A Generalization of SAGE Certificates for Constrained Optimization

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July 11, 2019

Joint work with Venkat Chandrasekaran and Adam Wierman (Caltech).

Nonnegativity and optimization

Caltech

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a set $X \subset \mathbb{R}^n$, we have

$$\begin{aligned} f_X^\star &= \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f(\boldsymbol{x}) \geq \gamma \text{ for all } \boldsymbol{x} \text{ in } X\} \\ &= \sup\{\gamma : f - \gamma \text{ is nonnegative over } X\}. \end{aligned}$$

How do we test if f (or $f - \gamma$) is nonnegative over X?

- This is usually NP-Hard, even for $X = \mathbb{R}^n$.
- Desire *sufficient conditions* that f is nonnegative over X.
- Different sufficient conditions for different function classes.

Signomials and polynomials

Caltech

Parameters $\boldsymbol{\alpha}_i$ in $\mathbb{R}^n$, c_i in $\mathbb{R}$.

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$$

Uncountable basis.

Complexity measured by number of terms m.

Nonnegativity certificates

- SAGE

Parameters $\boldsymbol{\alpha}_i$ in $\mathbb{N}^n$, c_i in $\mathbb{R}$.

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \prod_{j=1}^{n} x_j^{\alpha_{ij}}$$

Countable basis.

Complexity measured by degree $\max_{ij} \alpha_{ij}$.

Select nonnegativity certificates

- SOS
- SONC
- SAGE

The signomial nonnegativity cones

Write $f = \mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ to mean $f(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \exp(\boldsymbol{\alpha}_i \cdot \boldsymbol{x})$.

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Define the nonnegativity cone for signomials over exponents $\boldsymbol{\alpha}$:

$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}) \doteq \{\, \boldsymbol{c} \,:\, \mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n \}.$$

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$$C_{\mathrm{NNS}}(\boldsymbol{\alpha}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{u}) = C_{\mathrm{NNS}}(\boldsymbol{\alpha}\boldsymbol{V})$$

for all row vectors $\boldsymbol{u}$ in $\mathbb{R}^n$ and all invertible $\boldsymbol{V}$ in $\mathbb{R}^{n \times n}$.

SAGE certificates for signomials

Caltech

Definition. A nonnegative signomial with at most one negative coefficient is an "**AM/GM Exponential**," or an "AGE function."

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For each k, have cone of coefficients for AM/GM Exponentials

$$C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k) \doteq \{\boldsymbol{c} \ : \ \boldsymbol{c}_{\setminus k} \geq \mathbf{0} \text{ and } \boldsymbol{c} \text{ in } C_{\mathrm{NNS}}(\boldsymbol{\alpha})\}.$$

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$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = \sum_{k=1}^{m} C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k).$$

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Crucial question: How to represent the AGE cones?

The convex duality behind AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$.

Does $\boldsymbol{c}$ belong to $C_{\mathrm{NNS}}(\boldsymbol{\alpha})$?

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Appeal to convex duality. The nonnegativity condition

$$\inf_{\boldsymbol{x} \in \mathbb{R}^n} \mathrm{Sig}(\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k,\, \boldsymbol{c}_{\setminus k})(\boldsymbol{x}) \geq -c_k$$

holds **if and only if**

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$$D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k \text{ and } [\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{\nu} = \mathbf{0}.$$

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$$C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k) = \{\boldsymbol{c} : \boldsymbol{c}_{\setminus k} \geq \mathbf{0}, \text{ and there exists } \boldsymbol{\nu} \text{ in } \mathbb{R}^{m-1} \text{ satisfying} \\ D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k \text{ and } [\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]^\intercal \boldsymbol{\nu} = \mathbf{0}\}.$$

This allows us to optimize over

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}) = \sum_{k=1}^{m} C_{\mathrm{AGE}}(\boldsymbol{\alpha}, k)$$

with a convex program of size $O(m^2)$.

Handling conditional nonnegativity

Caltech

Want sufficient conditions that f is nonnegative over X.

