The SDP-exact region in quadratic optimization

Diego Cifuentes

Department of Mathematics
Massachusetts Institute of Technology

Joint work with **Corey Harris** (University of Oslo),
Bernd Sturmfels (MPI-MiS Leipzig & UC Berkeley)

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Quadratic programming

Consider a quadratically constrained quadratic program:

$$\min_{x\in\mathbb{R}^n} \quad g(x) := x^T C x + 2c^T x$$

$$f_i(x) := x^T A x + 2a_i^T x + \alpha_i = 0, \quad i = 1, \ldots, m \qquad (QP)$$

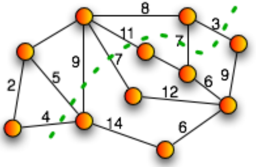
QP's are nonconvex and NP-hard. We consider convex relaxations based on *semidefinite programming* (SDP).

Example (Max-Cut)

Given a graph $G = (V, E)$ with weights c_{ij},

$$\min_{x} \quad \sum_{ij\in E} c_{ij} x_i x_j$$

$$\text{s.t.} \quad x_i^2 = 1 \text{ for } i \in V$$



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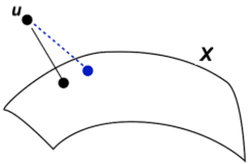
QP's are nonconvex and NP-hard. We consider convex relaxations based on *semidefinite programming* (SDP).

Example (Nearest point to quadratic variety)

Given a variety $X \subset \mathbb{R}^n$ defined by quadrics, and a point $u \in \mathbb{R}^n$,

$$\min_x \quad \|x - u\|^2$$
$$\text{s.t.} \quad x \in X$$

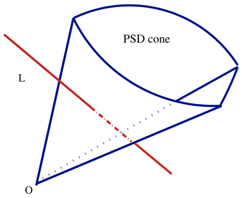


Semidefinite Programming

A semidefinite program (SDP) is

$$\begin{aligned} \min_{X \in \mathbb{S}^d} \quad & \mathcal{C} \bullet X \\ \text{s.t.} \quad & \mathcal{A}_i \bullet X = b_i \text{ for } i \in [m] \\ & X \succeq 0 \end{aligned}$$

where $\mathcal{C}, \mathcal{A}_i \in \mathbb{S}^d$, $b_i \in \mathbb{R}$ are given.



SDP relaxation

Consider the quadratic program

$$\min_{x\in\mathbb{R}^n} \quad \begin{array}{l} g(x) := x^T C x + 2c^T x \\ f_i(x) := x^T A x + 2a_i^T x + \alpha_i = 0, \text{ for } i \in [m] \end{array} \qquad (QP)$$

Let the following matrices in $\mathbb{S}^{n+1}$

$$\mathcal{C} := \begin{bmatrix} 0 & c^T \\ c & C \end{bmatrix}, \quad \mathcal{A}_0 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_i := \begin{bmatrix} \alpha_i & a_i^T \\ a_i & A_i \end{bmatrix}$$

The QP can be equivalently written as

$$\min_{x\in\mathbb{R}^n, X\in\mathbb{S}^{n+1}} \quad \begin{array}{l} C \bullet X \\ \mathcal{A}_0 \bullet X = 1 \\ \mathcal{A}_i \bullet X = 0 \text{ for } i \in [m] \\ X = \begin{bmatrix} 1 & x^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \end{array}$$

SDP relaxation

Consider the quadratic program

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The SDP relaxation of QP is

$$\min_{X\in\mathbb{S}^n} \quad \begin{array}{l} C \bullet X \\ \mathcal{A}_0 \bullet X = 1 \\ \mathcal{A}_i \bullet X = 0 \text{ for } i \in [m] \\ X \succeq 0 \end{array} \tag{SDP}$$

The relaxation is exact if the optimal solution is rank-one.

