Lindalg: Mathemagix Package for Symbolic Resolution
of
Linear Differential Systems with Singularities

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Lindalg is dedicated to the local analysis of nth-order linear differential equations and first order linear differential systems. At ordinary points, it suffices to consider Taylor series (power series). Any engineering student or scientist is familiar with their resolution procedure and popular computer systems always consider a package for this goal. However, singular points require further investigation based on an analysis of a Newton polygon and matricial manipulations. Differential equations with singularities arise from countless applications and encompass a vast body of contemporary academic literature. The package Isolde written in the computer algebra system Maple is dedicated to the symbolic resolution of such systems and more generally linear functional matrix equations (e.g. difference equations).

On the other hand, the new package Lindalg sets a first milestone in providing the two-decade span of Isolde content in an open source software.

Mathemagix provides under GNU General Public License, a new high level general purpose language, for symbolic and certified numeric algorithms, that can be both interpreted by a shell or compiled.

Before proceeding you might be interested in one of the following:

- Download Mathemagix and its tutorial: www.mathemagix.org

- Download the tutorial on formal reduction and remarks on implementation and

underlying algorithms and

- Download the set of example files for this manual:

  www.unilim.fr/pages_perso/suzy.maddah/Research_.html

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This manual is written in TeXmacs : www.texmacs.org .

Linear Singular Ordinary Differential Systems

Given such a system:

$$[A] \qquad x^{p+1} \frac{\mathrm{dF}}{\mathrm{dx}} = A(x)\, F$$

This pakcage deals with its local analysis:

$$A(x) = \sum_{k=0}^{\infty} A_k\, x^k, \quad A_k \in \mathcal{M}_n(\mathbb{C})$$

A Fundamental Matrix of Formal Solutions (Hukuhara- Turritin- Levelt) is givne by:

$$\Phi\big(x^{1/s}\big)\, x^C \exp\big(Q\big(x^{-1/s}\big)\big)$$

- $s \in \mathbb{N}^*$: *ramification index*

- $\Phi\big(x^{1/s}\big)$: root-meromorphic matrix in x over $\mathbb{C}$

- $Q\big(x^{-1/s}\big)$: *exponential part*,

  diagonal matrix of polynomials in $x^{-1/s}$ over $\mathbb{C}$ without constant terms

- C : constant matrix which commutes with the exponential part

- p is called the Poincaré rank

Construction of a fundamental matrix of formal solutions

- Existence: Turrittin, Hukuhara, Levelt, Balser-Jurkat-Lutz, ...

- Algorithms for related problems: Levelt, Hilali and Wazner (1980's), Sommeling (1993), Chen, Schaefke, Barkatou,(1990 - ...), Pfluegel (2000 - ...), ...

- Books: Wasow (1965), Balser(2000), Hsieh and Sibuya (1999), ...

- Package ISOLDE in MAPLE by M. A. Barkatou and E. Pfluegel (ISSAC'2012):

  http://sourceforge.net/projects/isolde

The algorithms implemented herein decouple the input system to system(s) of $n=1$ (scalar case) or $p=0$ (regular system $\rightarrow$ zero exponential part) by a change of basis $F=TG$ which yields an *equivalent* system

$$\begin{bmatrix} \tilde{A} \end{bmatrix} \qquad x^{\tilde{p}+1} \frac{dG}{dx} = \tilde{A}(x)\,G$$

such that

$$\frac{\tilde{A}}{x^{\tilde{p}+1}} = T^{-1} \frac{A}{x^{p+1}}\;T \;-\; T^{-1}\frac{dT}{dx}$$

LINDALG : Main Functionalities

Formal Reduction:

- Splitting of system into systems of lower dimension

- Rank Reduction (classification of singularity)

- $p=0$: Solutions of Regular System

- $p>0$: Computing the exponential part (Irregular systems)

- Solutions of nth-order linear singular differential equations (equivalence with first order systems)

$$[A] \qquad x^{p+1}\frac{dF}{dx} = A(x)\,F = \left(\sum_{k=0}^{\infty} A_k\,x^k \right) F$$

The discussion depends on the eigenvalues of A_0.

<div align="center">

Getting ready!

</div>

MATHEMAGIX consists of a collection of packages which partially depend on each other. Currently, you may download the complete sources of Mathemagix from svn server and specify the individual packages that you want to use at the configuration stage. Details on installation are available on the MATHEMAGIX's webpage.

