(x) \in \mathrm{K}[[x]][\vartheta]^{n\times n}$. Since $Q(0)$ is invertible, $Q^{-1}(x) \in \mathrm{K}[[x]]^{n\times n}$ and $\mathcal{L}_2(0,\lambda) = \overline{\mathcal{L}}(0,\lambda)\,Q^{-1}(0)$ is regular. Note that $\mathcal{L}_2(x,\vartheta) = \mathcal{L}_1(x,\vartheta)\,x^{\alpha} \in \mathrm{K}[[x]][\vartheta]^{n\times n}$. Let $m = \sum_{i=1}^{k} n_i$. Since for $i = 1,\ldots,k$, $\alpha_i < 0$, the valuations in x of the first m columns of $\mathcal{L}_1(x,\vartheta)$ are necessarily positive, which implies that, for $i = 1,\ldots,m$, $\mathcal{L}_1(0,\lambda)(.,i) = 0$. Consequently, the number of right minimal indices of $\mathcal{L}_1(0,\lambda)$ equal to zero is greater than or equal to m. If $m = n$ or equivalently $k = s$, then the proof ends. Otherwise, the columns $\mathcal{L}_2(0,\lambda)(.,i)$ for $i = m+1,\ldots,n$ cannot be zero since $\mathcal{L}_2(0,\lambda)$ is regular, which implies that $\alpha_{k+1} = \cdots = \alpha_s = 0$ and finally $\mathcal{L}_2(0,\lambda)(.,i) = \mathcal{L}_1(0,\lambda)(.,i)$ for $i = m+1,\ldots,n$. This proves that the dimension of the right nullspace of $\mathcal{L}_1(0,\lambda)$ is exactly equal to m since the columns $\mathcal{L}_2(0,\lambda)(.,i)$ for $i = m+1,\ldots,n$ are necessarily linearly independent. □

Given a non-simple linear differential system $\mathcal{L}(x,\vartheta)(y(x)) = 0$ of the form (1) with invertible leading coefficient $A_\ell(x) \in \mathrm{K}[x]^{n\times n}$, we now develop an algorithm that either computes a linear substitution $y(x) = T(x)\,z(x)$ such that the new system $(\mathcal{L}(x,\vartheta)T(x))(z(x)) = 0$ is simple or proves that such a linear substitution does not exist. It proceeds as follows. First, compute a right minimal basis of $\mathcal{L}(0,\lambda)$. If one of its elements is non-constant, *i.e.*, belongs to $\mathrm{K}[\lambda]^n \setminus \mathrm{K}^n$, then, by Theorem 1, such a linear substitution does not exist and we are done. Otherwise, let $\mathcal{B}$ denote the matrix whose columns are the elements of the computed right minimal basis. For every column $\mathcal{B}(.,k)$, select one of its nonzero entries, say $\mathcal{B}(i_k,k)$, in such a way that the degree of the i_kth column of $A_\ell(x)$ is maximal among the degrees of the columns of $A_\ell(x)$ of indices corresponding to the nonzero entries of $\mathcal{B}(.,k)$. Then, execute the following *reduction procedure*:

- Replace the i_kth column of $\mathcal{L}(x,\vartheta)$ by $\mathcal{L}(x,\vartheta)\,\mathcal{B}(.,k)$. This is equivalent to multiplying $\mathcal{L}(x,\vartheta)$ on the right by a constant matrix T_1 defined by the identity matrix of size n whose i_kth column is replaced by the column $\mathcal{B}(.,k)$. If $\widetilde{\mathcal{L}}(x,\vartheta)$ denotes the resulting operator, then the i_kth column of $\widetilde{\mathcal{L}}(0,\lambda)$ is zero and that of its leading coefficient $\widetilde{A}_\ell(x)$ is exactly $A_\ell(x)\,\mathcal{B}(.,k)$;
- Let $\gamma_{i_k} = v\left(\widetilde{\mathcal{L}}(x,\vartheta)(.,i_k)\right)$ be the valuation of the i_kth column of $\widetilde{\mathcal{L}}(x,\vartheta)$. Since the leading coefficient $A_\ell(x)$ of $\mathcal{L}(x,\vartheta)$ is assumed to be invertible, this guarantees that $\widetilde{\mathcal{L}}(x,\vartheta)(.,i_k)$ is a nonzero column and implies that γ_{i_k} is finite, positive and less than or equal to the degree of the i_kth column of $\widetilde{A}_\ell(x)$. Multiply each component of the i_kth column of $\widetilde{\mathcal{L}}(x,\vartheta)$ on the right by $x^{-\gamma_{i_k}}$ (note that $\vartheta^j x^{-\gamma_{i_k}} = x^{-\gamma_{i_k}}(\vartheta - \gamma_{i_k})^j$). This is equivalent to multiplying $\widetilde{\mathcal{L}}(x,\vartheta)$ on the right by a matrix T_2 defined by the identity matrix of size n whose i_kth diagonal entry is replaced by $x^{-\gamma_{i_k}}$. Let $\overline{\mathcal{L}}(x,\vartheta) = \widetilde{\mathcal{L}}(x,\vartheta)\,T_2$. By definition of γ_{i_k}, the i_kth column of $\overline{\mathcal{L}}(0,\lambda)$ is nonzero.

Now, we use $\mathcal{B}(i_k,k)$ as a pivot to eliminate all the elements $\mathcal{B}(i_k,j)$ for $j \neq k$. In this way, the new columns of $\mathcal{B}$ of index $j \neq k$ belong now to the right nullspace of $\overline{\mathcal{L}}(0,\lambda)$. We then repeat the reduction procedure on $\overline{\mathcal{L}}(x,\vartheta)$ using the new columns $\mathcal{B}(.,j)$ for $j \neq k$.

Proposition 5. *The reduction procedure described above strictly reduces the degree of one column of the leading coefficient while the degrees of the other columns remain unchanged.*

Proof. The i_kth column of $\overline{\mathcal{L}}(x, \vartheta)$ is given by

$$\overline{\mathcal{L}}(x, \vartheta)(., i_k) = \mathcal{L}(x, \vartheta)\mathcal{B}(., k)\, x^{-\gamma_{i_k}} = x^{-\gamma_{i_k}} \sum_{i=0}^{\ell} A_i(x)\mathcal{B}(., k)(\vartheta - \gamma_{i_k})^i. \tag{22}$$

From the relation (22), we can deduce that the degree of the i_kth column of the leading coefficient $\overline{A}_\ell(x)$ of $\overline{\mathcal{L}}(x, \vartheta)$ is less than or equal to that of the i_kth column of $A_\ell(x)$ minus γ_{i_k}. Indeed, let d_j, for $j = 1, \ldots, n$, denote the degree of the jth column of $A_\ell(x)$ and write $A_\ell(x) = \sum_{i=0}^{d} A_{\ell,i}\, x^i$ where $d = \max_{j=1,\ldots,n} d_j$. According to (22), the i_kth column of $\overline{A}_\ell(x)$ is defined as follows:

$$\overline{A}_\ell(x)(., i_k) = x^{-\gamma_{i_k}}\, A_\ell(x)\, \mathcal{B}(., k) = \sum_{i=0}^{d} A_{\ell,i}\, \mathcal{B}(., k)\, x^{i-\gamma_{i_k}}.$$

