Time-delay Example 1

Example from the paper by L.A. Marquez-Martinez (C.R. Acad-.Sci. Paris, t. 329, serie 1, pages 545-550, 1999.)

Let us consider the following differential time-delay nonlinear system:

```
eqs = {x1'[t] → x1[t] u[t] + u[t - 2],
    x2'[t] → u[t] + u[t - 1],
    x3'[t] → u[t - 1] - u[t - 2]};
vars = {x1[t], x2[t], x3[t], u[t]};
TableForm[eqs]
```

$x_1'[t] \to u[-2+t] + u[t]\, x_1[t]$

$x_2'[t] \to u[-1+t] + u[t]$

$x_3'[t] \to -u[-2+t] + u[-1+t]$

Let us introduce the following Ore algebra A of differential time-delay operators

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[Der[t], S[-1][t], replA]
```

$\mathbb{K}(t)\left[D_t;\, 1,\, D_t\right]\left[\left(S_t^{-1}\right);\, \#1\ /.\ t \to t-1\ \&,\ 0\ \&\right]$

and the matrix R of ordinary differential time-delay operators, which defines the generic linearization of the above nonlinear system

```
MatrixForm[R = ToOrePolynomialD[eqs, vars, A]]
```

$$\begin{pmatrix} D_t - u[t] & 0 & 0 & -\left(S_t^{-1}\right)^2 - x_1[t] \\ 0 & D_t & 0 & -\left(S_t^{-1}\right) - 1 \\ 0 & 0 & D_t & \left(S_t^{-1}\right)^2 - \left(S_t^{-1}\right) \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times 4} / A^{1\times 3}\, R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} D_t - u[-t] & 0 & 0 \\ 0 & D_t & 0 \\ 0 & 0 & D_t \\ -\left(S_t^{-1}\right)^2 - x_1[-t] & -\left(S_t^{-1}\right) - 1 & \left(S_t^{-1}\right)^2 - \left(S_t^{-1}\right) \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
MatrixForm /@ ({Ann, Rp, Q} = Exti[Radj, A, 1])
```

A very large output was generated. Here is a sample of it:

$$\left\{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&D_t\end{pmatrix},\ \begin{pmatrix}0&-D_t&0&\left(S_t^{-1}\right)+1\\-D_t+u[t]&0&-D_t&\left(S_t^{-1}\right)+x_1[t]\\0&0&D_t&\left(S_t^{-1}\right)^2-\left(S_t^{-1}\right)\\0&-\left(S_t^{-1}\right)^2+\left(S_t^{-1}\right)&-\left(S_t^{-1}\right)-1&0\end{pmatrix},\ (\ll 1\gg)\right\}$$

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Since the first matrix is not identity, we deduce that M admits nontrivial torsion elements and thus the corresponding system admits autonomous elements τ[1], ..., τ[4], defined by:

```
AutonomousElements[R, {dx1[t], dx2[t], dx3[t], du[t]}, τ, A, Relations → True]
```

```
{{τ[1][t] → du[-1 + t] + du[t] - dx2'[t],
  τ[2][t] → du[-1 + t] + u[t] dx1[t] + du[t] x1[t] - dx1'[t] - dx3'[t],
  τ[3][t] → du[-2 + t] - du[-1 + t] + dx3'[t],
  τ[4][t] → -dx2[-2 + t] + dx2[-1 + t] - dx3[-1 + t] - dx3[t]},
 {τ[1][t] == 0, τ[2][t] == 0, τ[3][t] == 0, τ[4]'[t] == 0},
 {-τ[2][t] - τ[3][t], -τ[1][t], τ[3][t],
  -τ[1][-2 + t] + τ[1][-1 + t] + τ[3][-1 + t] + τ[3][t] + τ[4]'[t]}}
```

We note that first three autonomous elements τ[1], τ[2], τ[3] are trivial. The only nontrivial autonomous element is τ[4] = $-dx_2[-2+t]+dx_2[-1+t]-dx_3[-1+t]-dx_3[t]$, which satisfies $\tau[4]'[t] == 0$. We find again the results of Example 1, studied in the paper L.A. Marquez-Martinez (C.R. Acad-.Sci. Paris, t. 329, serie 1, pages 545-550, 1999).