We first start an interactive session and change to the correct directory:

```
Shell session inside TeXmacs pid = 4539
```

```
Shell] cd /Users/maddah/Documents/mathemagix/mmx2
```

If you do not wish to install it, you can still use it via *development mode*. Hence, the following lines:

```
Shell] source set-devel-paths;./configure --enable-lindalg > /dev/null ;
        make > /dev/null
```

lindalg/mmx contains the source code of the package and lindalg/test contains the test files. You can access the former if you wish to read and/or modify the code. If more help is needed on the syntax of MATHEMAGIX then you can refer to mmx tutorial. However, if you only wish to use the package to compute with your own examples, then you can simply access the relevant test file and replace the input system by yours.

```
Shell] cd /Users/maddah/Documents/mathemagix/mmx2/lindalg/mmx;ls
```

```
algebraic_solve.mmx               matrix_manip.mmx
bad_spectrum.mmx                  moser_barkatou.mmx
block_diag.mmx                    moser_barkatou_trans.mmx
calgebraic                        newton_polygon.mmx
calgebraic.mmx                    ralgebraic.mmx
cball.mmx                         ramification.mmx
characteristic_polynomial.mmx     rball.mmx
column_operations.mmx             reduced_poly.mmx
exp_part.mmx                      reg_singular.mmx
exp_part_fns.mmx                  rho_elim.mmx
format_printing.mmx              shearing.mmx
gauss_elim.mmx                    smart_ramification.mmx
get_coeffs.mmx                    split_lemma
int_bezout.mmx                    split_lemma.mmx
katz_invariant.mmx               split_lemma_trans.mmx
lin_solve.mmx                    sylvester_solve.mmx
mat_pol_mult.mmx                 tensor_product.mmx
```

```
Shell] cd /Users/maddah/Documents/mathemagix/mmx2/lindalg/test;ls
```

block_diag_test.mmx   int_bezout_test.mmx
calgebraic_test.mmx   inv_calgebraic_test.mmx
calgebraic_test_print   katz_invariant_test.mmx
cball_test.mmx   lin_solve_test
characteristic_polynomial_test.mmx   lin_solve_test.mmx
column_operations_test.mmx   mat_pol_mult_test.mmx
exp_part_fns_test.mmx   matrix_calgebraic_test.mmx
exp_part_test   matrix_manip_test.mmx
exp_part_test.mmx   moser_barkatou_test
exp_part_test1a   moser_barkatou_test.mmx
exp_part_test1a.mmx   moser_barkatou_trans_test.mmx
exp_part_test1b   newton_polygon_test.mmx
exp_part_test1b.mmx   poly_test.mmx
exp_part_test2   presentation_greece_ACA2015
exp_part_test2.mmx   quotient_test.mmx
exp_part_test3   ralgebraic_test.mmx
exp_part_test3.mmx   rball_test.mmx
exp_part_test4   reduced_poly_test.mmx
exp_part_test4.mmx   reg_singular_test.mmx
exp_part_test5   rho_elim_test.mmx
exp_part_test5.mmx   roots_test
exp_part_test6   roots_test.mmx
exp_part_test6.mmx   series_test.mmx
exp_part_test7   shearing_test
exp_part_test7.mmx   shearing_test.mmx
exp_part_test8   smart_ramification_test.mmx
exp_part_test8.mmx   split_lemma_test.mmx
exp_part_test9.mmx   sylvester_solve_test.mmx
gauss_elim_test   tensor_product_test.mmx
gauss_elim_test.mmx   test
get_coeffs_test.mmx   test.mmx

```
Shell]
```

The example program can be run in two ways: by compiling it using the MATHEMAGIX compiler mmc and then running the resulting executable, or by interpreting it using the MATHEMAGIX interpreter (from mmx language tutorial, pp 9). In this manual, we use the compiler. All the following examples are grouped in a folder accompanying this manual.

```
Shell] cd /Users/maddah/Documents/mathemagix/mmx2/lindalg/test/
       presentation_greece_ACA2015;ls
```

```
exp_part_test1a                 pfaffA.mmx
exp_part_test1a.mmx             reg_singular_test
exp_part_test3                  reg_singular_test.mmx
exp_part_test3.mmx              split_lemma_trans_test
exp_part_test4                  split_lemma_trans_test.mmx
exp_part_test4.mmx              split_lemma_trans_test100
moser_barkatou_test             split_lemma_trans_test100.mmx
moser_barkatou_test.mmx         split_lemma_trans_test1000
moser_barkatou_trans_test1      split_lemma_trans_test1000.mmx
moser_barkatou_trans_test1.mmx  split_lemma_trans_test2000
moser_barkatou_trans_test2      split_lemma_trans_test2000.mmx
moser_barkatou_trans_test2.mmx  split_lemma_trans_test500
moser_barkatou_trans_test3      split_lemma_trans_test500.mmx
moser_barkatou_trans_test3.mmx  split_sys_test
moser_barkatou_trans_test4      split_sys_test.mmx
moser_barkatou_trans_test4.mmx  test
pfaffA                          test.mmx
```