By definition of $\gamma_{i_k} > 0$, we have $A_{\ell,i}\, \mathcal{B}(., k) = 0$, for $i = 0, \ldots, \gamma_{i_k} - 1$. Moreover, for $j \in \{1, \ldots, n\}$ such that $d_j > d_{i_k}$, we have $\mathcal{B}(j, k) = 0$ and for $j \in \{1, \ldots, n\}$ such that $d_j \leq d_{i_k}$ and $i > d_{i_k}$, we have $A_{\ell,i}(., j) = 0$. Consequently, for $d_{i_k} < i \leq d$, $A_{\ell,i}\, \mathcal{B}(., k) = \sum_{j=1}^{n} A_{\ell,i}(., j)\, \mathcal{B}(j, k) = 0$ and then $\deg(\overline{A}_\ell(x)(., i_k)) \leq d_{i_k} - \gamma_{i_k} < d_{i_k} = \deg(A_\ell(x)(., i_k))$. □

We illustrate the above approach with the following example:

Example 4. We consider the linear differential system given by

$$\mathcal{L}(x, \vartheta) \ = A_2(x)\, \vartheta^2 + A_1(x)\, \vartheta + A_0(x) = \begin{pmatrix} 1+x^2 & 2 & 1 \\ 0 & 3x & 4x \\ 0 & 0 & x \end{pmatrix} \vartheta^2$$
$$+ \begin{pmatrix} x & x^2 & 0 \\ 1 & 2 & 1 \\ 0 & x^2 & 0 \end{pmatrix} \vartheta + \begin{pmatrix} 1+x & 2 & 1 \\ 0 & x^2 & 0 \\ 2+x & 4 & 2 \end{pmatrix}.$$

The system is non-simple since its associated matrix polynomial

$$\mathcal{L}(0, \lambda) = \begin{pmatrix} \lambda^2+1 & 2\,\lambda^2+2 & \lambda^2+1 \\ \lambda & 2\,\lambda & \lambda \\ 2 & 4 & 2 \end{pmatrix}$$

is singular. A right minimal basis of $\mathcal{L}(0, \lambda)$ is given by the columns of the matrix

$$\mathcal{B} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the first column of $\mathcal{B}$ which corresponds to $k = 1$ with the previous notation. Its first two entries are nonzero but since $\deg(A_2(x)(., 1)) = 2 > \deg(A_2(x)(., 2)) = 1$, we select the first one, *i.e.*, $i_1 = 1$. We then apply our reduction procedure. It consists in first replacing $\mathcal{L}(x, \vartheta)(., 1)$ by $\mathcal{L}(x, \vartheta)\, \mathcal{B}(., 1)$. The operator obtained is given by

$$\widetilde{\mathcal{L}}(x, \vartheta) = \begin{pmatrix} -2x^2 & 2 & 1 \\ 3x & 3x & 4x \\ 0 & 0 & x \end{pmatrix} \vartheta^2 + \begin{pmatrix} -2x+x^2 & x^2 & 0 \\ 0 & 2 & 1 \\ x^2 & x^2 & 0 \end{pmatrix} \vartheta + \begin{pmatrix} -2x & 2 & 1 \\ x^2 & x^2 & 0 \\ -2x & 4 & 2 \end{pmatrix}.$$

The first column of $\widetilde{\mathcal{L}}(0, \lambda)$ is now zero and the degree of the first column of the leading coefficient has not increased. With the previous notation, we have $\gamma_1 = 1$, so we multiply $\widetilde{\mathcal{L}}(x, \vartheta)(., 1)$ on the right by x^{-1} to obtain the new matrix differential operator

$$\overline{\mathcal{L}}(x, \vartheta) = \begin{pmatrix} -2x & 2 & 1 \\ 3 & 3x & 4x \\ 0 & 0 & x \end{pmatrix} \vartheta^2 + \begin{pmatrix} 5x-2 & x^2 & 0 \\ -6 & 2 & 1 \\ x & x^2 & 0 \end{pmatrix} \vartheta + \begin{pmatrix} -3x & 2 & 1 \\ 3+x & x^2 & 0 \\ -2-x & 4 & 2 \end{pmatrix}.$$

Consequently, the first column of $\overline{\mathcal{L}}(0, \lambda)$ is nonzero and the degree of the first column of the leading coefficient has decreased by $\gamma_1 = 1$. Then, we use $\mathcal{B}(1, 1)$ as a pivot to eliminate $\mathcal{B}(1, 2)$ and obtain

$$\mathcal{B}_1 = \begin{pmatrix} -2 & 0 \\ 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

Now, we consider the second column of $\mathcal{B}_1$. Since the degrees in x of the second and third columns of the leading coefficient of $\overline{\mathcal{L}}(x, \vartheta)$ are equal, we choose one of these two columns to perform our reduction. Let us choose the second one. Replacing $\overline{\mathcal{L}}(x, \vartheta)(., 2)$ by $\overline{\mathcal{L}}(x, \vartheta)\, \mathcal{B}_1(., 2)$, we get

$$\mathcal{L}_1(x, \vartheta) = \begin{pmatrix} -2x & 0 & 1 \\ 3 & \frac{5}{2}x & 4x \\ 0 & x & x \end{pmatrix} \vartheta^2 + \begin{pmatrix} 5x-2 & -\frac{1}{2}x^2 & 0 \\ -6 & 0 & 1 \\ x & -\frac{1}{2}x^2 & 0 \end{pmatrix} \vartheta + \begin{pmatrix} -3x & 0 & 1 \\ 3+x & -\frac{1}{2}x^2 & 0 \\ -2-x & 0 & 2 \end{pmatrix}.$$

Now we have $v(\mathcal{L}_1(x, \vartheta)(., 2)) = 1$. Then, we multiply $\mathcal{L}_1(x, \vartheta)(., 2)$ on the right by x^{-1} and obtain a new differential system $\mathcal{L}_2(x, \vartheta)$ given by

$$\mathcal{L}_2(x, \vartheta) = \begin{pmatrix} -2x & 0 & 1 \\ 3 & \frac{5}{2} & 4x \\ 0 & 1 & x \end{pmatrix} \vartheta^2 + \begin{pmatrix} 5x-2 & -\frac{1}{2}x & 0 \\ -6 & -5 & 1 \\ x & -2-\frac{1}{2}x & 0 \end{pmatrix} \vartheta + \begin{pmatrix} -3x & \frac{1}{2}x & 1 \\ 3+x & \frac{5}{2}-\frac{1}{2}x & 0 \\ -2-x & 1+\frac{1}{2}x & 2 \end{pmatrix}.$$

The determinant of the matrix polynomial $\mathcal{L}_2(0, \lambda)$ is equal to $(\lambda - 1)^2\,(3\,\lambda^4 - 6\,\lambda^3 + 13\,\lambda^2 - 16\,\lambda + 8)$, so $\mathcal{L}_2(x, \vartheta)$ is simple. Note that $\mathcal{L}_2(x, \vartheta) = \mathcal{L}(x, \vartheta)\, T(x)$ where $T(x)$ is the invertible matrix given by

$$T(x) = \begin{pmatrix} -\frac{2}{x} & 0 & 0 \\ \frac{1}{x} & -\frac{1}{2x} & 0 \\ 0 & \frac{1}{x} & 1 \end{pmatrix}.$$

The following corollary follows from Proposition 5.