The SDP-exact region

$$\min_{x\in\mathbb{R}^n} \quad g(x) \quad \text{s.t.} \quad f_i(x) = 0, \text{ for } i \in [m] \tag{QP}$$

Let $\mathbf{f} = (f_1, \dots, f_m)$ be fixed quadrics. The SDP-exact region is

$$\mathcal{R}_{\mathbf{f}} \;=\; \{\; g \in \mathbb{R}[x]_{\leq 2} \;:\; \text{the QP is solved exactly by its SDP relaxation} \;\}.$$

The SDP-exact region is a semialgebraic set in $\mathbb{R}[x]_{\leq 2} \cong \mathbb{S}^{n+1}$.

Problem

Given $\mathbf{f}$, compute/characterize the SDP-exact region. If the computation is intractable, we would like to understand its degree.

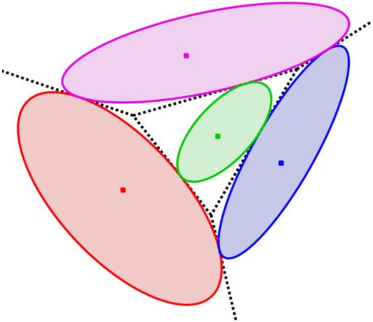
Nearest point to four points

For $u \in \mathbb{R}^n$, consider

$$\min_x \quad \|x - u\|^2 \quad s.t. \quad x \in X$$

$$X := \{ \text{ zero set of two polynomials in two variables } \}$$

The SDP-exact region is the union of four conics, each contained in a *Voronoi cell*.



Max-Cut

Given a graph $G = (V, E)$ with $V = [n]$ and weights c_{ij}

$$\begin{aligned} \min_{x} \quad & \sum_{ij \in E} c_{ij} x_i x_j \\ \text{s.t.} \quad & x_i^2 = 1 \text{ for } i \in [n] \end{aligned}$$

Let $S := \{C : \mathcal{L}(C) \succeq 0\}$ be the spectrahedron of the *Laplacian* matrix

$$\mathcal{L}(C) = \begin{pmatrix} -\sum_{j \neq 1} c_{1j} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & -\sum_{j \neq 2} c_{2j} & c_{23} & \cdots & c_{2n} \\ c_{13} & c_{23} & -\sum_{j \neq 3} c_{2j} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \cdots & -\sum_{j \neq n} c_{jn} \end{pmatrix}.$$

The SDP-exact region consists of 2^{n-1} copies of S.

Rank-one region in SDP

Consider the SDP

$$\min_{X \in \mathbb{S}^d} \quad \mathcal{C} \bullet X \quad \text{s.t.} \quad \mathcal{A}_i \bullet X = b_i \text{ for } i \in [m], \quad X \succeq 0 \tag{P}$$

Let $\mathcal{A}, b$ be fixed. We define the rank-one region as

$$\mathcal{R}_{\mathcal{A},b} \;=\; \{\; C \in \mathbb{S}^d \;:\; \text{ there is an optimal solution of rank one } \;\}.$$

This is a generalization of the SDP-exact region to arbitrary SDP's.

Rank-one region: primal-dual characterization

Consider the primal-dual pair

$$\min_{X \in \mathbb{S}^d} \quad \mathcal{C} \bullet X \quad \text{s.t.} \quad \mathcal{A}_i \bullet X = b_i \text{ for } i \in [m], \quad X \succeq 0 \qquad \text{(P)}$$

$$\max_{\lambda \in \mathbb{R}^\ell, Y \in \mathbb{S}^d} \quad b^T \lambda \quad \text{s.t.} \quad Y = \mathcal{C} - \textstyle\sum_i \lambda_i \mathcal{A}_i \quad Y \succeq 0 \qquad \text{(D)}$$

The rank-one region is defined by the *critical equations*

$$\mathcal{R}_{\mathcal{A},b} = \left\{ C \in \mathbb{S}^d : \begin{array}{l} \mathcal{A}_i \bullet X = b_i, \quad Y = \mathcal{C} - \sum_i \lambda_i \mathcal{A}_i, \quad X \cdot Y = 0, \\ X \succeq 0, \quad Y \succeq 0, \quad \operatorname{rank} X = 1, \quad \operatorname{rank} Y = d-1 \end{array} \right\}$$