We start with the famous `Hello World` example. One can optionally open and read the example file.

```
Shell] open test.mmx
```

To compile the program (In the examples which follow, we assume the programs are already compiled. We thus skip this step.):

```
Shell] mmc test.mmx
```

To run it:

```
Shell] ./test
```

```
Hello World!
```

```
Shell]
```

Example 1.

$$x^3 \, \frac{\mathrm{dF}}{\mathrm{dx}} = A(x) \, F$$

$$A(x) = \begin{pmatrix} -2x+2 & 6x^2-x & 2x^2-5x+5 & x^2 \\ x^2+x & x & x^2-x & 3x+3 \\ 3x^2+3x+3 & \frac{4}{5}x^2+7x & x^2+x+8 & 2x^2+2x+2 \\ \frac{2}{3}x^2-\frac{1}{5}x & \frac{1}{3}x^2+\frac{2}{3}x & -2x^2 & 2x-1 \end{pmatrix}$$

split_sys(A,p,m) where 2 is the Poincaré rank computes an equivalent block-diagonalized system (whenever possible) up to order m :

$$x^3 \, \frac{\mathrm{dG}}{\mathrm{dx}} = \tilde{A}(x)G \qquad \text{where} \quad \tilde{A}(x) \text{ is given by:}$$

```
Shell session inside TeXmacs pid = 9603
```

```
Shell] open split_sys_test.mmx
```

```
Shell] ./split_sys_test
```

```
 A block-diagonalized equivalent system is given (up to order 5) by:

[-15709552 / 46525 * x^4 - 15024 / 1861 * x^3 + 30509 / 9305 * x^2 -
18422 / 1861 * x + 18422 / 1861, -87731389 / 51949815 * x^4 - 854468 /
17316605 * x^3 - 37029 / 3463321 * x^2 + 22483 / 3463321 * x - 16900 /
3463321, 125443 / 51949815 * x^4 + 162 / 3463321 * x^3 + 6 / 3463321 *
x^2 - 5 / 3463321 * x + 5 / 3463321, 0; 35869394 / 75 * x^4 + 135823 / 15
* x^3 - 23936 / 5 * x^2 - 1861 / 5 * x, 41437294 / 16749 * x^4 + 1620016
/ 27915 * x^3 + 3987 / 1861 * x^2 - 1, -287156 / 83745 * x^4 - 376 / 5583
* x^3 - 14 / 5583 * x^2, 0; 955542377164 / 75 * x^4 + 18962033242 / 75 *
x^3 - 105740568 / 5 * x^2 + 68763947 * x + 3, 27429116257853 / 418725 *
x^4 + 44771091332 / 27915 * x^3 + 484224378 / 9305 * x^2 - 138193299 /
1861 * x + 1022 / 1861, -7646109619 / 83745 * x^4 - 49933258 / 27915 *
x^3 - 250981 / 5583 * x^2 + 16561 / 1861 * x + 188 / 1861, 0; 0, 0, 0,
20233903 / 225 * x^4 + 8694 / 5 * x^3 + 608 / 15 * x^2 + 3 * x]
```

```
Shell]
```

We rewrite the above output here:

$$x^3 \, \frac{\mathrm{dG}}{\mathrm{dx}} = \tilde{A}(x)G \qquad \text{where} \quad \tilde{A}(x) \text{ is given by}$$

$$\begin{pmatrix} \frac{18422}{1861} - \frac{18422}{1861}x + O(x^2) & -\frac{16900}{3463321} + \frac{22483}{3463321}x + O(x^2) & \frac{5-5x}{3463321} + O(x^2) & 0 \\ -\frac{1861}{5}x + O(x^2) & -1 + O(x^2) & O(x^2) & 0 \\ 3 + 68763947x + O(x^2) & \frac{1022 - 138193299x}{1861} + O(x^2) & \frac{188}{1861} + O(x) & 0 \\ 0 & 0 & 0 & 3x + O(x^2) \end{pmatrix}$$