Corollary 2. *The number of iterations of the reduction procedure described above does not exceed* $D = \sum_{j=1}^{n} \deg(A_\ell(x)(., j))$.

Thus, after applying the reduction procedure at most D times, we obtain a new matrix differential operator $\overline{\mathcal{L}}(x, \vartheta) = \mathcal{L}(x, \vartheta)\, T(x)$ with $T(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ such that either $\overline{\mathcal{L}}(x, \vartheta)$ is simple or a right minimal basis of $\overline{\mathcal{L}}(0, \lambda)$ contains non-constant elements.

Lemma 9. *The matrix $T(x)$ constructed by applying iteratively the reduction procedure described above is a matrix polynomial in x^{-1} satisfying $v(T) \geq -D$ and* $\mathrm{span}(T) \leq D$ *where* $D = \sum_{j=1}^{n} \deg(A_\ell(x)(., j))$.

Proof. The matrix $T(x)$ is the product of invertible matrices which are either constant matrices or diagonal matrices of the form $\mathrm{diag}(1, \ldots, 1, x^{-\gamma_{i_k}}, 1, \ldots, 1)$ where $1 \leq \gamma_{i_k} \leq \deg(A_\ell(x)(., i_k))$. Therefore, $T \in \mathrm{K}[x^{-1}]^{n \times n}$ with $v(T) \geq -\sum_{i_k} \gamma_{i_k}$ and $\mathrm{span}(T) \leq \sum_{i_k} \gamma_{i_k}$ (see Lemma 8). Now, from Proposition 5 and Corollary 2, $\sum_{i_k} \gamma_{i_k}$ cannot exceed D, which ends the proof. □

In practice, we deal with matrix differential operators with truncated matrix coefficients. Let $\mathcal{L}_N(x, \vartheta)$ denote the operator $\mathcal{L}(x, \vartheta)$ given by (1) truncated at order N, *i.e.*, $\mathcal{L}(x, \vartheta) = \mathcal{L}_N(x, \vartheta) \mod x^{N+1}$. The following proposition shows how to choose N so that if we apply iteratively the reduction procedure described above to $\mathcal{L}_N(x, \vartheta)$ and get a matrix T such that $\mathcal{L}_N(x, \vartheta)\, T$ is simple, then we can ensure that the whole system $\mathcal{L}(x, \vartheta)\, T$ is simple too.

Proposition 6. *With the above notation, we have*

$$\forall\, N \in \mathbb{N}, \quad \mathcal{L}(x, \vartheta)\, T = \mathcal{L}_N(x, \vartheta)\, T \mod x^{N+v(T)+1}.$$

Therefore, if we choose N greater than or equal to $-v(T)$, then $\mathcal{L}_N(x, \vartheta)\, T$ being simple implies that $\mathcal{L}(x, \vartheta)\, T$ is simple too.

Proof. The first assertion is obvious. Now, for $N \geq -v(T)$, we have, in particular, $\mathcal{L}(x, \vartheta)\, T = \mathcal{L}_N(x, \vartheta)\, T \mod x$. Therefore, the matrix polynomials associated respectively with $\mathcal{L}(x, \vartheta)\, T$ and $\mathcal{L}_N(x, \vartheta)\, T$ are equal. Hence, $\mathcal{L}_N(x, \vartheta)\, T$ being simple implies that $\mathcal{L}(x, \vartheta)\, T$ is simple too. □

From the discussion above, we derive the following algorithm:

Algorithm 2. Input: An integer $N \geq \sum_{j=1}^{n} \deg(A_\ell(x)(.,j))$ and $\mathcal{L}_N(x, \vartheta)$, the truncation of $\mathcal{L}(x, \vartheta)$ at order N, i.e., $\mathcal{L}_N(x, \vartheta) = \sum_{j=1}^{N} x^j L_j(\vartheta) + \mathcal{L}(0, \vartheta)$ where the $L_j(\vartheta)$ are given by (6).
Output: The empty list [ ] in the case where $\mathcal{L}(x, \vartheta)$ cannot be reduced to a simple operator by means of a linear substitution or an invertible matrix $T \in \mathrm{K}[x^{-1}]^{n\times n}$ such that $\mathcal{L}(x, \vartheta)\, T$ is simple.

1. **Initialization:** $T = I_n$ and $\overline{\mathcal{L}}(x, \vartheta) = \mathcal{L}_N(x, \vartheta)$;
2. **While** $\overline{\mathcal{L}}(0, \lambda)$ is singular **do**
   (a) Compute a right minimal basis of $\overline{\mathcal{L}}(0, \lambda)$;
   (b) **If** it contains a non-constant element **then**
      i. **Return** [ ];
   (c) **Else**
      i. Let $\mathcal{B}$ be the matrix whose columns are the elements of the right minimal basis;
      ii. **For** each column $\mathcal{B}(., k)$ **do**
         A. Let $\overline{A}_\ell$ be the leading coefficient of $\overline{\mathcal{L}}(x, \vartheta)$;
         B. Let $J_k = \{j \in \{1, \ldots, n\}$ such that $\mathcal{B}(j, k) \neq 0\}$;
         C. Choose $i_k \in J_k$ such that $\deg(\overline{A}_\ell(., i_k)) \geq \deg(\overline{A}_\ell(., j))\ \ \forall j \in J_k$;
         D. Apply the following reduction procedure:
            - Let $\overline{\mathcal{L}}(x, \vartheta)(., i_k) = \overline{\mathcal{L}}(x, \vartheta)\, \mathcal{B}(., k)$;
            - Let $\gamma_{i_k} = v\left(\overline{\mathcal{L}}(x, \vartheta)(., i_k)\right)$ and $\overline{\mathcal{L}}(x, \vartheta)(j, i_k) = \overline{\mathcal{L}}(x, \vartheta)(j, i_k)\, x^{-\gamma_{i_k}}$ for $j = 1, \ldots, n$;
         E. Let $T(., i_k) = x^{-\gamma_{i_k}}\, T\, \mathcal{B}(., k)$;
         F. Use $\mathcal{B}(i_k, k)$ as a pivot to eliminate all the elements $\mathcal{B}(i_k, j)$ with $j \neq k$;
         end do;
      end if;
   end do;
3. **Return** T.

Proposition 7. *Let $\mathcal{L}(x, \vartheta)$ be a non-simple matrix differential operator of the form (1) with invertible leading coefficient $A_\ell(x) \in \mathrm{K}[x]^{n\times n}$. Let $D = \sum_{j=1}^{n} \deg(A_\ell(x)(.,j))$ and $N \geq D$ be the order of truncation of the matrix coefficients $A_i(x)$ of $\mathcal{L}(x, \vartheta)$. Then, Algorithm 2 stops after at most D calls of the reduction procedure and uses at most $\widetilde{\mathcal{O}}(n^{\omega+1}\, \ell\, N\, D)$ operations in* K.

