On positive duality gaps in semidefinite programming

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A pair of Semidefinite Programs (SDP)

$$(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad \begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \; \forall i. \end{array} \quad (D)$$

Here

- A_i, B **are symmetric matrices,** $c, x \in \mathbb{R}^m$.
- $A \preceq B$ **means that** $B - A$ **is symmetric positive semidefinite (psd).**
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$.

SDP duality

$$\begin{array}{llll} & \sup_x \ c^T x & \inf_Y \ B \bullet Y & \\ (P) & s.t. \ \sum_{i=1}^m x_i A_i \preceq B & Y \succeq 0 & (D) \\ & & A_i \bullet Y = c_i \, \forall i. & \end{array}$$

SDP duality

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Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

SDP duality

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Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideally: $\exists x^*, \exists Y^* : c^T x^* = B \bullet Y^*$.

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But: pathologies occur, as nonattainment, positive gaps.

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Easy: If x and Y are feasible, then $c^T x \leq B \bullet Y$.

Ideally: $\exists x^*, \exists Y^* : c^T x^* = B \bullet Y^*$.

But: pathologies occur, as nonattainment, positive gaps.

$\rightarrow$ in such cases we cannot certify optimality.

Example: positive duality gap

Primal:

$$\sup \ x_2$$

$$s.t. \ x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$x_2 = 0$ **identically.**

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$x_2 = 0$ identically $\Rightarrow$ primal $= 0$.

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Dual matrix $Y \succeq 0$

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Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; y_{11} = 0 \Rightarrow Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$

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2nd dual constraint $\Rightarrow y_{22} = 1 \Rightarrow \text{dual opt} = 1$

Example: positive duality gap

Primal:

$$\sup \; x_2$$
$$s.t. \; x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Looks quite odd: x_1 "only exists" to create a zero block in the dual matrix.

Positive gaps

- **Maybe the "worst/most interesting" pathology.**
- **Solvers fail, or report a wrong solution.**
- **Good model of positive gaps in more general convex programs**

Literature

- **Pathological semidefinite systems P, 2011 –**
  "Bad semidefinite programs: they all look the same"

- **It does not distinguish among bad objective functions.**

Literature

- Pathological semidefinite systems P, 2011 –
  "Bad semidefinite programs: they all look the same"
- It does not distinguish among bad objective functions.
- Positive gaps related to complementarity in homogeneous systems: Tuncel, Wolkowicz, 2012
- Weak infeasibility: Lourenco, Muramatsu, Tsuchiya, 2014
- Infeasibility, weak infeasibility: Liu, P 2015, 2017

Literature: how to solve some pathological SDPs

- **Facial reduction of Borwein-Wolkowicz, Waki-Muramatsu, Pataki: implemented by Permenter, Parrilo 2014; Permenter,Friberg,Andersen 2015**

- **Very simple facial reduction (just inspect the constraints): Zhu, P, Tran Dinh (Sieve-SDP) 2017**

- **SPECTRA, exact arithmetic SDP solver Henrion-Naldi-El Din 2016**

- **Douglas-Rachford splitting: Liu, Ryu, Yin 2017**

- **Homotopy method Hauenstein, Liddell, Zhang 2018**

Main ideas

- **Look at small instances.**
- **Proposition: positive gap $\Rightarrow m \geq 2$.**
- **Fully characterize the $m = 2$ case.**

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  Proposition: positive gap $\Rightarrow m \geq 2$.

- **Fully characterize the $m = 2$ case.**

- **Show that the $m = 2$ case sheds light on larger m.**

- **Precisely: the structure that causes positive gap when $m = 2$ often does the same when $m > 2$.**

Main ideas

- Look at small instances.

  Proposition: positive gap $\Rightarrow m \geq 2$.

- Fully characterize the $m = 2$ case.

- Show that the $m = 2$ case sheds light on larger m.

- Precisely: the structure that causes positive gap when $m = 2$ does the same in many cases even if $m > 2$.

- Reformulate

- Borrow ideas from linear system of equations:
  to show $Ax = b$ is infeasible, we create an equation
  $\langle 0, x \rangle = 1$.