`split_lemma_trans(A,2,m)` outputs the two uncoupled subsystems and the transformation $F = TG = T\left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right)$ as well:

```
Shell] open split_lemma_trans_test.mmx
```
```
Shell] ./split_lemma_trans_test
```
```
Shell]
```

We rewrite the above output here:

$$T(x) = \begin{pmatrix} 1861 & 0 & 0 & \frac{149}{5583}x \\ 0 & -3 & 0 & 1+\frac{2}{3}x \\ 2940 & -\frac{3380}{1861} & \frac{1}{1861} & -\frac{541099}{3}x \\ -\frac{1109}{5}x & -\frac{1368}{1861}x+1 & -\frac{4}{1861}x & 0 \end{pmatrix} + O(x^2)$$

$$x^3\,\frac{\mathrm{d}G_1}{\mathrm{d}x} = \tilde{A}_1(x)\,G_1 \quad\text{and}\quad x^3\,\frac{\mathrm{d}G_2}{\mathrm{d}x} = \tilde{A}_2(x)\,G_2$$

where

$$\tilde{A}_1(x) = \begin{pmatrix} \frac{18422}{1861} - \frac{18422}{1861}x + O(x^2) & -\frac{16900}{3463321} + \frac{22483}{3463321}x + O(x^2) & \frac{5-5x}{3463321} + O(x^2) \\ -\frac{1861}{5}x + O(x^2) & -1 + O(x^2) & O(x^2) \\ 3 + 68763947x + O(x^2) & \frac{1022 - 138193299x}{1861} + O(x^2) & \frac{188}{1861} + O(x) \end{pmatrix}$$

$$\tilde{A}_2(x) = \begin{pmatrix} 3x + O(x^2) \end{pmatrix}$$

Rank Reduction (classification of singularity) - Moser'59 and Barkatou'95

Example 2. Irregular system (Poincaré rank drops)

$$x^8\,\frac{\mathrm{d}F}{\mathrm{d}x} = A(x)\,F$$

$$A(x) = \begin{pmatrix} 4x^7 & x^{10} & -2x^{13} & -x^{13} \\ 0 & -x^7 - x^6 & x^6 & 0 \\ 1 & 0 & x^6 - 2x^7 & x^6 \\ x^2 + x & -x^5 & x^9 + x^8 + x^5 & -3x^7 \end{pmatrix}$$

`moser_barkatou(A,p)` where 7 is the Poincaré rank computes an equivalent rank_reduced system. This command is to be used in case the true Poincaré rank is already known (which is 2 here):

$$x^3\,\frac{\mathrm{d}G}{\mathrm{d}x} = \tilde{A}(x)G$$

```
Shell] open moser_barkatou_test.mmx
```

```
Shell] ./moser_barkatou_test
```

 Moser-barkatou reduction computes an equivalent system:

```
[-2 * x^2 - x, 0, x, 0; 1, -2 * x^2, -2 * x^3, -x^2; 0, 1, -3 * x^2 + x,
1; -x, x^3 + x^2, x^5 + x^4 + x, -3 * x^2]
```

```
Shell]
```

We rewrite the output here:

$$\tilde{A}(x) = \begin{pmatrix} -2x^2 - x & 0 & x & 0 \\ 1 & -2x^2 & -2x^3 & -x^2 \\ 0 & 1 & -3x^2 + x & 1 \\ -x & x^3 + x^2 & x^5 + x^4 + x & -3x^2 \end{pmatrix}$$

`moser_barkatou_trans(A,p)` outputs an equivalent rank-reduced system, the transformation $F = TG$ and the true Poincaré rank as well:

```
Shell] open moser_barkatou_trans_test1.mmx
```

```
Shell]
```

```
Shell] ./moser_barkatou_trans_test1
```

 Moser-barkatou reduction computes an equivalent system, a
 transformation, and the true Poincare rank to be:

```
[[-2 * x^2 - x, 0, x, 0; 1, -2 * x^2, -2 * x^3, -x^2; 0, 1, -3 * x^2 + x,
1; -x, x^3 + x^2, x^5 + x^4 + x, -3 * x^2], [0, x^6, 0, 0; x, 0, 0, 0; 0,
0, x, 0; 0, 0, 0, 1], 2]
```

```
Shell]
```

We rewrite the above output here:

$$\tilde{A}(x) = \begin{pmatrix} -2x^2 - x & 0 & x & 0 \\ 1 & -2x^2 & -2x^3 & -x^2 \\ 0 & 1 & -3x^2 + x & 1 \\ -x & x^3 + x^2 & x^5 + x^4 + x & -3x^2 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 0 & x^6 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