Rank-one region: primal-dual characterization

Consider the primal-dual pair

$$\min_{X\in\mathbb{S}^d} \quad \mathcal{C}\bullet X \quad \text{s.t.} \quad \mathcal{A}_i\bullet X = b_i \text{ for } i\in[m], \quad X\succeq 0 \tag{P}$$

$$\max_{\lambda\in\mathbb{R}^\ell, Y\in\mathbb{S}^d} \quad b^T\lambda \quad \text{s.t.} \quad Y=\mathcal{C}-\textstyle\sum_i \lambda_i\mathcal{A}_i \quad Y\succeq 0 \tag{D}$$

The rank-one region is defined by the *critical equations*

$$\mathcal{R}_{\mathcal{A},b} = \left\{ C\in\mathbb{S}^d : \begin{array}{c} \mathcal{A}_i\bullet X = b_i, \quad Y=\mathcal{C}-\sum_i\lambda_i\mathcal{A}_i, \quad X\cdot Y=0, \\ X\succeq 0, \quad Y\succeq 0, \quad \operatorname{rank}X=1, \quad \operatorname{rank}Y=d-1 \end{array} \right\}$$

Computation: For generic $\mathcal{A}, b$, the boundary hypersurface $\partial_{\text{alg}}\mathcal{R}_{\mathcal{A},b}$ can be computed by eliminating X, Y, λ in the equations.

$$\mathcal{A}_i\bullet X = b_i, \quad Y=\mathcal{C}-\textstyle\sum_i\lambda_i\mathcal{A}_i, \quad X\cdot Y=0, \quad \text{minors}_2(X)=0$$

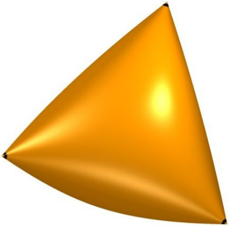
Rank-one region in MaxCut SDP

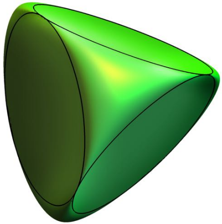
$$\min_{X \in \mathbb{S}^d} \quad \mathcal{C} \bullet X \quad \text{s.t.} \quad X_{ii} = 1 \text{ for } i \in [n], \quad X \succeq 0 \tag{P}$$

$$\max_{\lambda \in \mathbb{R}^\ell} \quad \sum_i \lambda_i \quad \text{s.t.} \quad Y = \mathcal{C} - \mathrm{Diag}(\lambda) \quad Y \succeq 0 \tag{D}$$

The rank-one region has 2^{n-1} pieces, one for each $X = xx^T$, $x \in \{\pm 1\}^n$.

$$\mathcal{R}_{\mathcal{A},b} \;=\; \bigcup_{X \text{ vertex}} \left\{ C \in \mathbb{S}^d : \;\; X \cdot Y = 0, \quad Y = \mathcal{C} - \mathrm{Diag}(\lambda), \quad Y \succeq 0, \right\}$$





Generic degree of the rank-one region

Theorem

Consider an SDP with generic $\mathcal{A}$ and b. The boundary degree $\partial_{\mathrm{alg}}\mathcal{R}_{\mathcal{A},b}$ is

$$2^{\ell-1}(d-1)\binom{d}{\ell} - 2^{\ell}\binom{d}{\ell+1} \qquad \text{for } 3 \le \ell \le d$$

If $\ell = 2$ then $\mathcal{R}_{\mathcal{A},b}$ is dense and the degree is $\binom{d+1}{3}$.

The proof follows the strategy by [Nie-Ranestad-Sturmfels] to compute the algebraic degree of SDP.