Typical approach: search for a function $\mathcal{L}$ for which we can prove

$$0 \leq \mathcal{L}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n \quad \text{and} \quad \mathcal{L}(\boldsymbol{x}) \leq f(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \text{ in } X$$

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Concretely, we fix a representation $X = \{\boldsymbol{x} : g(\boldsymbol{x}) \geq \mathbf{0}\}$, and use

$$\mathcal{L} = f - \textstyle\sum_i \lambda_i g_i$$

with dual variables $\lambda_i : \mathbb{R}^n \to \mathbb{R}_+$. Apply your favorite (tractable!) proof system to ensure that λ_i and $\mathcal{L}$ are nonnegative functions.

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We will be taking a different route.

X-SAGE Signomials

The signomial X-nonnegativity cones

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X-SAGE certificates for signomials

Caltech

Definition. A signomial which is nonnegative over X and which has at most one negative coefficient is called an "X-AGE function."

For each k, have cone of coefficients for X-AGE functions

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We take sums of X-AGE cones to obtain the **X-SAGE cone**

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This definition assumes nothing of the set X!

Basic properties

Caltech

Conditional SAGE cones ...

- are order-reversing in the set-argument.

  If $X_2 \subset X_1$, then $C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X_1) \subset C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X_2)$.

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- *are tractable whenever X is a tractable convex set.*

Representing X-AGE cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, and $\boldsymbol{c}$ in $\mathbb{R}^m$ satisfying $\boldsymbol{c}_{\setminus k} \geq \mathbf{0}$. Convex $X \subset \mathbb{R}^n$.

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$$\begin{aligned}&\sigma_X(\boldsymbol{\lambda}) + D(\boldsymbol{\nu}, \boldsymbol{c}_{\setminus k}) - \boldsymbol{\nu}^\intercal \mathbf{1} \leq c_k, \text{ and} \\ &[\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]^\intercal \boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}.\end{aligned}$$

Representing X-AGE cones

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The preceding slides established that when X is convex,

$$
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& \text{satisfying } \ [\boldsymbol{\alpha}_{\setminus k} - \mathbf{1}\boldsymbol{\alpha}_k]^\intercal \boldsymbol{\nu} + \boldsymbol{\lambda} = \mathbf{0}, \\
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For implementations, suppose $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \in K\}$. Then

$$\sigma_X(\boldsymbol{\lambda}) \doteq \sup_{\boldsymbol{x} \in X} \boldsymbol{\lambda}^\intercal \boldsymbol{x} \ \leq \ \inf\{\boldsymbol{b}^\intercal \boldsymbol{\eta} : \boldsymbol{A}^\intercal \boldsymbol{\eta} + \boldsymbol{\lambda} = \mathbf{0}, \ \boldsymbol{\eta} \in K^\dagger\},$$

and equality holds generically.

Dual SAGE relaxations

Caltech

Let $f = \mathrm{Sig}(\boldsymbol{\alpha}, \boldsymbol{c})$ have $\boldsymbol{\alpha}_1 = \mathbf{0}$.

The primal and dual SAGE relaxations for $f_X^\star$ are

$$\begin{aligned} f_X^{\mathrm{SAGE}} &= \sup\{\, \gamma \,:\, \boldsymbol{c} - \gamma(1, 0, \ldots, 0) \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)\} \\ &= \inf\{\boldsymbol{c}^\intercal \boldsymbol{v} \,:\, v_1 = 1 \text{ and } \boldsymbol{v} \text{ in } C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger\}. \end{aligned}$$

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When X is convex, the dual X-SAGE cone can be expressed as

$$\begin{aligned} C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger = \mathrm{cl}\{\boldsymbol{v} : \ & \text{some } \boldsymbol{z}_1, \ldots, \boldsymbol{z}_m \text{ in } \mathbb{R}^n \text{ satisfy} \\ & v_k \log(\boldsymbol{v}/v_k) \geq [\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{z}_k \\ & \text{and } \boldsymbol{z}_k/v_k \in X \text{ for all } k \text{ in } [m]\}. \end{aligned}$$

