

*AppSing is a Maple Package for removing apparent singularities of systems of linear differential equations with rational function coefficients.<sup>1</sup> In what follows, we describe its two main procedures: Desing and Desingp.*

## 1 Desingularization at p

**Desingp** — constructs an equivalent system with rational function coefficients, which is a desingularization of the input system at p.

### Calling sequence

Desingp(A, z, p)

### Parameters

- A - the matrix of the system  $\frac{d}{dz} X = A(z) X$
- z - the independent variable
- p - a polynomial in z

### Returned Values

T, invT, B, t := Desingp(A, z, p);

- B - the matrix of the resulting equivalent system  $\frac{d}{dz} Y = B(z) Y$
- T - the transformation matrix (gauge transformation) corresponding to this desingularization ( $Y = T X$ )
- invT - the inverse of the transformation matrix T
- t - the execution time

### Examples

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<sup>1</sup> M.A. Barkatou, S.S. Maddah (2015). Removing Apparent Singularities of Systems of Linear Differential Equations with Rational Function Coefficients. To appear in Proceedings of the 40th International Symposium on Symbolic and Algebraic Computation, ACM Press.

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Maple 1] with(LinearAlgebra): with(DETools):
Maple 2] libname := "/Users/maddah/Documents/Implementations/
AppSing/AppSing.lib", "/Library/Frameworks/Maple.framework/
Versions/17/lib";
Maple 21] march('list', "/Users/maddah/Documents/Implementations/
AppSing/AppSing.lib");

```

### Example 1:

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Maple 18] A := Matrix(2, 2, [0, 1, (4*z^2-2)/(z^2+2), -(3*z^2-4)/
(z*(z^2+2))]);

```

$$\begin{bmatrix} 0 & 1 \\ \frac{4z^2-2}{z^2+2} & -\frac{3z^2-4}{z(z^2+2)} \end{bmatrix}$$

The roots of  $z^2 + 2$  are not apparent singularities. In fact, the eigenvalues of the corresponding residue matrix don't fulfill the requirements.

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Maple 19] T, invT, B, t := Desingp(A, z, z^2+2);

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 \frac{2z^2-1}{z^2+2} & -\frac{3z^2-4}{z(z^2+2)} \end{bmatrix}, 0.064$$

$z=0$  is an apparent singularity: It is removed in the equivalent system.

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Maple 20] T, invT, B, t := Desingp(A, z, z);

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$$\begin{bmatrix} 0 & 1 \\ z^2 & z \end{bmatrix}, \begin{bmatrix} -z^{-1} & z^{-2} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{z(z^2+7)}{z^2+2} & -1 \\ z^2 & z \end{bmatrix}, 0.230$$

## Example 2<sup>2</sup>

Maple 7]  $A := \text{Matrix}(2, 2, [0, 1, (2*x^4-x^3-35*x^2+25*x+45)/((x+1)*(2*x^3-x^2-20*x+23)), -(2*(-x^3-3*x^2-9*x+33))/((x+1)*(2*x^3-x^2-20*x+23))]);$

$$\begin{bmatrix} 0 & 1 \\ \frac{2x^4 - x^3 - 35x^2 + 25x + 45}{(x+1)(2x^3 - x^2 - 20x + 23)} & -2 \frac{-x^3 - 3x^2 - 9x + 33}{(x+1)(2x^3 - x^2 - 20x + 23)} \end{bmatrix}$$

The roots of  $2x^3 - x^2 - 20x + 23$  are apparent singularities. They are removed in the equivalent system.

