The set of separable states has no semidefinite representation except in dimension 3×2

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Separable states

$$\mathtt{Sep}(n,m) = \mathrm{conv}\left\{ \ (x\otimes y)(x\otimes y)^\dagger \ : x\in\mathbb{C}^n, y\in\mathbb{C}^m \ \right\}.$$

- $x^\dagger = \bar{x}^T$
- Full-dim convex cone in $\mathrm{Herm}(nm) \simeq \mathbb{C}^{n^2 m^2}$
- Plays a fundamental role in quantum information. $\mathtt{Sep}$ = set of *non-entangled* bipartite states on $\mathbb{C}^n \otimes \mathbb{C}^m$

Polynomials, duality

Linear form nonnegative on $\mathtt{Sep}(n,m)$:

$$\underbrace{\langle M, (x \otimes y)(x \otimes y)^\dagger \rangle}_{\sum_{ijkl} M_{ij,kl} x_i \bar{x}_k y_j \bar{y}_l} \;\geq\; 0 \quad \forall (x,y) \in \mathbb{C}^n \times \mathbb{C}^m.$$

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- Dual of $\mathtt{Sep}$ = cone of nonnegative Hermitian biquadratic polynomials
- *Hermitian polynomial*: $f(z,\bar{z})$ polynomial in $(z,\bar{z})$ such that $f(z,\bar{z})\in\mathbb{R}$ for all $z\in\mathbb{C}^N$

$$f(z,\bar{z}) = \sum_{\alpha,\beta} f_{\alpha\beta} z^\alpha \bar{z}^\beta, \qquad f_{\alpha\beta} = \overline{f_{\beta\alpha}}$$

Sums of squares

- Hermitian polynomial $f(z, \bar{z})$ is a sum of squares if

$$f(z, \bar{z}) = \sum_i g_i(z, \bar{z})^2$$

for some Hermitian polynomials $g_i(z, \bar{z})$

Sos relaxation of $\mathtt{Sep}$

$$p_M(x, \bar{x}, y, \bar{y}) = \sum_{ijkl} M_{ijkl} x_i \bar{x}_k y_j \bar{y}_l$$

- $\mathtt{Sep}^* = \{M \in \mathrm{Herm}(nm) : p_M$ is nonnegative$\}$.

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$$\begin{array}{ccc} \mathtt{Sep} & \subset & \mathtt{PPT} \\ \text{(duality)} \updownarrow & & \updownarrow \text{(duality)} \\ \text{Nonnegative polynomials } p_M & \supset & \text{Sum-of-squares polynomials } p_M \end{array}$$

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- Størmer–Woronowicz: $\mathtt{Sep}(n, m) = \mathtt{PPT}(n, m)$ iff $n + m \leq 5$

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- Set $C \subseteq \mathbb{R}^N$ has a *SDP representation* if

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- Does $\mathtt{Sep}(n, m)$ have an SDP representation outside the range $n + m \leq 5$?

Theorem (Fawzi)

If $\mathcal{Sep}(n, m) \neq \mathcal{PPT}(n, m)$ then $\mathcal{Sep}(n, m)$ has no SDP representation.

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- Scheiderer (2016) gave examples of convex semialgebraic sets that do not have any semidefinite representation, solving a long-standing open problem
- This talk: another look at Scheiderer's proof, and application to the set of separable states

General result in the real case

Theorem (Main, real case)

Let $p \in \mathbb{R}[x]$ *be a nonnegative polynomial that is not sos. Let*

$$A = \{\alpha \in \mathbb{N}^n : \alpha \leq \beta \text{ for some } \beta \in \mathrm{support}(p)\}$$

be the "staircase" under support(p)*. Then*

$$C_A = \mathrm{conv}\left\{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\right\}$$

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- Application: Take $p =$ Motzkin (inhomogeneous) polynomial. Associated A is $\subseteq \left\{\alpha \in \mathbb{N}^2 : |\alpha| \leq 6\right\}$. Shows that $P^*_{2,6}$ has no SDP representation (where $P_{2,6}$ is set of nonneg. polynomials in 2 vars. of degree ≤ 6)

Proof

$$C_A = \operatorname{conv}\{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\}$$

- Linear functions nonnegative on $C_A \leftrightarrow$ nonnegative polynomials supported on A

Characterization of SDP lifts [Gouveia, Parrilo, Thomas]:

Theorem

C_A has an SDP representation iff there are functions $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, k)$ such that any nonnegative polynomial supported on A can be written as a sum of squares of functions from $\operatorname{span}(f_1, \ldots, f_k)$.

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- **Semialgebraic functions are tame:** They are smooth (C^∞) almost everywhere (except on a set of measure 0)

Proof of main theorem

p nonnegative polynomial not sos, $A =$ staircase under support(p)

$$C_A = \text{conv}\left\{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\right\}.$$

- Assume C_A has an SDP representation, and let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be the semialgebraic functions associated to this representation
- Since A is a staircase, I can translate the f_i as I want, i.e., the $\tilde{f}_i(x) = f_i(x - a)$ (for any $a \in \mathbb{R}^n$) are also "valid"
- Since the $(f_i)_{i=1,\ldots,k}$ are smooth almost everywhere, there is a point $a \in \mathbb{R}^n$ such that the f_i are all smooth at a. By shifting, can assume wlog that $a = 0$

Smooth sums of squares

Proposition

Assume p is a homogeneous polynomial such that $p = \sum_j f_j^2$ for some arbitrary functions f_j that are C^∞ at the origin. Then p is a sum of squares of polynomials.

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- Proves theorem when p is homogeneous
- Additional technical argument based on Puiseux expansions is needed for general p

Main result, complex case

Theorem (Main, complex case)

Let p be a nonnegative Hermitian polynomial that is not sos. Let

$$A = \{(\alpha, \alpha') \in \mathbb{N}^n \times \mathbb{N}^n : (\alpha, \alpha') \leq (\beta, \beta'),\ \textit{for some } (\beta, \beta') \in \mathrm{support}(p)\}$$

be the "staircase" under support(p). *Then*

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Thank you!

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