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When X is convex, the dual X-SAGE cone can be expressed as

$$\begin{aligned} C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)^\dagger = \mathrm{cl}\{\boldsymbol{v} : \ & \text{some } \boldsymbol{z}_1, \ldots, \boldsymbol{z}_m \text{ in } \mathbb{R}^n \text{ satisfy} \\ & v_k \log(\boldsymbol{v}/v_k) \geq [\boldsymbol{\alpha} - \mathbf{1}\boldsymbol{\alpha}_k]\boldsymbol{z}_k \\ & \text{and } \boldsymbol{z}_k/v_k \in X \text{ for all } k \text{ in } [m]\}. \end{aligned}$$

Solution recovery? Consider vectors $\boldsymbol{x}_k = \boldsymbol{z}_k/v_k$ for k in $[m]$.

A Signomial Example

Caltech

$$\inf_{\boldsymbol{x}\in\mathbb{R}^3} \; f(\boldsymbol{x}) \doteq 0.5\exp(x_1 - x_2) - \exp x_1 - 5\exp(-x_2)$$

$$\text{s.t. } 100 - \exp(x_2 - x_3) - \exp x_2 - 0.05\exp(x_1 + x_3) \geq 0$$

$$\exp \boldsymbol{x} - (70,\, 1,\, 0.5) \geq \mathbf{0}$$

$$(150,\, 30,\, 21) - \exp \boldsymbol{x} \geq \mathbf{0}$$

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$$(150,\ 30,\ 21) - \exp \boldsymbol{x} \geq \mathbf{0}$$

Compute $f_X^{\mathrm{SAGE}} = -147.85713 \leq f_X^\star$, recover feasible

$$\boldsymbol{x}^\star = (5.01063529,\ 3.40119660, -0.48450710)$$

satisfying $f(\boldsymbol{x}^\star) = -147.66666$. *This is actually optimal!*

X-SAGE polynomials

Geometric-form signomials

Caltech

If $\boldsymbol{x} > 0$, then

$$\boldsymbol{x} \mapsto \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}$$

is defined for any real $\boldsymbol{\alpha}_i$.

For $X \subset \mathbb{R}^n_{++}$ and $\boldsymbol{\alpha}$ in $\mathbb{R}^{m \times n}$, define

$$\mathsf{C}^{\mathsf{GEOM}}_{\mathsf{NNS}}(\boldsymbol{\alpha}, X) = \{\boldsymbol{c} : \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X\}.$$

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$$\mathsf{C}_{\mathsf{NNS}}^{\mathsf{GEOM}}(\boldsymbol{\alpha}, X) = C_{\mathrm{NNS}}(\boldsymbol{\alpha}, \log X).$$

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Thus we naturally define

$$\mathsf{C}_{\mathsf{SAGE}}^{\mathsf{GEOM}}(\boldsymbol{\alpha}, X) \doteq C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, \log X).$$

The polynomial X-nonnegativity cones

Caltech

Fix $\boldsymbol{\alpha}$ in $\mathbb{N}^{m \times n}$. Write $f = \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})$ to mean

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i}, \quad \text{where} \quad \boldsymbol{x}^{\boldsymbol{\alpha}_i} \doteq \prod_{j=1}^{n} x_j^{\alpha_{ij}}.$$

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The matrix $\boldsymbol{\alpha}$ and the set X induce a nonnegativity cone

$$C_{\mathrm{NNP}}(\boldsymbol{\alpha}, X) = \{\boldsymbol{c} \,:\, \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X\}.$$

X-SAGE certificates for polynomials

Caltech

We call $f = \text{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})$ an X-**AGE polynomial** if

1. $\boldsymbol{c}$ belongs to $C_{\text{NNP}}(\boldsymbol{\alpha}, X)$, and
2. at most one "i" has $c_i \boldsymbol{x}^{\boldsymbol{\alpha}_i} < 0$ for some $\boldsymbol{x} \in X$.