Proof. The first assertion follows from Corollary 2. Now, let us study the complexity of the algorithm. As we have already seen, computing a right minimal basis of a matrix polynomial of size n and entries of degree bounded by ℓ can be done in $\widetilde{\mathcal{O}}(n^{\omega+1}\, \ell)$ arithmetic operations (see Section 1.2). In the algorithm, we compute at most D right minimal bases, so the total cost of minimal bases computations is bounded by $\widetilde{\mathcal{O}}(n^{\omega+1}\, \ell\, D)$ operations. In the first step of the reduction procedure, write $\overline{\mathcal{L}}(x, \vartheta) = \sum_{j=0}^{\overline{N}} x^j\, \overline{L}_j(\vartheta)$ with $\overline{N} \leq N$. The cost of computing the product $\overline{\mathcal{L}}(x, \vartheta)\, \mathcal{B}(., k)$ is bounded by $(\overline{N} + 1)$ times the cost of computing one product $\overline{L}_j(\vartheta)\, \mathcal{B}(., k)$. Now, $\overline{L}_j(\vartheta)$ is a constant matrix operator of order at most ℓ, so computing $\overline{L}_j(\vartheta)\,\mathcal{B}(.,k)$ can be done using at most $\mathcal{O}(n^2\,\ell)$ operations in K. Consequently, the first step of the reduction procedure can be done in at most $\mathcal{O}(n^2\,\ell\,N)$ operations in K. Now, multiplying each component of the column $\overline{\mathcal{L}}(x,\vartheta)(.,i_k)$ on the right by $x^{-\gamma_{i_k}}$ can be done by substituting ϑ by $\vartheta-\gamma_{i_k}$ in the i_kth column of each matrix $\overline{L}_j(\vartheta)$. The latter operation uses at most $\mathcal{O}(n\,\ell)$ operations in K. Since we have at most $N+1$ matrices $\overline{L}_j(\vartheta)$, then the total cost of the second step of the reduction procedure is at most $\mathcal{O}(n\,\ell\,N)$ operations in K. Consequently, the reduction procedure can be done using at most $\mathcal{O}(n^2\,\ell\,N)$ operations in K. As it is repeated at most D times, the total cost is then $\mathcal{O}(n^2\,\ell\,N\,D)$ operations in K. Note that Steps E and F can be done in at most $\mathcal{O}(n^2)$ operations in K and are repeated at most D times, so we get $\mathcal{O}(n^2\,D)$ operations in K. Consequently, the total cost of the algorithm is at most $\widetilde{\mathcal{O}}(n^{\omega+1}\,\ell\,N\,D)$ operations in K. □

Computation of the general regular solution of $\mathcal{L}(x,\vartheta)(y(x))=0$ of a given order. The following proposition shows how to choose N in Algorithm 2 and ν in Algorithm 1 in order to get the general regular solution of $\mathcal{L}(x,\vartheta)(y(x))=0$ of a fixed order in the sense of Definition 6.

Proposition 8. *Let $\mathcal{L}(x,\vartheta)(y(x))=0$ be a non-simple system of the form (1) with invertible leading matrix coefficient $A_\ell(x)\in \mathrm{K}[x]^{n\times n}$. Suppose that $\mathcal{L}(x,\vartheta)$ can be reduced to a simple operator by means of a linear substitution. Let $D=\sum_{j=1}^{n}\deg(A_\ell(x)(.,j))$ and $\nu_1\in\mathbb{N}$. The general regular solution of order ν_1 of $\mathcal{L}(x,\vartheta)(y(x))=0$ can be computed as follows:*

1. *apply Algorithm 2 to $\mathcal{L}_N(x,\vartheta)$ with $N=\nu_1+2D$ and let T be the computed matrix; then*
2. *apply Algorithm 1 to $\mathcal{L}_N(x,\vartheta)T$ with $\nu=\nu_1+D$, and*
3. *finally, multiply the output of Algorithm 1 on the left by T.*

Proof. We shall start by proving that to compute the regular solutions of $\mathcal{L}(x,\vartheta)(y(x))=0$ up to order ν_1, it suffices to compute those of $\overline{\mathcal{L}}(x,\vartheta)(z(x))=0$ up to order ν_1+D. Write $T=P(x)\,x^\alpha\,Q(x)$ where $P(x)\in\mathrm{K}[x]^{n\times n}$ and $Q(x)\in\mathrm{K}[[x]]^{n\times n}$ are both unimodular and $\alpha=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$ with $\alpha_1\le\cdots\le\alpha_n$ and $\alpha_1<0$ because $v(T)<0$. Put $\mathcal{L}_1(x,\vartheta)=\mathcal{L}(x,\vartheta)\,P(x)$ and $\mathcal{L}_2(x,\vartheta)=\overline{\mathcal{L}}(x,\vartheta)\,Q^{-1}(x)$; then $\mathcal{L}_2(x,\vartheta)=\mathcal{L}_1(x,\vartheta)\,x^\alpha$. Since $P(x)$ is unimodular, to each regular solution y of $\mathcal{L}(x,\vartheta)$ associated with the exponent $\lambda_0\in\overline{\mathrm{K}}$ and of order ν_1 there corresponds a regular solution $u=P^{-1}(x)\,y$ of $\mathcal{L}_1(x,\vartheta)$ associated with the same exponent λ_0 and of the same order ν_1. Similarly, to each regular solution z of $\overline{\mathcal{L}}(x,\vartheta)$ there corresponds a regular solution $w=Q(x)\,z$ of $\mathcal{L}_2(x,\vartheta)$ with the same exponent and order. Additionally, u and w are related by $u=x^\alpha w$. Our problem is now reduced to showing that in order to compute u up to order ν_1 it suffices to truncate w at order ν_1+D. For this, write $w=x^{\lambda_1}\widetilde{w}$ with $\lambda_1\in\overline{\mathrm{K}}$ and

$$\widetilde{w}=\widetilde{w}_0+\widetilde{w}_1x+\cdots+\widetilde{w}_kx^k+\cdots\quad\text{where } \widetilde{w}_k\in\overline{\mathrm{K}}[\log(x)]^n \text{ and } \widetilde{w}_0\neq 0.$$

Let $\widetilde{w}^{(i)}$ denote the ith component of $\widetilde{w}$ and $t_i=v\left(\widetilde{w}^{(i)}\right)\ge 0$. Put $\widetilde{u}=x^\alpha\widetilde{w}$ and $m=v(\widetilde{u})=\min_{i=1,\ldots,n}(\alpha_i+t_i)$. Since $\widetilde{w}_0\neq 0$, then there exists $i_0\in\{1,\ldots,n\}$ such that the i_0th component of $\widetilde{w}_0$ is nonzero, *i.e.*, for which $t_{i_0}=0$. Therefore, m is well defined and $\alpha_1\le m\le\alpha_{i_0}+t_{i_0}=\alpha_{i_0}\le\alpha_n$. Since $u=x^{\lambda_1}\widetilde{u}$, u is then a regular solution of $\mathcal{L}_1(x,\vartheta)$ with exponent λ_1+m. Write

$$\widetilde{u}=x^m\left(\widetilde{u}_0+\cdots+\widetilde{u}_kx^k+\cdots\right)\quad\text{where } \widetilde{u}_k\in\overline{\mathrm{K}}[\log(x)]^n \text{ and } \widetilde{u}_0\neq 0.$$