Algebraic degree of SDP

$\ell \backslash d$	3	4	5	6	7
2	6	12	20	30	42
3	4	16	40	80	140
4		8	40	120	280
5			16	96	336
6				32	224

Rank-one boundary degrees

$\ell \backslash d$	3	4	5	6	7
2	4	10	20	35	66
3	8	40	120	280	560
4		24	144	504	1344
5			64	448	1792
6				160	1280

Generic degree of the SDP-exact region

$$\min_{x\in\mathbb{R}^n} \quad g(x) := x^T Cx + 2c^T x \qquad (QP)$$

$$f_i(x) := x^T Ax + 2a_i^T x + \alpha_i = 0, \quad i = 1, \dots, m$$

Consider the Hessian of the Lagrangian

$$H(\lambda) = \nabla^2 L(x, \lambda) = C - \textstyle\sum_i \lambda_i A_i \in \mathbb{S}^n$$

The SDP-exact region is

$$\mathcal{R}_{\mathbf{f}} \;=\; \bigcup_{x\in V_{\mathbf{f}}} \Big\{ (c, C) \;:\; \; c - \textstyle\sum_i \lambda_i a_i + H(\lambda)x = 0, \quad H(\lambda) \succeq 0 \;\Big\}$$

Generic degree of the SDP-exact region

$$\min_{x \in \mathbb{R}^n} \quad g(x) := x^T C x + 2c^T x \qquad (QP)$$

$$f_i(x) := x^T A x + 2a_i^T x + \alpha_i = 0, \quad i = 1, \ldots, m$$

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The SDP-exact region is

$$\mathcal{R}_{\mathbf{f}} \;=\; \bigcup_{x \in V_{\mathbf{f}}} \left\{ (c, C) : \;\; c - \textstyle\sum_i \lambda_i a_i + H(\lambda) x = 0, \quad H(\lambda) \succeq 0 \;\right\}$$

Theorem

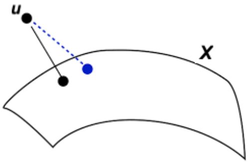
Consider a QP with generic equations $\mathbf{f}$*. The boundary degree* $\partial_{\mathrm{alg}} \mathcal{R}_{\mathbf{f}}$ *is*

$$2^m \left(n \binom{n}{m} - \binom{n}{m+1} \right).$$

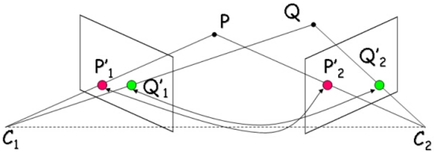
Nearest point problems

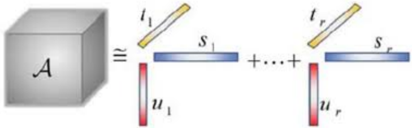
Let $X \subset \mathbb{R}^n$ quadratic and $u \in \mathbb{R}^n$.

$$\begin{array}{rl} \min_x & \|x-u\|^2 \\ \text{s.t.} & x \in X \end{array}$$



Many different applications





Nearest point problems

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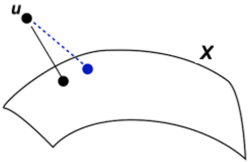


The SDP-exact region for the Euclidean Distance (ED) problem is

$$\mathcal{R}_{\mathbf{f}}^{ed} \;=\; \{\; u \in \mathbb{R}^n \;:\; \text{problem is solved exactly by its SDP relaxation} \,\}.$$

This is an affine slice of $\mathcal{R}_{\mathbf{f}}$.

Nearest point problems

Theorem

The SDP-exact region satisfies

$$\mathcal{R}_{\mathbf{f}}^{ed} \;=\; \bigcup_{x\in V_{\mathbf{f}}} \left(x - \tfrac{1}{2}\mathrm{Jac}_{\mathbf{f}}(x)\cdot \mathrm{S}_{\mathbf{f}}^{ed}\right)$$

where

$$\mathrm{S}_{\mathbf{f}}^{ed} = \left\{ \lambda \in \mathbb{R}^m \,:\, \sum_i \lambda_i A_i \prec I_n \right\}, \qquad \mathrm{Jac}_{\mathbf{f}}(x) : \mathbb{R}^m \to \mathrm{NormalSpace}(x)$$

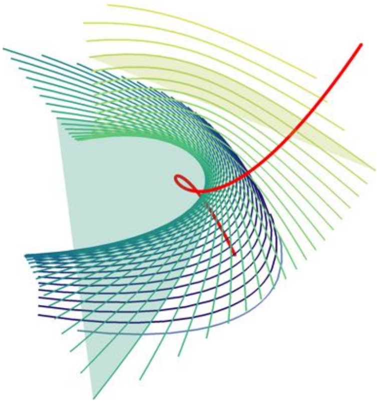
This is a bundle of *spectrahedral shadows*.