True Poincaré rank $= 2$

Example 3. Irregular system (rank of leading coefficient matrix drops)

$$x^2 \frac{dF}{dx} = A(x) F$$

$$A(x) = \begin{pmatrix} 4 & 2x & 0 & 0 & 0 & 2x \\ 0 & 2 & 2x-1 & 0 & x & 0 \\ 4x & -2 & 1 & 0 & -x & 8x \\ x & 2 & 0 & 2x & 0 & 0 \\ 8 & x+4 & -1 & 2x & x & -4x \\ 0 & 2x & 0 & 0 & 0 & x \end{pmatrix}$$

`moser_barkatou_trans(A,1)` outputs an equivalent rank-reduced system, the transformation $F = TG$ and the true Poincaré rank as well:

```
Shell] open moser_barkatou_trans_test4.mmx
```

```
Shell] ./moser_barkatou_trans_test4
```

```
 Moser-barkatou reduction computes an equivalent system, a
transformation, and the true Poincare rank to be:

[[-x + 4, 2 * x, x, 0, 0, 2 * x; -2 * x, -x + 3, 2 * x, 3 / 2, 3 / 2 * x,
-4 * x; 4 * x, -2, -x, -1, -x, 8 * x; x^2, 2 * x, x, 2 * x, 0, 0; -x + 8,
x + 2, 1 / 2 * x, 1, 0, -4 * x; 0, 2 * x, x, 0, 0, 0], [x, 0, 0, 0, 0, 0;
0, x, 1 / 2 * x, 0, 0, 0; 0, 0, x, 0, 0, 0; 0, 0, 0, 1, 0, 0; 0, 0, 0, 1,
x, 0; 0, 0, 0, 0, 0, x], 1]
```

```
Shell]
```

$$x^2 \frac{dG}{dx} = \tilde{A}(x)G = \begin{pmatrix} -x+4 & 2x & x & 0 & 0 & 2x \\ -2x & -x+3 & 2x & \frac{3}{2} & \frac{3}{2}x & -4x \\ 4x & -2 & -x & -1 & -x & 8x \\ x^2 & 2x & x & 2x & 0 & 0 \\ -x+8 & x+2 & \frac{1}{2}x & 1 & 0 & -4x \\ 0 & 2x & x & 0 & 0 & 0 \end{pmatrix} G$$

$$T(x) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & \frac{1}{2}x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}$$

True Poincaré rank $= 1$

Example 4. Regular system (Poincaré rank drops to zero)

$$x^3 \frac{dF}{dx} = A(x)\, F$$

$$A(x) = \begin{pmatrix} 6x^3 & 27x^4 & 6x^2 & 7x^5 \\ 0 & x^4 & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}$$

`moser_barkatou_trans(A,2)` outputs an equivalent rank-reduced system, the transformation $F = TG$ and the true Poincaré rank as well:

```
Shell] ./moser_barkatou_trans_test3
```

```
 Moser-barkatou reduction computes an equivalent system, a
transformation, and the true Poincare rank to be:

[[-1, 0, 0, 0; 6, 6 * x - 2, 27, 7 * x; 0, 0, x^2, 0; 0, -6, 0, 0], [0,
x^2, 0, 0; 0, 0, 1, 0; x^2, 0, 0, 0; 0, 0, 0, 1], 0]
```

```
Shell]
```

$$x\frac{\mathrm{dG}}{\mathrm{dx}} = \tilde{A}(x)G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 6 & 6x-2 & 27 & 7x \\ 0 & 0 & x^2 & 0 \\ 0 & -6 & 0 & 0 \end{pmatrix} G$$

$$T(x) = \begin{pmatrix} 0 & x^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

True Poincaré rank $= 0$

$p = 0$: Solutions of a regular system - Classical

Example 5.

$$x\frac{\mathrm{dF}}{\mathrm{dx}} = A(x)\,F$$

$$A(x) = \begin{pmatrix} a(x) & 0 & 0 & 0 \\ 0 & 2a(x) & 0 & 0 \\ 0 & 0 & 2a(x) & -2a(x) \\ \frac{-1+a(x)+x^6}{2} & \frac{1-a(x)-x^6}{2} & \frac{-1+a(x)+x^6}{2} & 1-\frac{a(x)+x^6}{2} \end{pmatrix}$$

where $a(x) = x - x^2 + x^3 - x^4 + x^5 - x^6$.

