Noncommutative polynomial optimization and quantum graph parameters

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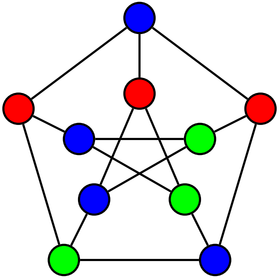
Centrum Wiskunde & Informatica

Classical coloring number

$\chi(G) =$ min number of colors needed for proper coloring of V



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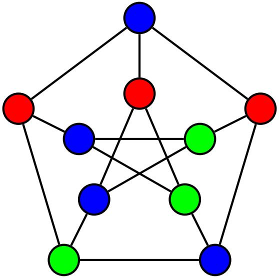

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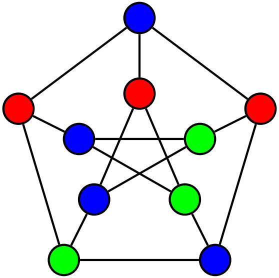

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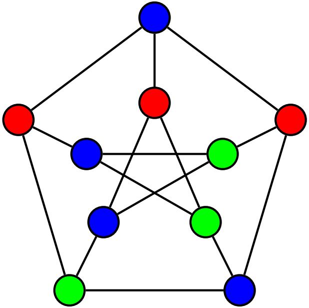

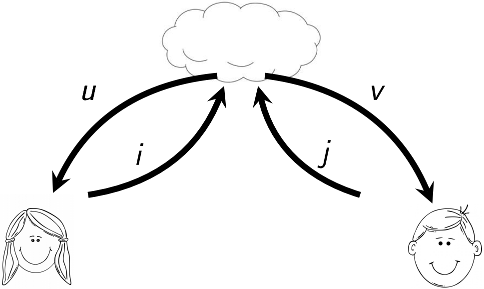

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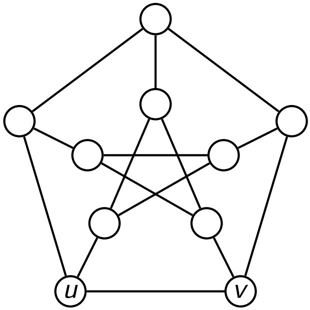

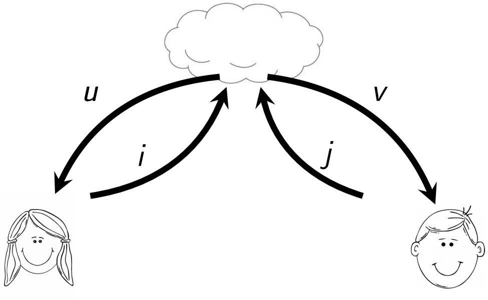

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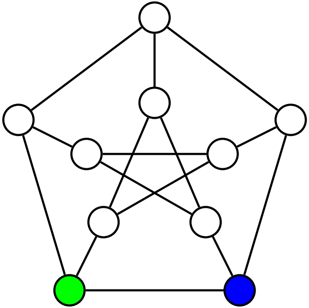

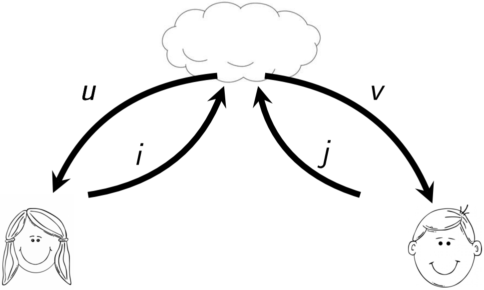

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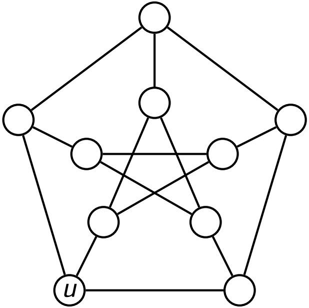

