This maple file illustrates the use of the main procedures of the package *RationalFirstIntegrals* writen by *A. Bostan, G. Chèze, T. Cluzeau, and J.-A. Weil.*
For explanations and details about the algorithms behind the procedures, see our related paper:

Efficient algorithms for computing rational first integrals of planar polynomial vector fields.

```
> restart:
```

We load our package and the ***gfun*** package needed for the procedure ***HeuristicRationalFirstIntegral***:

```
> with(gfun);
  with(RationalFirstIntegrals);
```

[*Laplace*, *NumGfun*, *Parameters*, *algebraicsubs*, *algeqtodiffeq*, *algeqtoseries*, *algfuntoalgeq*, *borel*, *cauchyproduct*, *diffeq**diffeq*, *diffeq*+*diffeq*, *diffeqtohomdiffeq*, *diffeqtorec*, *firstnonzero*, *getname*, *gfun_pade*, *guesseqn*, *guessgf*, *hadamardproduct*, *holexprtodiffeq*, *invborel*, *listprimpart*, *listtoalgeq*, *listtodiffeq*, *listtohypergeom*, *listtolist*, *listtoratpoly*, *listtorec*, *listtoseries*, *makediffeq*, *maxdegcoeff*, *maxdegeqn*, *maxindex*, *mindegcoeff*, *mindegeqn*, *minindex*, *myisolve*, *nth_term*, *pade2*, *poltodiffeq*, *poltorec*, *ratpolytocoeff*, *rec**rec*, *rec*+*rec*, *rectodiffeq*, *rectohomrec*, *rectoproc*, *seriestoalgeq*, *seriestodiffeq*, *seriestohypergeom*, *seriestolist*, *seriestoratpoly*, *seriestorec*, *seriestoseries*, *systomatrix*]

[*DarbouxPolynomials*, *DeterministicRationalFirstIntegral*, *GenericRationalFirstIntegral*, *GuessMinimalPolynomial*, *HeuristicRationalFirstIntegral*, *PadeHermiteHeuristic*, *ProbabilisticRationalFirstIntegral*, *SimplifyRFI*, *simpli1*, *simpli2*] **(1)**

We define the planar polynomial vector field {x'=A(x,y), y'=B(x,y)} by giving the polynomials A and B:

```
> A := -18*x^8*y^8-20*x^6*y^9-6*x^2*y^12+24*x^10*y^3-6*x^4*y^9-4*y^13
  -3*x^12-7*x^2*y^10;
> B := 2*x*(-16*x^6*y^9+8*x^14-18*x^4*y^10-2*y^13+10*x^8*y^4-2*x^2*
  y^10-2*x^10*y-3*y^11) ;
```

$$A := -18\,x^8\,y^8 - 20\,x^6\,y^9 - 6\,x^2\,y^{12} + 24\,x^{10}\,y^3 - 6\,x^4\,y^9 - 4\,y^{13} - 3\,x^{12} - 7\,x^2\,y^{10}$$

$$B := 2\,x\,\left(-16\,x^6\,y^9 + 8\,x^{14} - 18\,x^4\,y^{10} - 2\,y^{13} + 10\,x^8\,y^4 - 2\,x^2\,y^{10} - 2\,x^{10}\,y - 3\,y^{11}\right) \quad \textbf{(2)}$$

We need a point *pt* for which A(pt,y) <> 0:

```
> pt:=0;
  check_pt:=eval(A,x=pt);
```

$$pt := 0$$

$$check\_pt := -4\,y^{13} \quad \textbf{(3)}$$

COMPUTATION OF RATIONAL FIRST INTEGRALS

We can now use our procedures to search for rational first integrals of degree less than or equal to a given *N:*

```
> N:=3;
```

$$N := 3 \quad (4)$$

The first procedure is called *GenericRationalFirstIntegral*:

```
> GenericRationalFirstIntegral(A,B,pt,N);
```

$$\text{"None"} \quad (5)$$

This means that the planar polynomial vector field {x'=A(x,y), y'=B(x,y)} does not have any rational first integral of degree
less than or equal to 4 but we can of course search for a rational first integral of greater degree:

In practice, this first procedure is not very efficient so it is better to use one of the next procedures to search for rational first
integrals of degree higher than 3:

For example, one may use our probabilistic algorithm *ProbabilisticRationalFirstIntegral.*
This procedure requires two rational numbers *c1* and *c2* that can be chosen arbitrarily (so that A (pt,ci) <> 0):

```
> c1:=-1;
  c2:=1;
```

$$c1 := -1$$

$$c2 := 1 \quad (6)$$

Warning the next computation for N=15 and 18 takes some time.

```
> for N from 3 to 18 by 3 do
  F[N]:=ProbabilisticRationalFirstIntegral(A,B,pt,[c1,c2],N);
  od;
```

$$F_3 := [\text{"I don't know"}, [1]]$$

$$F_6 := \text{"None"}$$

$$F_9 := \text{"None"}$$

$$F_{12} := \text{"None"}$$

$$F_{15} := \text{"None"}$$

$$F_{18} := \left(24\left(-x^2 y^9 + x^{10} - y^{10}\right)\right) \Big/ \left(8 x^{18} - 24 x^{12} y^4 + 12 x^{14} y + 24 x^6 y^8 - 24 x^8 y^5 + 6 x^{10} y^2 - 8 y^{12} + 44 x^2 y^9 - 32 x^{10} - 6 x^4 y^6 + 32 y^{10} + x^6 y^3\right) \quad (7)$$