`reg_singular(A,m)` computes a fundamental matrix of formal solutions up to order m (whether the leading matrix coefficient has good or bad spectrum). For m=4 we have:

```
Shell] ./reg_singular_test
```

This system is regular singular and a FMFS up to order 4 is given by:
[[algebraic (0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic
(0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic (0, Field (x,
0e-323228496 + 0e-323228496 * i)), algebraic (0, Field (x, 0e-323228496 +
0e-323228496 * i)); algebraic (0, Field (x, 0e-323228496 + 0e-323228496 *
i)), algebraic (0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic
(0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic (0, Field (x,
0e-323228496 + 0e-323228496 * i)); algebraic (0, Field (x, 0e-323228496 +
0e-323228496 * i)), algebraic (0, Field (x, 0e-323228496 + 0e-323228496 *
i)), algebraic (0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic
(0, Field (x, 0e-323228496 + 0e-323228496 * i)); algebraic (1 / 2, Field
(x, 0e-323228496 + 0e-323228496 * i)), algebraic (-1 / 2, Field (x, 0e-
323228496 + 0e-323228496 * i)), algebraic (1 / 2, Field (x, 0e-323228496
+ 0e-323228496 * i)), algebraic (0, Field (x, 0e-323228496 + 0e-323228496
* i))], [algebraic (1, Field (x, 0e-323228496 + 0e-323228496 * i)) * x +
algebraic (1, Field (x, 0e-323228496 + 0e-323228496 * i)), 0, 0, 0; 0,
algebraic (1, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2 +
algebraic (2, Field (x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic
(1, Field (x, 0e-323228496 + 0e-323228496 * i)), 0, 0; algebraic (-1 /
18, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^3 + algebraic (1 / 4,
Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (0, Field
(x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic (0, Field (x, 0e-
323228496 + 0e-323228496 * i)), algebraic (7 / 18, Field (x, 0e-323228496
+ 0e-323228496 * i)) * x^3 + algebraic (-1 / 4, Field (x, 0e-323228496 +
0e-323228496 * i)) * x^2 + algebraic (0, Field (x, 0e-323228496 + 0e-
323228496 * i)) * x + algebraic (0, Field (x, 0e-323228496 + 0e-323228496
* i)), algebraic (-7 / 18, Field (x, 0e-323228496 + 0e-323228496 * i)) *
x^3 + algebraic (5 / 4, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2
+ algebraic (2, Field (x, 0e-323228496 + 0e-323228496 * i)) * x +
algebraic (1, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic (1 /
3, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^3 + algebraic (-1,
Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (0, Field
(x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic (0, Field (x, 0e-
323228496 + 0e-323228496 * i)); algebraic (7 / 1728, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^4 + algebraic (-17 / 288, Field (x,
0e-323228496 + 0e-323228496 * i)) * x^3 + algebraic (3 / 8, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^2 + algebraic (1 / 2, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x + algebraic (1 / 2, Field (x, 0e-
323228496 + 0e-323228496 * i)), algebraic (-91 / 1728, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^4 + algebraic (101 / 288, Field (x,
0e-323228496 + 0e-323228496 * i)) * x^3 + algebraic (-11 / 8, Field (x,
0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (-1, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x + algebraic (-1 / 2, Field (x, 0e-
323228496 + 0e-323228496 * i)), algebraic (91 / 1728, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^4 + algebraic (-101 / 288, Field (x,
0e-323228496 + 0e-323228496 * i)) * x^3 + algebraic (11 / 8, Field (x,
0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (1, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x + algebraic (1 / 2, Field (x, 0e-
323228496 + 0e-323228496 * i)), algebraic (-7 / 144, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^4 + algebraic (7 / 24, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^3 + algebraic (-1, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^2 + algebraic (1, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x + algebraic (0, Field (x, 0e-323228496
+ 0e-323228496 * i))]]

We rerwrite the above output:

$$F = \Phi(x) \ \ x^C$$

where

$$C \quad = \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \end{pmatrix} \qquad \text{and}$$

$$\Phi(x) = \begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & x^2+2x+1 & 0 & 0 \\ \frac{1}{4}x^2 & -\frac{1}{4}x^2 & \frac{5}{4}x^2+2x+1 & -x^2 \\ \frac{3}{8}x^2+\frac{1}{2}x+\frac{1}{2} & -\frac{11}{8}x^2-x-\frac{1}{2} & \frac{11}{8}x^2+x+\frac{1}{2} & -x^2+x \end{pmatrix} + O(x^3)$$