Recall: a pair of Semidefinite Programs (SDPs)

$$(P) \quad \begin{array}{ll} \sup_x & c^T x \\ s.t. & \sum_{i=1}^m x_i A_i \preceq B \end{array} \qquad \begin{array}{ll} \inf_Y & B \bullet Y \\ & Y \succeq 0 \\ & A_i \bullet Y = c_i \, \forall i. \end{array} \quad (D)$$

Reformulations of $(P)-(D)$ are obtained by

- Choose T invertible, and

$$B \leftarrow T^T B T,\ A_i \leftarrow T^T A_i T\,\forall i.$$

- Elementary row operations on (D) : e.g., exchange two constraints $A_i \bullet Y = c_i$ and $A_j \bullet Y = c_j$.

- Choose $\mu \in \mathbb{R}^m$ and

$$B \leftarrow B + \sum_{i=1}^{m} \mu_i A_i.$$

Reformulations preserve positive gaps (if any).

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$
\begin{array}{ll}
\sup & c_2' x_2 \\
s.t. & x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array} \right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \\ \hline \end{array} \right)
\end{array}
$$

where $\Lambda \succ 0,\ M \neq 0,\ c_2' > 0, s \geq 0.$

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where $\Lambda \succ 0,\ M \neq 0,\ c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$

This is the easy direction.

Essentially reuse the argument from before.

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where $\Lambda \succ 0$, $M \neq 0$, $c_2' > 0, s \geq 0$.

Proof of $\Leftarrow M \neq 0 \Rightarrow x_2 = 0 \Rightarrow$ primal $= 0$.

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$
\begin{array}{ll}
\sup & c_2' x_2 \\
s.t. & x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array} \right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array} \right)
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where $\Lambda \succ 0,\ M \neq 0,\ c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$ Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow\ \Lambda \bullet Y(1:p, 1:p) = 0$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \; c_2' x_2$$

$$s.t. \; x_1 \left(\begin{array}{c|cc|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right) + x_2 \left(\begin{array}{c|c|c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array}\right) \preceq \left(\begin{array}{c|c|c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right)$$

where $\Lambda \succ 0, \; M \neq 0, \; c_2' > 0, s \geq 0.$

Proof of $\Leftarrow$ **Dual matrix** $Y \succeq 0$

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Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

$$\sup \; c_2' x_2$$

$$s.t. \;\; x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

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Proof of $\Leftarrow$ Dual matrix $Y \succeq 0$

1st dual constraint $\Rightarrow \; \Lambda \bullet Y(1:p, 1:p) = 0$

$\Rightarrow \; Y(1:p, 1:p) = 0$

$\Rightarrow$ 1st p rows and columns of Y are zero.

Suppose $m = 2$.
Then positive gap $\Leftrightarrow \exists$ reformulation

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$$s.t.\; x_1 \begin{pmatrix} \Lambda & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} + x_2 \begin{pmatrix} \times & \times & \times & M \\ \times & \Sigma & & \\ \times & & -I_s & \\ M^T & & & \end{pmatrix} \preceq \begin{pmatrix} I_p & & & \\ & I_{r-p} & & \\ & & & \\ & & & \end{pmatrix}$$

where $\Lambda \succ 0,\; M \neq 0,\; c_2' > 0, s \geq 0.$

$$\Rightarrow \text{dual is} \quad \begin{array}{rl} \inf & \begin{pmatrix} I_{r-p} & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' \\ s.t. & \begin{pmatrix} \Sigma & 0 \\ 0 & -I_s \end{pmatrix} \bullet Y' = c_2' > 0 \\ & Y' \succeq 0, \end{array}$$

Suppose $m = 2$.
Then positive gap $\Leftrightarrow$ $\exists$ reformulation

$$
\begin{array}{ll}
\sup & c_2' x_2 \\
s.t. & x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array}\right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right)
\end{array}
$$

where $\Lambda \succ 0$, $M \neq 0$, $c_2' > 0$, $s \geq 0$.

$\Rightarrow$ dual optimal value > 0.

Simple certificate of the positive pap

When does the underlying system admit a gap?

Given

$$(P_{SD}) \quad \sum_{i=1}^{m} x_i A_i \preceq B$$

is there $c \in \mathbb{R}^m$ such that there is a positive gap?