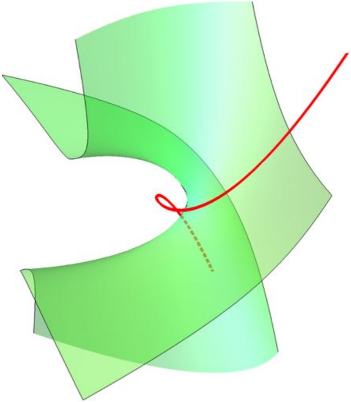
Corollary (C.-Agarwal-Parrilo-Thomas)

If x be a smooth point, then $\mathcal{R}_{\mathbf{f}}^{ed}$ contains an open ball around x.

Applications: Triangulation (vision), Rank-1 tensor approximation.

Nearest point to the twisted cubic

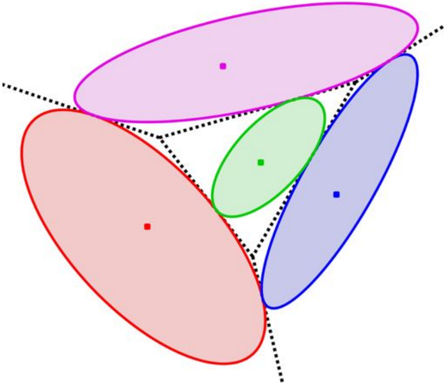


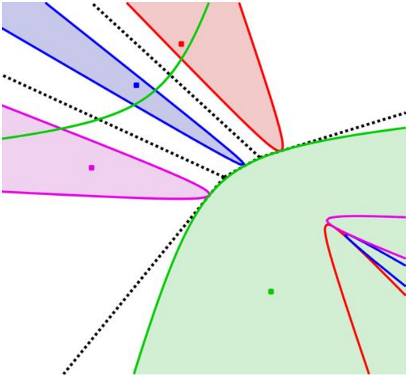


$$X := \{(x_1, x_2, x_3) : x_2 = x_1^2,\ x_3 = x_1 x_2\}$$

Nearest point to 2^n points

Theorem: Let $m = n$ and $\mathbf{f}$ generic, so $X = V_{\mathbf{f}}$ is finite. The SDP-exact region consists of 2^n spectrahedra. The boundaries are pairwise tangent.





$X =$ (four points in the plane)

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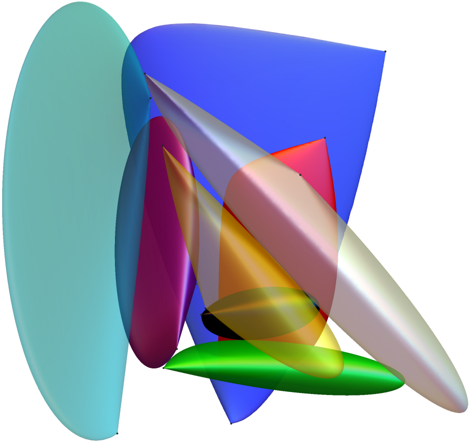
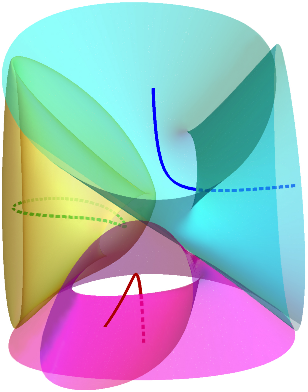


$X =$ (eight points in 3D space)

Nearest point to a complete intersection

Theorem: Let $m \leq n$, $\mathbf{f}$ generic. The degree of $\partial_{\mathrm{alg}} \mathcal{R}_{\mathbf{f}}^{ed}$ equals $2^m n \binom{n-2}{m-2}$.