Example 6. Irregular system

$$x^3 \frac{dF}{dx} = A(x)F = \begin{pmatrix} -2x^2 - x & 0 & x & 0 \\ 1 - x & -x + x^3 & x - 2x^3 + x^4 + x^5 & -3x^{23} - x^4 \\ 0 & 1 & -3x^2 + x & 1 \\ -x & x^3 + x^2 & x^4 + x^3 + x & -4x^2 - x^3 \end{pmatrix} F$$

`exp_part_clag(A,2)` where 2 is the Poincaré rank computes the exponential part

```
Shell] ./exp_part_test1a
```

```
 The exponential part in a Fundamental Matrix of Formal Solutions of A
with p=2 is:


[[algebraic (-1, Field (x, 0e-323228496 + 0e-323228496 * i)) / (algebraic
(2, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (0,
Field (x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic (0, Field (x,
0e-323228496 + 0e-323228496 * i))), (algebraic (-590 / 81, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^4 + algebraic (25 / 27, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^3 + algebraic (5 / 3, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x^2 + algebraic (-5 / 4, Field (x, 0e-
323228496 + 0e-323228496 * i)) * x + algebraic (-3, Field (x, 0e-
323228496 + 0e-323228496 * i))) / (algebraic (5, Field (x, 0e-323228496 +
0e-323228496 * i)) * x^5 + algebraic (0, Field (x, 0e-323228496 + 0e-
323228496 * i)) * x^4 + algebraic (0, Field (x, 0e-323228496 + 0e-
323228496 * i)) * x^3 + algebraic (0, Field (x, 0e-323228496 + 0e-
323228496 * i)) * x^2 + algebraic (0, Field (x, 0e-323228496 + 0e-
323228496 * i)) * x + algebraic (0, Field (x, 0e-323228496 + 0e-323228496
* i)))], [algebraic (1, Field (x, 0e-323228496 + 0e-323228496 * i)) * x +
algebraic (0, Field (x, 0e-323228496 + 0e-323228496 * i)), algebraic (1,
Field (x - 1, 1.000000000000000000)) * x^3 + algebraic (0, Field (x - 1,
1.000000000000000000)) * x^2 + algebraic (0, Field (x - 1,
1.000000000000000000)) * x + algebraic (0, Field (x - 1,
1.000000000000000000))], [1, 1]]
```

```
Shell]
```

We rewrite the above output:

$$Q(t): \left[\frac{-1}{2t^2}, \frac{-118}{81t} + \frac{5}{27t^2} + \frac{1}{3t^3} - \frac{1}{2t^2} - \frac{3}{5t^5} \right]$$

$$x = [t, t^3]$$

$$\text{multiplicity} = [1, 1]$$

As you can notice, the computations are always up to conjugates!

Example 7. Irregular system

$$x^2 \frac{\mathrm{dF}}{\mathrm{dx}} = A(x)F = \begin{pmatrix} -x & \frac{2}{3}(x-1) & \frac{16}{9}x \\ 0 & -x & \frac{4}{3}x \\ \frac{9}{4} & \frac{3}{4}(x+1) & -x \end{pmatrix} F$$

`exp_part_clag(A,2)` where 2 is the Poincaré rank computes the exponential part

```
Shell] ./exp_part_test4


 The exponential part in a FMFS of A with p=1 is:

[[(algebraic (-10, Field (x, 0e-323228496 + 0e-323228496 * i)) * x +
algebraic (6, Field (x, 0e-323228496 + 0e-323228496 * i))) / (algebraic
(2, Field (x, 0e-323228496 + 0e-323228496 * i)) * x^2 + algebraic (0,
Field (x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic (0, Field (x,
0e-323228496 + 0e-323228496 * i)))], [algebraic (-1 / 2, Field (x + 2, -
2.000000000000000000)) * x^3 + algebraic (0, Field (x + 2, -
2.000000000000000000)) * x^2 + algebraic (0, Field (x + 2, -
2.000000000000000000)) * x + algebraic (0, Field (x + 2, -
2.000000000000000000))], [1]]

Shell]
```

We rewrite the above output:

$$Q(t): \left[\frac{-10t+6}{2t^2} \right]$$

$$x = \left[-\frac{1}{2}t^3 \right]$$

$$\text{multiplicity} = [1]$$

Example 8. Irregular system associated to the equation:

$$f^{(6)} - x^6 \, f^{(5)} - x^3 f'' + f = 0$$

$$x^9 \frac{dF}{dx} = A(x)F = \begin{pmatrix} 0 & x^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^9 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^9 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^9 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^9 \\ -1 & 0 & x^3 & 0 & x^6 & 0 \end{pmatrix} F$$