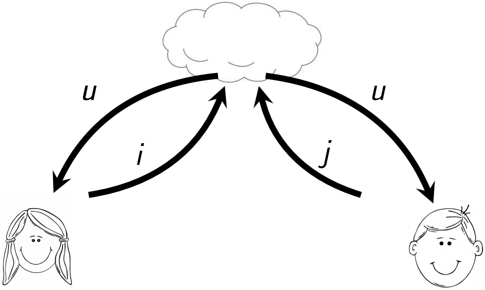

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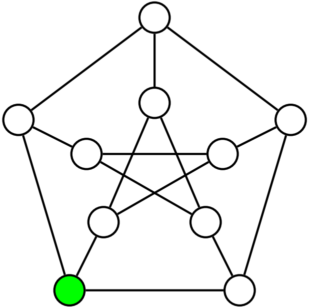

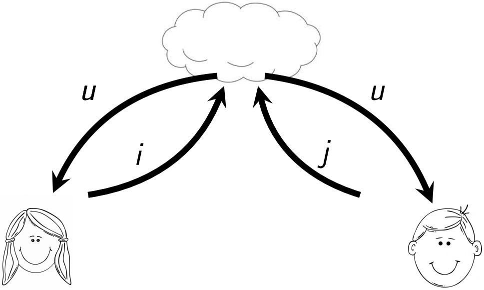

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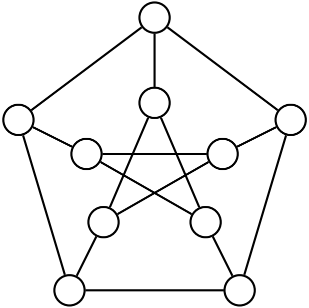

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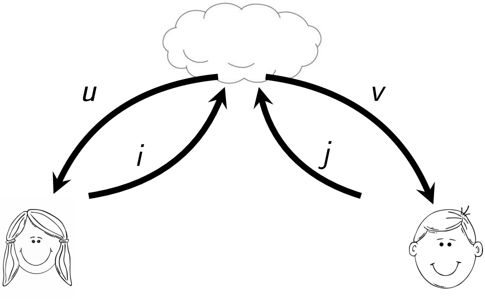

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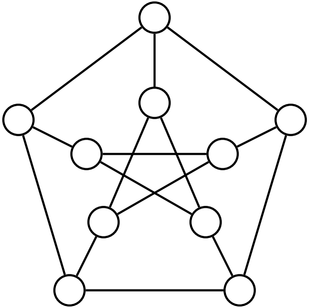

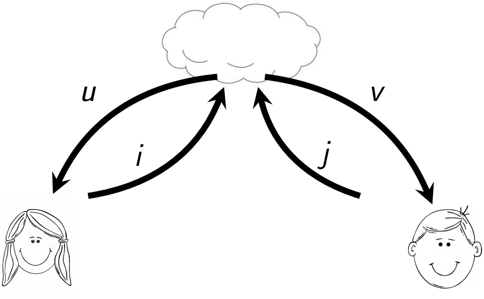

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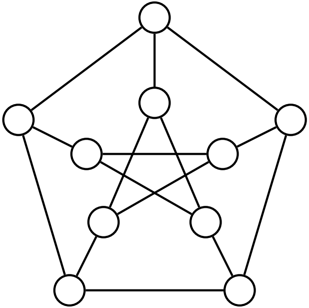

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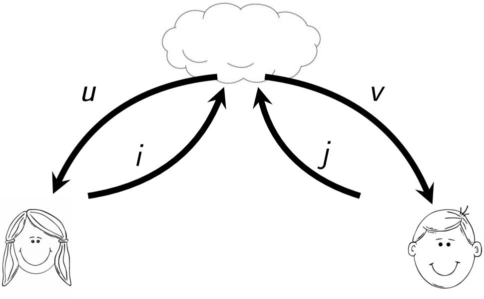