Suppose $m = 2$. Then $\exists(c_1, c_2)$ with positive gap
$\Leftrightarrow$ (P_{SD}) has reformulation

$$x_1 \left(\begin{array}{c|c||c|c} \Lambda & & & \\ \hline & & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right) + x_2 \left(\begin{array}{c|c||c|c} \times & \times & \times & M \\ \hline \times & \Sigma & & \\ \hline\hline \times & & -I_s & \\ \hline M^T & & & \end{array}\right) \preceq \left(\begin{array}{c|c||c|c} I_p & & & \\ \hline & I_{r-p} & & \\ \hline\hline & & & \\ \hline & & & \end{array}\right),$$

where $\Lambda \succ 0$, $M \neq 0$, $s \geq 0$.

How about $m > 2$?

Similar example with $m = 3$

$$
\begin{aligned}
\sup \quad & x_3 \\
s.t. \quad & x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.
\end{aligned}
$$

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$$s.t. \; x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

Primal $= 0.$

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\end{aligned}$$

Primal $= 0.$

Dual : Variable $Y = (y_{ij}) \succeq 0$

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Primal $= 0.$

Dual : Variable $Y = (y_{ij}) \succeq 0$

1st two dual constraints $\Rightarrow$ **1st two rows and columns of** Y **are zero.**

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\end{array}
$$

Primal $= 0.$

$\Rightarrow$ Dual is equivalent to:

$$
\begin{array}{ll}
\inf & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' \\
s.t. & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet Y' = 1 \\
& Y' \succeq 0,
\end{array}
$$

Same structure as in the 2 variable case

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$$\begin{array}{ll} \sup & x_3 \\ s.t. & x_1 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \end{array}$$

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$\Rightarrow$ Dual is equivalent to:

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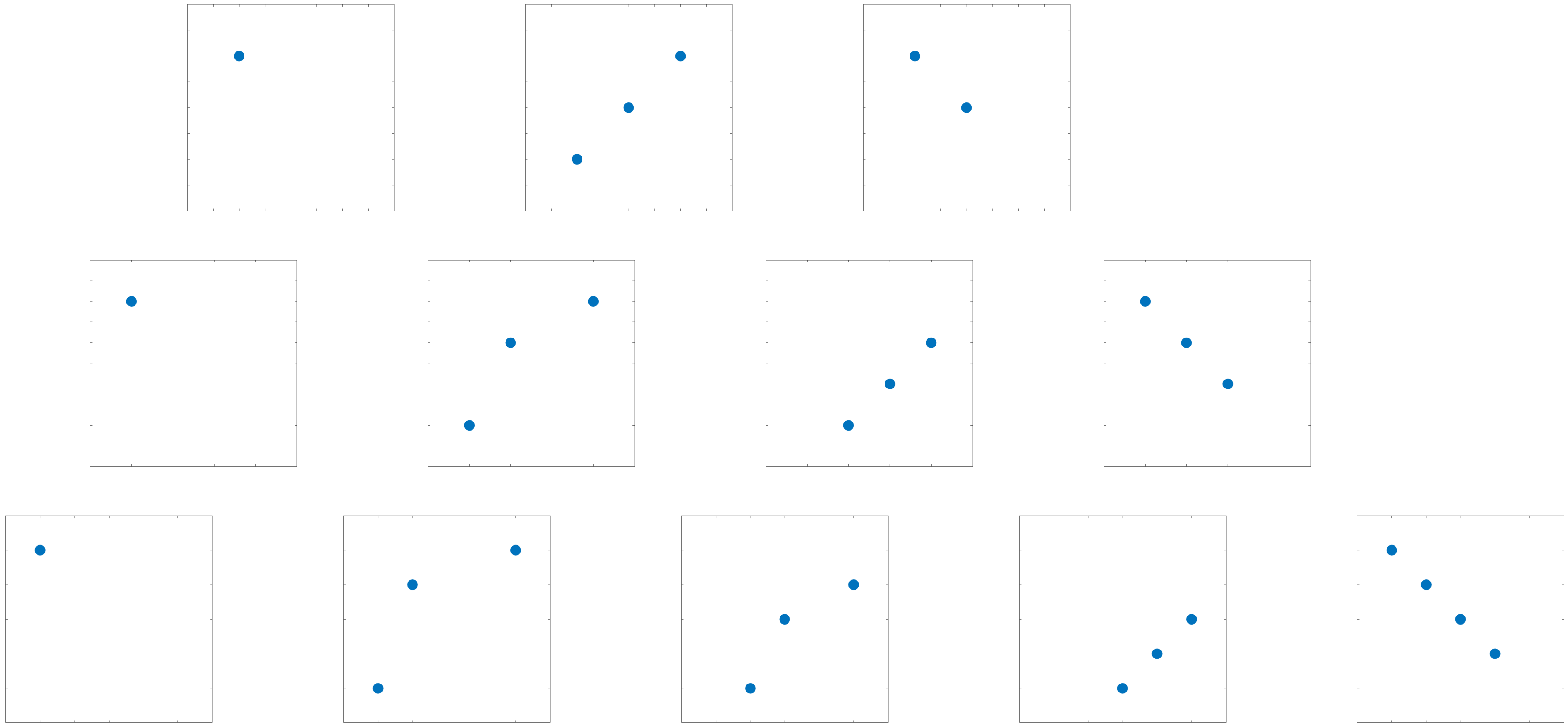
Same structure as in the 2 variable case