`exp_part_clag(A,2)` where 2 is the Poincaré rank computes the exponential part

```
Shell] ./exp_part_test3
```

```
The exponential part in a FMFS of A with p=8 is:


[[algebraic (-2, Field (x, 0e-323228496 + 0e-323228496 * i)) / (algebraic
(1, Field (x, 0e-323228496 + 0e-323228496 * i)) * x + algebraic (0, Field
(x, 0e-323228496 + 0e-323228496 * i)))], [algebraic (1, Field (x - 1,
1.000000000000000000)) * x^2 + algebraic (0, Field (x - 1,
1.000000000000000000)) * x + algebraic (0, Field (x - 1,
1.000000000000000000))], [1]]
```

```
Shell]
```

We rewrite the above output:

$$Q(t) : \left[\frac{-2}{t} \right]$$

$$x = [t^2]$$

$$\text{multiplicity} = [1]$$

Example 9.

$$x^9 \, f''' - (-6x^8 + x^7 + 4x^6) \, f'' - (-6x^7 + 2x^6 + 3x^5 - 3x^4 - 5x^3) \, f' - (2x^3 + 6x^2 + 2x + 2) \, f = 0$$

$$\text{Let } F = [f, f', f'']^T \quad \text{then} \quad x^9 \frac{dF}{dx} = A(x) \, F$$

$$A(x) = \begin{pmatrix} 0 & x^9 & 0 \\ 0 & 0 & x^9 \\ 2x^3 + 6x^2 + 2x + 2 & -6x^7 + 2x^6 + 3x^5 - 3x^4 - 5x^3 & -6x^8 + x^7 + 4x^6 \end{pmatrix}$$

`moser_barkatou_trans(A,p)` outputs an equivalent rank-reduced system, the transformation $F = TG$ and the true Poincaré rank:

```
Shell]  ./moser_barkatou_trans_test2
```

```
 Moser-barkatou reduction computes an equivalent system, a
transformation, and the true Poincare rank to be:

[[-3 * x^2, 0, 1; 4 * x^3 - 15 / 2 * x^2 + 1, -6 * x^2, x - 5 / 2; -8 *
x^4 + x^3 + 16 * x^2, 2 * x^3 + 6 * x^2 + 2 * x + 2, -6 * x^2 + x + 4],
[-x^7 + 5 / 2 * x^6, x^6, 0; x^3, 0, 0; 0, 0, 1], 2]
```

```
Shell]
```

We rewrite the above output:

$$x^3\frac{\mathrm{dG}}{\mathrm{dx}} = \tilde{A}(x)G = \begin{pmatrix} -3x^2 & 0 & 1 \\ 4x^3 - \frac{15}{2}x^2 + 1 & -6x^2 & x - \frac{5}{2} \\ -8x^4 + x^3 + 16x^2 & 2x^3 + 6x^2 + 2x + 2 & -6x^2 + x + 4 \end{pmatrix} G$$

$$T(x) = \begin{pmatrix} -x^7 + \frac{5}{2}x^6 & x^6 & 0 \\ x^3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What else to experiment with LINDALG ?

- Each function is implemented for the most general algebraic structure (e.g. shearing trasformations are over rings, Block-diagonalization over fields,...)

- Distinct fundamental transformations are implemented in separate files.

- The functionalities of LINDALG can be used to compute some formal invariants of completely integrable Pfaffian systems and singularly-perturbed linear differential systems (see PhD thesis'2015, maddah)

- Code can be re-used for other problems specific to linear algebra by restricting the transformation to the similarity term.

   Example 10. Writing an algebraic version of Moser-Barkatou algorithm for the algebraic eigenvalue problem.

- Compute 500, 800, 1000, ... terms in the series expansion of the equivalent system and for reduction of matrices of higher dimensions

Example 11.

Shell session inside TeXmacs pid = 4452

```
Shell] ./split_lemma_trans_test500
```

To do...

So far...

- LINDALG comes with the current release of MATHEMAGIX

- Accompanied with examples and can serve for research and pedagogical goals.

- This manual and a tutorial of formal reduction are available at:

  www.unilim.fr/pages_perso/suzy.maddah/Research_.html

To do ...

- Expand to include all functionalities of ISOLDE (e.g. rational solutions, ...)

- Build up on it to rewrite other packages which we are being written in MAPLE: apparent singularities, perturbed systems, pfaffian systems.