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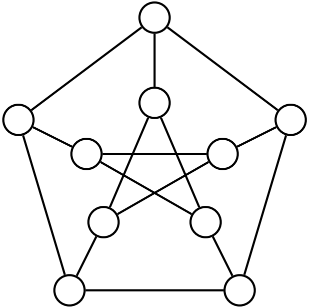

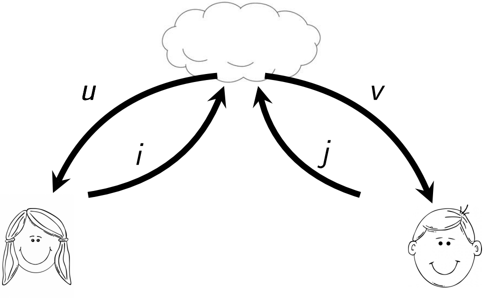

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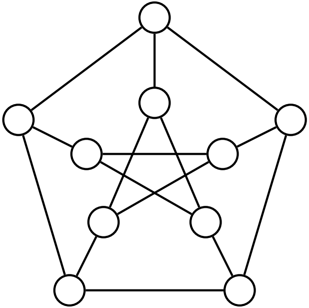

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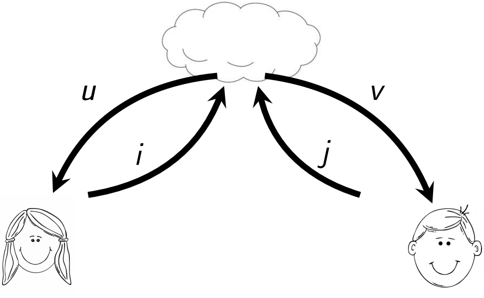

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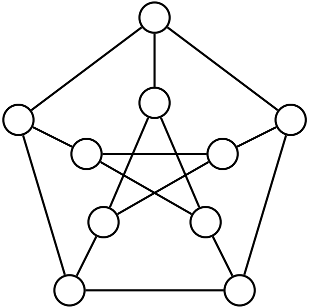

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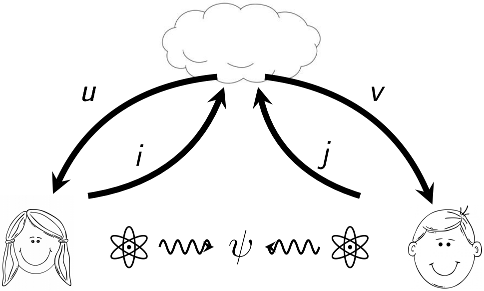

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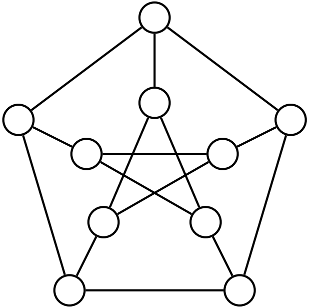

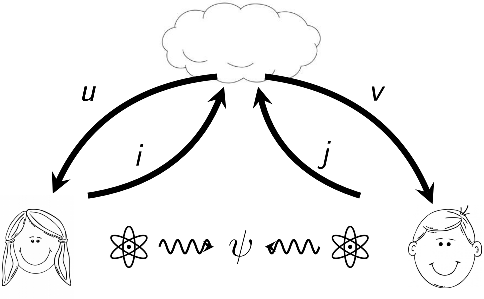

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Synchronous quantum correlations & CPSD

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Write $\xi_*(G)$ for "$\xi_\infty(G)$ plus a finite rank constraint"

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- We show $\xi_\infty(G) =$ tracial rank of G [Paulsen et al. '14]
  and $\xi_*(G) =$ projective rank of G [Mančinska-Roberson'12]
  $= \inf \frac{d}{r}$ s.t. $\exists$ rk-r $d \times d$ projectors X_u s.t. $X_u X_v = 0 \ (uv \in E)$

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- The tracial rank and $\overline{\vartheta}$ are multiplicative wrt the OR product and lexicographical product; what about ξ_t?