Our procedure finds a rational first integral of degree 18. We can check that this is a rational first integral:

```
> check_RFI:=simplify(A*diff(F[18],x)+B*diff(F[18],y));
```

$$check\_RFI := 0 \quad (8)$$

One may also use our deterministic algorithm *DeterministicRationalFirstIntegral* which runs recursively *ProbabilisticRationalFirstIntegral.*
This procedure requires an interval [a,b] in which the rational numbers needed for *ProbabilisticRationalFirstIntegral* are chosen.

```
> a:=-1;
  b:=1;
```

$$a := -1$$

$$b := 1 \quad (9)$$

```
> N:=4;DeterministicRationalFirstIntegral(A,B,pt,[a,b],N);
```

$$N := 4$$

"None" (10)

```
> N:=18;
  DeterministicRationalFirstIntegral(A,B,pt,[a,b],N);F:=%:
  check_RFI:=simplify(A*diff(F,x)+B*diff(F,y));
```

$$N := 18$$

$$\left(24\left(-x^2 y^9+x^{10}-y^{10}\right)\right) \Big/ \left(8 x^{18}-24 x^{12} y^4+12 x^{14} y+24 x^6 y^8-24 x^8 y^5+6 x^{10} y^2 - 8 y^{12}+44 x^2 y^9-32 x^{10}-6 x^4 y^6+32 y^{10}+x^6 y^3\right)$$

$$check\_RFI := 0 \qquad (11)$$

The procedures *ProbabilisticRationalFirstIntegral* and *DeterministicRationalFirstIntegral* needs a lot of time (e.g., 2 or 3 minutes) to find the rational first integral.

One may use our heuristic *HeuristicRationalFirstIntegral* which is in general faster that the previous ones when there exists a rational first integrals.
This procedure requires the *gfun* package.
This procedure requires two rational numbers *c1* and *c2* that can be chosen arbitrarily (so that A (pt,ci) <> 0):

```
> c1:=-1;
  c2:=1;
```

$$c1 := -1$$

$$c2 := 1 \qquad (12)$$

```
> N:=4;HeuristicRationalFirstIntegral(A,B,pt,[c1,c2],N);
```

$$N := 4$$

"I don't know" (13)

Here our heuristic can not decide about the existence of rational first integral of degree smaller than 4.

```
> N:=18;
  HeuristicRationalFirstIntegral(A,B,pt,[c1,c2],N);F:=%:
  check_RFI:=simplify(A*diff(F,x)+B*diff(F,y));
```

$$N := 18$$

$$\left(24\left(-x^2 y^9+x^{10}-y^{10}\right)\right) \Big/ \left(8 x^{18}-24 x^{12} y^4+12 x^{14} y+24 x^6 y^8-24 x^8 y^5+6 x^{10} y^2 - 8 y^{12}+44 x^2 y^9-32 x^{10}-6 x^4 y^6+32 y^{10}+x^6 y^3\right)$$

$$check\_RFI := 0 \qquad (14)$$

Note that *HeuristicRationalFirstIntegral* finds this rational first integral faster (only 20 or 30 seconds) than *ProbabilisticRationalFirstIntegral* and *DeterministicRationalFirstIntegral*.

SIMPLIFICATION OF RATIONAL FIRST INTEGRALS

Our package includes a procedure called *SimplifyRFI* which can be used to obtain a rational first integral written in more compact (simpler) form than the ones returned by the previous procedures.

```
> F1:=SimplifyRFI(A,B,F);
```

(15)

$$F1 := \frac{-x^2 y^9 + x^{10} - y^{10}}{\left(2 x^6 - 2 y^4 + x^2 y\right)^3} \quad \textbf{(15)}$$

```
> check_RFI:=simplify(A*diff(F1,x)+B*diff(F1,y));
```

$$check\_RFI := 0 \quad \textbf{(16)}$$

COMPUTATION OF DARBOUX POLYNOMIALS

Our package also inlcude a last procedure for computing all the irreducible Darboux polynomials of the planar polynomial vector field { x'=A(x,y), y'=B(x,y)} of degree lower than or equal to a given bound N.

```
> N:=3;
  Darboux:=[DarbouxPolynomials(A,B,pt,N)];
```

$$N := 3$$

$$Darboux := [\text{"None"}] \quad \textbf{(17)}$$

Here we have no Darboux polynomial of degree lower than or equal to 3.

Note that, as *GenericRationalFirstIntegral,* the procedure *DarbouxPolynomial* is not very efficient so that, in this example, running it for greater values of N may take a long time.