Computing u up to order ν_1 requires the knowledge of the coefficients $\widetilde{u}_0,\ldots,\widetilde{u}_{\nu_1}$. Now, the ith component $\widetilde{u}_k^{(i)}$ of $\widetilde{u}_k$ is the coefficient of x^{k+m} in $\widetilde{u}^{(i)}=x^{\alpha_i}\widetilde{w}^{(i)}$, and hence, $\widetilde{u}_k^{(i)}=\widetilde{w}_{k+m-\alpha_i}^{(i)}$. Thus, the coefficients $\widetilde{u}_0,\ldots,\widetilde{u}_{\nu_1}$ depend on $\widetilde{w}_0,\ldots,\widetilde{w}_{\nu_1+m-\alpha_1}$. Hence, it suffices to compute w up to order ν_1+D since $\nu_1+m-\alpha_1\le\nu_1+\alpha_n-\alpha_1=\nu_1+\mathrm{span}(T)\le\nu_1+D$ (see Lemma 9).

Now, since $\overline{\mathcal{L}}(x,\vartheta)$ is simple, we have only to consider $\overline{\mathcal{L}}(x,\vartheta)=\mathcal{L}(x,\vartheta)T$ truncated at order ν_1+D. Thus, according to Proposition 6, we only need to truncate $\mathcal{L}(x,\vartheta)$ at order $\nu_1+D-v(T)$, so we take $N=\nu_1+2D\ge\nu_1+D-v(T)$. □

6. A differential variant of the *EG′-algorithm*

In this section, we consider a non-simple linear differential system of the form (1) with invertible leading coefficient $A_\ell(x)$ and we suppose here that all matrix coefficients $A_i(x)$ are **matrix polynomials** of size n. Inspired by the *EG′-algorithm* proposed by Abramov et al. in Abramov et al. (2003) (see also Abramov, 1999; Abramov and Bronstein, 2001, 2002), we shall develop an algorithm which consists in carrying out elementary operations on the rows of $\mathcal{L}(x,\vartheta)$ and always yields a simple operator $\overline{\mathcal{L}}(x,\vartheta)$ of the form $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{P}(x,\vartheta)\mathcal{L}(x,\vartheta)$ where $\mathcal{P}(x,\vartheta) \in \mathrm{K}((x))[\vartheta]^{n\times n}$. We then explain how to recover the regular solutions of $\mathcal{L}(x,\vartheta)$ from those of $\overline{\mathcal{L}}(x,\vartheta)$ which can be computed by Algorithm 1.

6.1. Preliminaries

In the sequel, we use definitions and terminologies defined in Beckermann et al. (2006) for matrices of Ore polynomials and we adapt them to matrix differential operators.

Definition 7. Let $\mathcal{L}(x,\vartheta) \in \mathrm{K}((x))[\vartheta]^{n\times n}$ be a matrix differential operator and $J \subseteq \{1,\ldots,n\}$. The rows $\mathcal{L}(x,\vartheta)(i,.)$ with $i \in J$ are said to be $\mathrm{K}((x))[\vartheta]$-*linearly dependent* if there exist differential operators $\{W_i\}_{i\in J}$ in $\mathrm{K}((x))[\vartheta]$ not all zero such that $\sum_{i\in J} W_i\,\mathcal{L}(x,\vartheta)(i,.) = 0$. Otherwise, they are called $\mathrm{K}((x))[\vartheta]$-*linearly independent*. The *rank of* $\mathcal{L}(x,\vartheta)$ is the maximum number of $\mathrm{K}((x))[\vartheta]$-linearly independent rows of $\mathcal{L}(x,\vartheta)$.

We are merely interested in applying elementary row operations of two types to a matrix differential operator $\mathcal{L}(x,\vartheta)$. The elementary row operations of the first type include:

(E1) interchange two rows of $\mathcal{L}(x,\vartheta)$;
(E2) multiply a row of $\mathcal{L}(x,\vartheta)$ on the left by a nonzero scalar differential operator with coefficients in $\mathrm{K}((x))$;
(E3) add to a row of $\mathcal{L}(x,\vartheta)$ another one multiplied on the left by a scalar differential operator with coefficients in $\mathrm{K}((x))$.

Those of the second type include the elementary row operations (E1) and (E3) and:

(E2′) multiply a row of $\mathcal{L}(x,\vartheta)$ on the left by a nonzero element of $\mathrm{K}((x))$.

Note that each elementary row operation can be performed by multiplying $\mathcal{L}(x,\vartheta)$ on the left by a square matrix differential operator.

Definition 8 (*Miyake, 1980*)**.** A square matrix differential operator $\mathcal{P}(x,\vartheta)$ of size n is said to be *invertible* if there exists another matrix differential operator $\mathcal{Q}(x,\vartheta)$ such that $\mathcal{Q}(x,\vartheta)\,\mathcal{P}(x,\vartheta) = \mathcal{P}(x,\vartheta)\,\mathcal{Q}(x,\vartheta) = I_n$. Two matrix differential operators $\mathcal{L}(x,\vartheta)$ and $\bar{\mathcal{L}}(x,\vartheta)$ are said to be *left-equivalent* if there exists an invertible matrix differential operator $\mathcal{P}(x,\vartheta)$ such that $\bar{\mathcal{L}}(x,\vartheta) = \mathcal{P}(x,\vartheta)\,\mathcal{L}(x,\vartheta)$.

Two left-equivalent matrix differential operators have the same regular formal solutions space.

Lemma 10 (*Miyake, 1980, Theorem III*)**.** *A matrix differential operator $\mathcal{P}(x,\vartheta)$ is invertible if and only if it can be expressed as a product of elementary row operations of the second type.*

Lemma 11 (*Beckermann et al., 2006, Lemma A.3*)**.** *The rank of a matrix differential operator $\mathcal{L}(x,\vartheta)$ does not change if we apply elementary row operations of the first type or of the second type to $\mathcal{L}(x,\vartheta)$.*

Proposition 9. *The rank of a matrix differential operator of the form $\mathcal{L}(x,\vartheta) = \sum_{i=0}^{\ell} A_i(x)\vartheta^i$ where for $i = 0,\ldots,\ell$, $A_i(x) \in \mathrm{K}((x))^{n\times n}$ and $A_\ell(x) \in \mathrm{GL}_n(\mathrm{K}((x)))$ equals n.*