Maple 8]  $T, \text{inv}T, B, t := \text{Desingp}(A, x, 2*x^3-x^2-20*x+23);$

$$\begin{bmatrix} 2x^3 - x^2 - 20x + 23 & -\frac{14}{143}x^2 + \frac{3}{11}x + \frac{153}{143} \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} (2x^3 - x^2 - 20x + 23)^{-1} & \frac{1}{143} \frac{14x^2 - 39x - 153}{2x^3 - x^2 - 20x + 23} \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \frac{1}{143} \frac{14x^3 - 39x^2 - 258x - 175}{x+1} & -\frac{1}{20449} \frac{98x^2 - 497x - 1385}{x+1} \\ \frac{2x^4 - x^3 - 35x^2 + 25x + 45}{x+1} & -\frac{1}{143} \frac{14x^3 - 39x^2 - 258x + 111}{x+1} \end{bmatrix}, 0.303$$

## 2 Full desingularization

**Desing** — constructs an equivalent system with rational function coefficients, whose finite singularities are exactly the non-apparent singularities of the input system.

### Calling sequence

$\text{Desing}(A, z)$

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2. Shaoshi Chen, M. Jaroschek, M. Kauers, M. F. Singer (2013). Desingularization Explains Order-Degree Curves for Ore Operators, Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, pp 157-164, ACM, U.S.A.

## Parameters

→  $A$  - the matrix of the system  $\frac{d}{dz} X = A(z) X$

→  $z$  - the independent variable

## Returned Values

$T, \text{inv}T, B, t := \text{Desing}(A, z);$

→  $B$  - the matrix of the resulting equivalent system  $\frac{d}{dz} Y = B(z) Y$

→  $T$  - the transformation matrix (gauge transformation) corresponding to this desingularization ( $Y = T X$ )

→  $\text{inv}T$  - the inverse of the transformation matrix  $T$

→  $t$  - the execution time

## Examples

### Example 3

Maple 9]  $A := \text{Matrix}(2, 2, [0, 1, (4*z^2-2)/(z^2+2), -(3*z^2-4)/(z*(z^2+2))]);$

$$\begin{bmatrix} 0 & 1 \\ \frac{4z^2-2}{z^2+2} & -\frac{3z^2-4}{z(z^2+2)} \end{bmatrix}$$

Maple 10]  $T, \text{inv}T, B, t := \text{Desing}(A, z);$

$$\begin{bmatrix} 0 & 1 \\ z^2 & z \end{bmatrix}, \begin{bmatrix} -z^{-1} & z^{-2} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{z(z^2+7)}{z^2+2} & -1 \\ z^2 & z \end{bmatrix}, 0.307$$

### Example 4<sup>3</sup>

Maple 11]  $A := \text{Matrix}(2, 2, [0, 1, -(x-2)*(2*x^2-3*x+3)/((x-1)*(x^2-3*x+3)*x), (x^2-3)*(x^2-2*x+2)/((x-1)*(x^2-3*x+3)*x)]);$

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<sup>3</sup>. Shaoshi Chen, M. Kauers, M. F. Singer (2014). Desingularization of Ore Operators, Available at: arXiv:1408.5512v1, (2014)

$$\begin{bmatrix} 0 & 1 \\ -\frac{(x-2)(2x^2-3x+3)}{(x-1)(x^2-3x+3)x} & \frac{(x^2-3)(x^2-2x+2)}{(x-1)(x^2-3x+3)x} \end{bmatrix}$$

The roots of  $(x^2 - 3x + 3)$  and  $x$  are apparent singularities. They are removed in the equivalent system. Whereas  $x = 1$  is not an apparent singularity.

Maple 12] `T, invT, B, t := Desing(A, x);`

$$\begin{bmatrix} x^4 - 3x^3 + 3x^2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2(x^2-3x+3)} & -\frac{1}{x^2(x^2-3x+3)} \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -2(x-1)^{-1} & 0 \\ -\frac{x(2x^3-7x^2+9x-6)}{x-1} & 1 \end{bmatrix}, 0.642$$

### 3 Other procedures

- Partial desingularization can be attained via the Maple procedure `DEtools[moser_reduce]`.
- The intermediate steps in the algorithm are displayed with the two AppSing procedures: `DesingpSteps(A,z,p)` and `DesingSteps(A,z)`.

**More Examples:**

Please refer to the Maple sheet on the [webpage](#).

This document is written in [TeXmacs](#).

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TeXmacs interface by Joris van der Hoeven