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In conic form, we can express $C_{\mathrm{AGE}}^{\mathrm{POLY}}(\boldsymbol{\alpha}, i, X) =$

$$\begin{aligned}\{\boldsymbol{c} : \ & \mathrm{Poly}(\boldsymbol{\alpha}, \boldsymbol{c})(\boldsymbol{x}) \geq 0 \text{ for all } \boldsymbol{x} \text{ in } X, \\ & c_j \geq 0 \ \text{ if } \ j \neq i \text{ and } \boldsymbol{x}^{\boldsymbol{\alpha}_j} > 0 \text{ for some } \boldsymbol{x} \text{ in } X, \\ & c_j \leq 0 \ \text{ if } \ j \neq i \text{ and } \boldsymbol{x}^{\boldsymbol{\alpha}_j} < 0 \text{ for some } \boldsymbol{x} \text{ in } X \ \}.\end{aligned}$$

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Define the X-SAGE polynomial cone in the natural way:

$$C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X) = \textstyle\sum_{i=1}^{m} C_{\mathrm{AGE}}^{\mathrm{POLY}}(\boldsymbol{\alpha}, i, X).$$

Representing $C_{\text{SAGE}}^{\text{POLY}}(\boldsymbol{\alpha}, X)$

Caltech

Consider the case when X is contained within a single orthant.

W.l.o.g, take $X \subset \mathbb{R}^n_+$.

Representing $C_{\mathrm{SAGE}}^{\mathrm{POLY}}(\boldsymbol{\alpha}, X)$

Caltech

Consider the case when X is contained within a single orthant.

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If $X \subset \mathbb{R}^n_+$ is representable as

$$X = \mathrm{cl}\{\boldsymbol{x} : \mathbf{0} < \boldsymbol{x}, H(\boldsymbol{x}) \leq \mathbf{1}\}$$

for a continuous map $H : \mathbb{R}^n_{++} \to \mathbb{R}^r$,

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$$Y = \{\boldsymbol{y} : H(\exp \boldsymbol{y}) \leq \mathbf{1}\},$$

we have

$$C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X) = C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, Y).$$

Representing $C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)$

Caltech

Define the set of signomial-representative coefficient vectors

$$\mathrm{SR}(\boldsymbol{\alpha}, \boldsymbol{c}) = \{\hat{\boldsymbol{c}} : \hat{c}_i = c_i \text{ whenever } \boldsymbol{\alpha}_i \text{ is in } 2\mathbb{N}^n, \text{ and } \hat{c}_i \leq -|c_i| \text{ whenever } \boldsymbol{\alpha}_i \text{ is not in } 2\mathbb{N}^n\}.$$

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$$C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X) = \{\boldsymbol{c} \ : \ \mathrm{SR}(\boldsymbol{\alpha}, \boldsymbol{c}) \cap C_{\mathrm{SAGE}}(\boldsymbol{\alpha}, Y) \text{ is nonempty}\}.$$

Working with $C^{\mathrm{POLY}}_{\mathrm{SAGE}}(\boldsymbol{\alpha}, X)$

Caltech

How can we formulate a problem to appeal to previous theorems?

Examples include

$$-a \leq x_j \leq a, \qquad \|\boldsymbol{x}\|_p \leq a, \qquad |\boldsymbol{x}^{\boldsymbol{\alpha}_i}| \geq a, \quad \text{and} \quad x_j^2 = a$$

where $a > 0$ is a fixed constant.

A polynomial example

Caltech

Minimize

$$f(\boldsymbol{x}) = -64 \sum_{i=1}^{7} \prod_{j \in [7] \setminus \{i\}} x_j$$

over the box $X = [-1/2, 1/2]^7$.

A polynomial example

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Symbolically, we have

- $f_i(\boldsymbol{x}) \doteq 1 - 64 \prod_{j \neq i} x_i$ are X-AGE, hence $f_X^{\mathrm{SAGE}} \geq -7$,
- and $f(\mathbf{1}/2) = f(-\mathbf{1}/2) = -7$.

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For numeric computation (with MOSEK, on a large workstation)

- the X-SAGE relaxation takes 0.01 seconds to solve
- the earliest tight SOS relaxation takes 90 seconds to solve.

Discussion

Caltech

The content of this presentation was a small fraction of

Signomial and Polynomial Optimization via Relative Entropy and Partial Dualization

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Refer to that article for ...

- Mixed X-SAGE and Lagrangian relaxations.

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