Proof. Since $A_\ell(x)$ is an invertible matrix, we may suppose, without loss of generality, that it is the identity matrix I_n. If the rows of $\mathcal{L}(x,\vartheta)$ are $\mathrm{K}((x))[\vartheta]$-linearly dependent, then there exist $W_1,\ldots,W_n \in \mathrm{K}((x))[\vartheta]$ not all zero such that $\sum_{i=1}^{n} W_i\,\mathcal{L}(x,\vartheta)(i,.) = 0$. Since the leading coefficient of $\mathcal{L}(x,\vartheta)$ is supposed to be the identity matrix, the order of each diagonal entry $\mathcal{L}(x,\vartheta)(i,i)$ is greater than those of other entries $\mathcal{L}(x,\vartheta)(i,j)$ for $j \neq i$. Now, choose $j_0 \in \{1,\ldots,n\}$ such that the order of the differential operator W_{j_0} is greater than or equal to the orders of all the W_j for $j \neq j_0$. We have $W_{j_0}\,\mathcal{L}(x,\vartheta)(j_0,j_0) = -\sum_{i\neq j_0} W_i\,\mathcal{L}(x,\vartheta)(i,j_0)$ which is impossible since the order of the left-hand side of the latter equality is greater than the order of its right-hand side. □

6.2. Algorithm

The following algorithm consists in applying elementary row operations of the first or second type to a non-simple matrix differential operator $\mathcal{L}(x,\vartheta) \in K[x][\vartheta]^{n\times n}$ with invertible leading coefficient and yields another operator $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{P}(x,\vartheta)\mathcal{L}(x,\vartheta)$ whose rank is equal to that of $\overline{\mathcal{L}}(0,\lambda)$. We know, from Proposition 9, that the rank of $\mathcal{L}(x,\vartheta)$ is equal to n. Consequently, according to Lemma 11, the rank of $\overline{\mathcal{L}}(x,\vartheta)$ is also n, so $\operatorname{rank}(\overline{\mathcal{L}}(0,\lambda)) = n$ and $\overline{\mathcal{L}}(x,\vartheta)$ is simple. Note that the regular formal solutions space of the original system $\mathcal{L}(x,\vartheta)(y(x)) = 0$ is a subspace of that of $\overline{\mathcal{L}}(x,\vartheta)(y(x)) = 0$. However, depending on the elementary row operations performed on $\mathcal{L}(x,\vartheta)$, it may be that the two systems share the same regular formal solutions space. The steps of the following algorithm are very close to those of Algorithm 2 developed in the previous section. The main changes are the following:

- We work on the rows of the matrix differential operator instead of working on its columns. In particular, we act on the left and compute left minimal bases. Note that a consequence is that the termination criterion of the algorithm changes slightly.
- We consider all the rows of the left minimal bases and not only the constant ones.
- As our goal is to find the regular formal solutions of the non-simple system, at each step of the reduction, we look at the type of the elementary row operation performed. If we apply an elementary row operation of the first and not the second type, then we keep the index of the corresponding row in a set $\mathcal{K}$ (see Algorithm 3 below) that will be needed in Section 6.3 to reconstruct the regular solutions of the original system.

Algorithm 3. Input: a non-simple matrix differential operator $\mathcal{L}(x,\vartheta) \in K[x][\vartheta]^{n\times n}$ with invertible leading matrix coefficient.
Output: a simple matrix differential operator $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{P}(x,\vartheta)\,\mathcal{L}(x,\vartheta)$ and a set $\mathcal{K}$.

1. **Initialization:** $\overline{\mathcal{L}}(x,\vartheta) = \mathcal{L}(x,\vartheta)$ and $\mathcal{K} = \{\}$;
2. **While** $\overline{\mathcal{L}}(0,\vartheta)$ is singular **do**
   (a) Compute a left minimal basis of $\overline{\mathcal{L}}(0,\vartheta)$;
   (b) Let $\mathcal{B}$ be the matrix whose rows are the elements of the left minimal basis;
   (c) **For** each row $\mathcal{B}(k,.)$ **do**
      i. Let $J_k = \{j \in \{1,\ldots,n\}$ such that $\mathcal{B}(k,j) \neq 0\}$;
      ii. Choose $i_k \in J_k$ such that $\deg_x(\overline{\mathcal{L}}(x,\vartheta)(i_k,.)) \geq \deg_x(\overline{\mathcal{L}}(x,\vartheta)(j,.))\ \ \forall j \in J_k$;
      iii. Apply the following reduction procedure:
         - Let $\overline{\mathcal{L}}(x,\vartheta)(i_k,.) = \mathcal{B}(k,.)\,\overline{\mathcal{L}}(x,\vartheta)$;
         - Let $\beta_{i_k} = v\left(\overline{\mathcal{L}}(x,\vartheta)(i_k,.)\right)$ and $\overline{\mathcal{L}}(x,\vartheta)(i_k,.) = x^{-\beta_{i_k}}\,\overline{\mathcal{L}}(x,\vartheta)(i_k,.)$;
      iv. **If** $\deg_\vartheta(\mathcal{B}(k,i_k)) \neq 0$ **then** $\mathcal{K} = \mathcal{K} \cup \{i_k\}$ **end if**;
      v. **If** $\deg_\vartheta(\mathcal{B}(k,.)) = 0$ **then** use $\mathcal{B}(k,i_k)$ as a pivot to eliminate all the elements $\mathcal{B}(j,i_k)$ with $j \neq k$; **else** go back to step 2 **end if**;

      end do

   end do
3. **Return** $\overline{\mathcal{L}}(x,\vartheta)$ and $\mathcal{K}$.

Proposition 10. *Let $\mathcal{L}(x,\vartheta) = \sum_{i=0}^{\ell} A_i(x)\vartheta^i \in K[x][\vartheta]^{n\times n}$ be a non-simple matrix differential operator with invertible leading coefficient $A_\ell(x)$ and $N = \max_{i=0,\ldots,\ell} \deg(A_i(x))$. Let ℓ_{simple} ($\ell_{\text{simple}} \leq n^{nN}\,\ell$) denote the order of the output operator of Algorithm 3. Then, Algorithm 3 stops after at most $n\,N$ calls of the reduction procedure and uses at most $\widetilde{\mathcal{O}}(n^4\,N\,\ell_{\text{simple}}^2)$ operations in* K.

Proof. Each time we execute the reduction procedure, the degree in x of one row of the operator $\overline{\mathcal{L}}(x,\vartheta)$ decreases by at least 1 and those of the other rows are unchanged (adapt the proof of Proposition 5: the columns of the leading coefficient are replaced by the rows of the matrix differential operator). Consequently, either the algorithm stops before performing $n\,N$ times the reduction procedure or after the $(n\,N)$th reduction, in which case, the output operator has constant coefficients and it is of rank n (by Lemma 11); therefore, it is necessarily simple. This proves the first claim. Now, we study the arithmetic complexity of the algorithm. Computing a left minimal basis of a singular matrix polynomial of size n having entries of degree bounded by an integer d can be done using $\widetilde{\mathcal{O}}(n^{\omega+1}\, d)$ operations in K and the degrees of the elements of a left minimal basis are bounded by $r\,.\,d < n\,d$ where r denotes the rank of the singular matrix polynomial (see Section 1.2). Consequently, if we suppose that, after running i times the loop **while**, the operator $\overline{\mathcal{L}}(x,\vartheta)$ is still non-simple and if $\ell_i \in \mathbb{N}^*$ ($\ell_0 = \ell$) denotes its order and r_i denotes the rank of $\overline{\mathcal{L}}(0,\vartheta)$, then we have $\ell_i \le \ell_{i+1} \le \ell_i + r_i\,\ell_i \le n\,\ell_i$. Now in the worst case, we run $n\,N$ times the loop **while**, so if ℓ_{simple} denotes the order of the output operator then we have $\ell_{\text{simple}} \le n^{n\,N}\,\ell$. Then, the cost of computing all the left minimal bases is bounded by $\widetilde{\mathcal{O}}(n^{\omega+1}\,\ell_{\text{simple}})$ operations in K. Now applying the same analysis as in the proof of Proposition 7 and taking into account the degrees of the elements of the computed left minimal bases, we obtain that the cost of a reduction procedure and of Step v in the ith passage in the loop **while** is at most $\mathcal{O}(n^4\,N\,\ell_{i-1}^2)$ operations in K. The total cost of all reduction procedures is thus bounded by $\mathcal{O}(n^4\,N\,\ell_{\text{simple}}^2)$ operations in K which ends the proof. □