We can create such an instance for any $m = 2, 3, 4, \ldots$ with $n = m + 1$

Sparsity structure when $m = 2, 3, 4$ $(n = m + 1)$



How do we get these instances? Background: facial reduction

Given H affine subspace, K closed convex cone, a facial reduction algorithm (FRA) works as:

(1) If $\text{ri}\ K \cap H = \emptyset$, find $y \in H^\perp \cap (K^* \setminus K^\perp)$.

(2) Replace K by $K \cap y^\perp$. Goto (1).

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Facial reduction sequence:

The sequence $y_1, y_2, \ldots$ generated by the FRA.

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Facial reduction sequence:

The sequence $y_1, y_2, \ldots$ generated by the FRA.

Singularity degree:

Is the smallest number of FRA steps, until the FRA stops.

Back to $m = 2$ example:

$$\sup \; x_2$$

$$s.t. \; x_1 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A_1} + x_2 \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_2} \preceq \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{B}$$

Here (A_1) is a facial reduction sequence for (D).

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Here (A_1) is a facial reduction sequence for (D).

$A_1 \bullet Y = 0$ proves that dual matrix must look like

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{23} & y_{33} \end{pmatrix}$$

Back to larger example:

$$\sup \; x_3$$

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Here (A_1, A_2) is a facial reduction sequence for (D).

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Here (A_1, A_2) is a facial reduction sequence for (D).

$A_1 \bullet Y = A_2 \bullet Y = 0$ proves that dual matrix must look like

$$Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y_{33} & y_{34} \\ 0 & 0 & y_{43} & y_{44} \end{pmatrix}$$

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Theorem:

- $\mathrm{sing}(D) \leq m$.
- $= m \Rightarrow$ no gap. (Easy)
- $\forall m \geq 2$, $\forall g > 0$ $\exists$ instance s.t.

  $\mathrm{sing}(D) = m - 1$ and gap is g. (See examples)

Analogous results for the homogeneous dual

$$\begin{aligned} A_i \bullet Y &= 0\,\forall i \\ B \bullet Y &= 0 \quad (HD) \\ Y &\succeq 0 \end{aligned}$$

See paper . . .

Problem library and computational results

- Instances with $m = 2, 3, \ldots, 11$; gap = 10.

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- Messy means we applied a similarity transformation $T^T()T$.

Results

	GAP, SINGLE, FINITE		GAP, SINGLE, INFINITE	
	CLEAN	MESSY	CLEAN	MESSY
MOSEK	1	1	0	0
SDPA-GMP	1	1	0	0
PP+MOSEK	10	1	10	0
SIEVE-SDP + MOSEK	10	1	10	0

- **PP: preprocessor of Permenter and Parrilo**
- **Sieve-SDP: preprocessor of Zhu, Pataki, Tran-Dinh**

Note: SPECTRA works when $m = 2$.

Summary:

- **Positive gaps: "worst/most interesting" pathology of SDPs.**
- **Complete characterization for $m = 2$ by reformulation.**
- **Complete characterization of positive gap systems with $m = 2$.**
- **Similarly structured positive gap SDPs in any dimension.**
- **Highest singularity degree that leads to a positive gap.**
- **Challenging problem library.**

Thank you!