$X =$ (curve defined by two quadrics)

Beyond complete intersections

We developed symbolic algorithms to compute $\partial_{\mathrm{alg}}\mathcal{R}_{\mathbf{f}}^{ed}$ for arbitrary $\mathbf{f}$, even if $m > n$. We have a conjecture for the degree.

Example (Twisted cubic)

We now use three defining equations:

$$X := \{(x_1, x_2, x_3) : x_2 = x_1^2,\ x_3 = x_1 x_2,\ x_1 x_3 = x_2^2\}$$

The region $\mathcal{R}_{\mathbf{f}}^{ed}$ is dense, and its bounded by

$$\begin{gathered}
5832u_2^3u_3^6 + 27648u_2^6u_3^2 - 62208u_1u_2^4u_3^3 - 2916u_1^2u_2^2u_3^4 + 15552u_2^4u_3^4 - 5832u_1^3u_3^5 + 8748u_1^2u_3^6 - 5832u_2^2u_3^6 - 4374u_1u_3^7 \\
+729u_3^8 - 41472u_1^2u_2^5 + 86400u_1^3u_2^3u_3 + 27648u_1u_2^5u_3 + 60750u_1^4u_2u_3^2 - 41472u_1^2u_2^3u_3^2 - 62208u_2^5u_3^2 - 106920u_1^3u_2u_3^3 \\
+85536u_1u_2^3u_3^3 + 71442u_1^2u_2u_3^4 - 19656u_2^3u_3^4 - 19440u_1u_2u_3^5 + 3888u_2u_3^6 - 84375u_1^6 - 54000u_1^4u_2^2 + 72576u_1^2u_2^4 \\
+202500u_1^5u_3 - 19440u_1^3u_2^2u_3 - 48384u_1u_2^4u_3 - 220725u_1^4u_3^2 + 6912u_1^2u_2^2u_3^2 + 58032u_2^4u_3^2 + 140454u_1^3u_3^3 - 35424u_1u_2^2u_3^3 \\
-54027u_1^2u_3^4 + 8424u_2^2u_3^4 + 11178u_1u_3^5 - 1161u_3^6 + 40050u_1^4u_2 - 50760u_1^2u_2^3 - 21132u_1^3u_2u_3 + 33840u_1u_2^3u_3 \\
+11880u_1^2u_2u_3^2 - 28744u_2^3u_3^2 + 3708u_1u_2u_3^3 - 1314u_2u_3^4 - 7431u_1^4 + 17736u_1^2u_2^2 + 6112u_1^3u_3 - 11824u_1u_2^2u_3 - 3246u_1^2u_3^2 \\
+7976u_2^2u_3^2 + 312u_1u_3^3 + 37u_3^4 - 3096u_1^2u_2 + 2064u_1u_2u_3 - 1176u_2u_3^2 + 216u_1^2 - 144u_1u_3 + 72u_3^2.
\end{gathered}$$

Summary

- We characterized the rank-one region (for SDPs) and the SDP-exact region (for QPs).
- We computed the degree for generic instances.
- The SDP-exact region for distance problems has even more structure (a bundle of spectrahedral shadows).

Summary

- We characterized the rank-one region (for SDPs) and the SDP-exact region (for QPs).
- We computed the degree for generic instances.
- The SDP-exact region for distance problems has even more structure (a bundle of spectrahedral shadows).

If you want to know more:

- D. Cifuentes, C. Harris, B. Sturmfels, *The geometry of SDP-exactness in quadratic optimization*, Math. Prog. (2019). `arXiv:1804.01796`.
- D. Cifuentes, S. Agarwal, P. Parrilo, R. Thomas, *On the local stability of semidefinite relaxations*, `arXiv:1710.04287`.

Thanks for your attention!

More SDP-exact regions