We shall make a few comments on Algorithm 3:

1. We use $\mathcal{B}(k,i_k)$ as a pivot to eliminate all the elements $\mathcal{B}(j,i_k)$ with $j \neq k$ only if $\mathcal{B}(k,.)$ is a constant row; otherwise we may increase the degrees of the elements of $\mathcal{B}$ and, consequently, we cannot ensure in the proof of Proposition 10 that $\ell_{i+1} \le \ell_i + r_i\,\ell_i$.
2. Algorithm 3 can be applied more generally to any non-simple matrix differential operator $\mathcal{L}(x,\vartheta) = \sum_{i=0}^{\ell} A_i(x)\,\vartheta^i \in \mathrm{K}[x][\vartheta]^{n\times n}$ of full rank and not necessarily with invertible leading coefficient $A_\ell(x)$.
3. The complexity result that we give is a worst case estimate. In practice, the potentially exponential growth of the order of the operator does not seem to be a serious limitation.

Example 5. Consider the matrix differential operator defined by

$$\mathcal{L}(x,\vartheta) = \begin{pmatrix} x & 0 & 0\\ 0 & x & x\\ 0 & 0 & 1\end{pmatrix}\vartheta^3 + \begin{pmatrix} 0 & x^3 & 1\\ 0 & x^2 & 1\\ x & 1 & 0\end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 1 & x^2\\ 0 & 1 & x\\ 1 & 0 & 2x\end{pmatrix}\vartheta + \begin{pmatrix} 1 & 0 & 0\\ 1 & 0 & 0\\ 0 & x^2 & 4x^3\end{pmatrix}.$$

The associated matrix polynomial $\mathcal{L}(0,\vartheta)$ given by

$$\mathcal{L}(0,\vartheta) = \begin{pmatrix} 1 & \vartheta & \vartheta^2\\ 1 & \vartheta & \vartheta^2\\ \vartheta & \vartheta^2 & \vartheta^3\end{pmatrix}$$

is singular; thus the operator $\mathcal{L}(x,\vartheta)$ is non-simple. A right minimal basis of $\mathcal{L}(0,\vartheta)$ is given by the columns of the matrix $\begin{pmatrix} -\vartheta & 0\\ 1 & -\vartheta\\ 0 & 1\end{pmatrix}$; hence, according to Theorem 1, there exists no linear substitution yielding a simple operator. Consequently, we shall apply Algorithm 3 above. A left minimal basis of $\mathcal{L}(0,\vartheta)$ is given by the rows of the matrix

$$\mathcal{B} = \begin{pmatrix} -1 & 1 & 0\\ \vartheta & 0 & -1\end{pmatrix}.$$

We start by considering the first row $\mathcal{B}(1,.) = \begin{pmatrix} -1 & 1 & 0\end{pmatrix}$ of $\mathcal{B}$. Its first two entries are nonzero and since $\deg_x(\mathcal{L}(x,\vartheta)(1,.)) = 3 > \deg_x(\mathcal{L}(x,\vartheta)(2,.)) = 2$, we have $i_1 = 1$. Replacing

$\mathcal{L}(x,\vartheta)(1,.)$ by $\mathcal{B}(1,.)\mathcal{L}(x,\vartheta)$, we get the new operator

$$\mathcal{L}_1(x,\vartheta) = \begin{pmatrix} -x & x & x \\ 0 & x & x \\ 0 & 0 & 1 \end{pmatrix}\vartheta^3 + \begin{pmatrix} 0 & x^2-x^3 & 0 \\ 0 & x^2 & 1 \\ x & 1 & 0 \end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 0 & x-x^2 \\ 0 & 1 & x \\ 1 & 0 & 2x \end{pmatrix}\vartheta + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & x^2 & 4x^3 \end{pmatrix}.$$

Now $\beta_1 = v(\mathcal{L}_1(x,\vartheta)(1,.)) = 1$ so we multiply $\mathcal{L}_1(x,\vartheta)(1,.)$ on the left by x^{-1} and obtain

$$\mathcal{L}_2(x,\vartheta) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & x & x \\ 0 & 0 & 1 \end{pmatrix}\vartheta^3 + \begin{pmatrix} 0 & x-x^2 & 0 \\ 0 & x^2 & 1 \\ x & 1 & 0 \end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 0 & 1-x \\ 0 & 1 & x \\ 1 & 0 & 2x \end{pmatrix}\vartheta + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & x^2 & 4x^3 \end{pmatrix}.$$

Since $\deg_\vartheta(\mathcal{B}(1,1)) = 0$, $\mathcal{L}_2(x,\vartheta)$ is then left-equivalent to $\mathcal{L}(x,\vartheta)$. Now we use $\mathcal{B}(1,1) = -1$ as a pivot to eliminate $\mathcal{B}(2,1)$. Consequently, $\mathcal{B}$ becomes

$$\mathcal{B}_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & \vartheta & -1 \end{pmatrix}.$$

Let us consider the second row $\mathcal{B}_1(2,.)$ which has the second and third components both nonzero, but since $\deg_x(\mathcal{L}_2(x,\vartheta)(3,.)) = 3 > \deg_x(\mathcal{L}_2(x,\vartheta)(2,.)) = 2$, we have $i_2 = 3$; we replace $\mathcal{L}_2(x,\vartheta)(3,.)$ by $\mathcal{B}_1(2,.)\mathcal{L}_2(x,\vartheta)$ and obtain

$$\mathcal{L}_3(x,\vartheta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x & x \end{pmatrix}\vartheta^4 + \begin{pmatrix} -1 & 1 & 1 \\ 0 & x & x \\ 0 & x+x^2 & x \end{pmatrix}\vartheta^3 + \begin{pmatrix} 0 & x-x^2 & 0 \\ 0 & x^2 & 1 \\ -x & 2x^2 & x \end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 0 & 1-x \\ 0 & 1 & x \\ 0 & 0 & -x \end{pmatrix}\vartheta + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -x^2 & -4x^3 \end{pmatrix}.$$

Since $\deg_\vartheta(\mathcal{B}_1(2,3)) = 0$, $\mathcal{L}_3(x,\vartheta)$ is also left-equivalent to $\mathcal{L}(x,\vartheta)$. Now $\beta_3 = 1$, so we multiply the third row of $\mathcal{L}_3(x,\vartheta)$ on the left by x^{-1} and get

$$\overline{\mathcal{L}}(x,\vartheta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\vartheta^4 + \begin{pmatrix} -1 & 1 & 1 \\ 0 & x & x \\ 0 & 1+x & 1 \end{pmatrix}\vartheta^3 + \begin{pmatrix} 0 & x-x^2 & 0 \\ 0 & x^2 & 1 \\ -1 & 2x & 1 \end{pmatrix}\vartheta^2 + \begin{pmatrix} 0 & 0 & 1-x \\ 0 & 1 & x \\ 0 & 0 & -1 \end{pmatrix}\vartheta + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -x & -4x^2 \end{pmatrix}.$$

This matrix differential operator is simple and left-equivalent to $\mathcal{L}(x,\vartheta)$. Consequently, the regular formal solutions spaces of $\mathcal{L}(x,\vartheta)$ and $\overline{\mathcal{L}}(x,\vartheta)$ are exactly the same, so applying Algorithm 1 to $\overline{\mathcal{L}}(x,\vartheta)$ yields the general regular formal solution of $\mathcal{L}(x,\vartheta)$.

