

The set of separable states has no semidefinite
representation except in dimension 3×2

Hamza Fawzi

University of Cambridge

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Separable states

$$\text{Sep}(n, m) = \text{conv} \{ (x \otimes y)(x \otimes y)^\dagger : x \in \mathbb{C}^n, y \in \mathbb{C}^m \}.$$

- $x^\dagger = \bar{x}^T$
- Full-dim convex cone in $\text{Herm}(nm) \simeq \mathbb{C}^{n^2 m^2}$
- Plays a fundamental role in quantum information. Sep = set of *non-entangled* bipartite states on $\mathbb{C}^n \otimes \mathbb{C}^m$

Polynomials, duality

Linear form nonnegative on $\text{Sep}(n, m)$:

$$\underbrace{\langle M, (x \otimes y)(x \otimes y)^\dagger \rangle}_{\sum_{ijkl} M_{ij,kl} \bar{x}_i y_j \bar{y}_l} \geq 0 \quad \forall (x, y) \in \mathbb{C}^n \times \mathbb{C}^m.$$

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- Dual of Sep = cone of nonnegative Hermitian biquadratic polynomials
- *Hermitian polynomial*: $f(z, \bar{z})$ polynomial in (z, \bar{z}) such that $f(z, \bar{z}) \in \mathbb{R}$ for all $z \in \mathbb{C}^N$

$$f(z, \bar{z}) = \sum_{\alpha, \beta} f_{\alpha\beta} z^\alpha \bar{z}^\beta, \quad f_{\alpha\beta} = \overline{f_{\beta\alpha}}$$

Sums of squares

- Hermitian polynomial $f(z, \bar{z})$ is a **sum of squares** if

$$f(z, \bar{z}) = \sum_i g_i(z, \bar{z})^2$$

for some Hermitian polynomials $g_i(z, \bar{z})$

Sos relaxation of Sep

$$p_M(x, \bar{x}, y, \bar{y}) = \sum_{ijkl} M_{ijkl} x_i \bar{x}_k y_j \bar{y}_l$$

- $\text{Sep}^* = \{M \in \text{Herm}(nm) : p_M \text{ is nonnegative}\}$.

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$$\text{PPT} = \{\rho \in \text{Herm}(nm) : \rho \geq 0 \text{ and } (I \otimes T)(\rho) \geq 0\}$$

with $T =$ transpose map

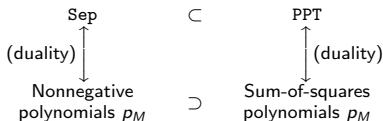
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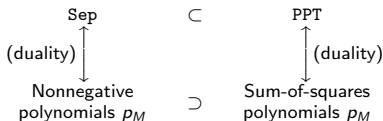
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- Størmer–Woronowicz: $\text{Sep}(n, m) = \text{PPT}(n, m)$ iff $n + m \leq 5$

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- Set $C \subseteq \mathbb{R}^N$ has a *SDP representation* if

$$C = \{x \in \mathbb{R}^N : \exists y \in \mathbb{R}^M \text{ s.t. } A(x, y) \geq 0\}$$

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- Does $\text{Sep}(n, m)$ have an SDP representation outside the range $n + m \leq 5$?

Theorem (Fawzi)

If $\text{Sep}(n, m) \neq \text{PPT}(n, m)$ then $\text{Sep}(n, m)$ has no SDP representation.

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- Scheiderer (2016) gave examples of convex semialgebraic sets that do not have any semidefinite representation, solving a long-standing open problem
- This talk: another look at Scheiderer's proof, and application to the set of separable states

General result in the real case

Theorem (Main, real case)

Let $p \in \mathbb{R}[x]$ be a nonnegative polynomial that is not sos. Let

$$A = \{\alpha \in \mathbb{N}^n : \alpha \leq \beta \text{ for some } \beta \in \text{support}(p)\}$$

be the “staircase” under $\text{support}(p)$. Then

$$C_A = \text{conv} \{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\}$$

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- Application: Take $p =$ Motzkin (inhomogeneous) polynomial. Associated A is $\subseteq \{\alpha \in \mathbb{N}^2 : |\alpha| \leq 6\}$. Shows that $P_{2,6}^*$ has no SDP representation (where $P_{2,6}$ is set of nonneg. polynomials in 2 vars. of degree ≤ 6)

$$C_A = \text{conv} \{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\}$$

- Linear functions nonnegative on $C_A \leftrightarrow$ nonnegative polynomials supported on A

Characterization of SDP lifts [Gouveia, Parrilo, Thomas]:

Theorem

C_A has an SDP representation iff there are functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ($i = 1, \dots, k$) such that any nonnegative polynomial supported on A can be written as a sum of squares of functions from $\text{span}(f_1, \dots, f_k)$.

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- $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *semialgebraic* if its graph $\{(x, f(x)) : x \in \mathbb{R}^n\}$ is a semialgebraic subset of \mathbb{R}^{n+1}
- **Semialgebraic functions are tame:** They are smooth (C^∞) almost everywhere (except on a set of measure 0)

Proof of main theorem

p nonnegative polynomial not sos, $A = \text{support}(p)$

$$C_A = \text{conv} \{(x^\alpha)_{\alpha \in A} : x \in \mathbb{R}^n\}.$$

- Assume C_A has an SDP representation, and let $f_1, \dots, f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ be the semialgebraic functions associated to this representation
- Since A is a staircase, I can translate the f_i as I want, i.e., the $\tilde{f}_i(x) = f_i(x - a)$ (for any $a \in \mathbb{R}^n$) are also “valid”
- Since the $(f_i)_{i=1, \dots, k}$ are smooth almost everywhere, there is a point $a \in \mathbb{R}^n$ such that the f_i are all smooth at a . By shifting, can assume wlog that $a = 0$

Smooth sums of squares

Proposition

Assume p is a homogeneous polynomial such that $p = \sum_j f_j^2$ for some arbitrary functions f_j that are C^∞ at the origin. Then p is a sum of squares of *polynomials*.

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- Proves theorem when p is homogeneous
- Additional technical argument based on Puiseux expansions is needed for general p

Main result, complex case

Theorem (Main, complex case)

Let p be a nonnegative Hermitian polynomial that is not sos. Let

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Thank you!

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