Example 6. Consider the matrix differential operator $\mathcal{L}(x,\vartheta)$ given by (21). According to Example 3, a simple operator cannot be obtained from $\mathcal{L}(x,\vartheta)$ by means of a linear substitution. Consequently, we shall apply Algorithm 3 to $\mathcal{L}(x,\vartheta)$. A left minimal basis of $\mathcal{L}(0,\vartheta)$ is composed of one vector $v = (1 \quad -\vartheta)$. Since $\deg_x(\mathcal{L}(x,\vartheta)(2,.)) > \deg_x(\mathcal{L}(x,\vartheta)(1,.))$, we replace $\mathcal{L}(x,\vartheta)(2,.)$ by

$v\, \mathcal{L}(x, \vartheta)$; this yields

$$\mathcal{L}_1(x, \vartheta) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \vartheta^3 + \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \vartheta^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vartheta.$$

Since the second component of v depends on ϑ, the new operator $\mathcal{L}_1(x, \vartheta)$ is not left-equivalent to $\mathcal{L}(x, \vartheta)$ and we set $\mathcal{K} = \{2\}$. Finally, we multiply the second row of $\mathcal{L}_1(x, \vartheta)$ on the left by x^{-1} to obtain

$$\overline{\mathcal{L}}(x, \vartheta) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \vartheta^3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vartheta^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vartheta. \tag{23}$$

The latter system is simple but not left-equivalent to $\mathcal{L}(x, \vartheta)$. Hence the regular formal solutions space of $\mathcal{L}(x, \vartheta)(y(x)) = 0$, where $\mathcal{L}(x, \vartheta)$ is given by (21), is a subspace of that of $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$.

6.3. *Reconstruction of the regular solutions*

Now, we shall explain how to reconstruct the general regular solution of the non-simple system $\mathcal{L}(x, \vartheta)(y(x)) = 0$ of order $\nu \in \mathbb{N}$ from that of the output of Algorithm 3.

Two cases have to be considered. If the output operator $\overline{\mathcal{L}}(x, \vartheta)$ of Algorithm 3 is left-equivalent to the input one $\mathcal{L}(x, \vartheta)$ (this corresponds to $\mathcal{K} = \{\}$), then the general regular solution of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ is exactly that of $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$. Consequently, to get the general regular solution of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ of order ν, it suffices to compute that of $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$ of order ν by applying Algorithm 1.

Otherwise, *i.e.*, if $\mathcal{K}$ is a nonempty set, $\overline{\mathcal{L}}(x, \vartheta)$ is not left-equivalent to $\mathcal{L}(x, \vartheta)$ and we can proceed as follows. First, we compute the general regular solution $z(x)$ of $\overline{\mathcal{L}}(x, \vartheta)(z(x)) = 0$ of order ν by applying Algorithm 1. Write $z(x) = \sum_{i=1}^{r} x^{\lambda_i} z_i$ where $z_i \in \overline{\mathbb{K}}[x][\log(x)]^n$ is of degree in x at most ν and $x^{\lambda_i} z_i$ is the general regular solution associated with the set σ_i and truncated at order ν (see Section 2). Then, we consider the subsystem of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ formed by equations given by the rows of $\mathcal{L}(x, \vartheta)$ of indices $j \in \mathcal{K}$ and plug $z(x)$ into it. Now, $z(x)$ is a general regular solution of order ν of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ if and only if the coefficients of $x^{\lambda_i + k}$ in $\mathcal{L}(x, \vartheta)(j, .)(z(x))$ where $j \in \mathcal{K}$, $1 \le i \le r$ and $0 \le k \le \nu$ are all equal to zero. This yields a system of linear equations in the parameters appearing in $z(x)$. Finally, solving this system and substituting the solution into $z(x)$, we get the general regular solution of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ of order ν.

Example 7. We are interested in computing the general regular solution of $\mathcal{L}(x, \vartheta)(y(x)) = 0$, where $\mathcal{L}(x, \vartheta)$ is given by (21). As we have already seen in Example 6, Algorithm 3 returns a non-left-equivalent operator $\overline{\mathcal{L}}(x, \vartheta)$ given by (23) and $\mathcal{K} = \{2\}$. The operator $\overline{\mathcal{L}}(x, \vartheta)$ has constant matrix coefficients; then the regular solutions of system $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$ are of the form given by Lemma 2. On the other hand, we have $\sigma(\overline{\mathcal{L}}(0, \lambda)) = \{-1, 0\}$, and hence the general regular solution of $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$ is of the form $x^{-1}\,(U_0 + U_1\, x)$ with $U_0,\ U_1 \in \mathbb{Q}[\log(x)]^2$. Consequently, we apply Algorithm 1 to $\overline{\mathcal{L}}(x, \vartheta)$ with $\nu = 1$. It returns the general regular solution of $\overline{\mathcal{L}}(x, \vartheta)(y(x)) = 0$ given by

$$z(x) = \begin{pmatrix} C_1\, x^{-1} + C_2 + C_3\, \log(x) - \frac{C_4}{2}\, \log^2(x) \\ C_1\, x^{-1} + C_5 + C_4\, \log(x) \end{pmatrix},$$

where the C_i, for $i = 1, \ldots, 5$, are arbitrary constants in $\mathbb{Q}$. Since $\mathcal{K} = \{2\}$, we plug $z(x)$ into the second equation of the original system and we find

$$\mathcal{L}(x, \vartheta)(2, .)(z(x)) = C_1 + C_3 + C_5.$$

Therefore, $z(x)$ is the general regular solution of $\mathcal{L}(x, \vartheta)(y(x)) = 0$ if and only if $C_1 = -C_3 - C_5$. Hence, the general regular solution of the system $\mathcal{L}(x, \vartheta)(y(x)) = 0$ given by (21) is

$$\begin{pmatrix} C_2 + C_3\, (\log(x) - x^{-1}) - \frac{C_4}{2}\, \log^2(x) - C_5\, x^{-1} \\ -C_3\, x^{-1} + C_5\, (1 - x^{-1}) + C_4\, \log(x) \end{pmatrix},$$

where the C_i, for $i = 2, \ldots, 5$, are arbitrary constants in $\mathbb{Q}$.

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