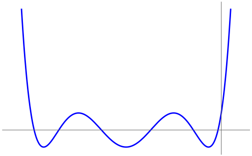
Completely log-concave polynomials and matroids



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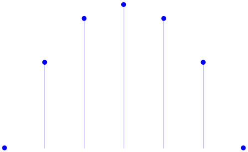

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joint work with

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Complete log-concavity

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$f(x) \geq 0$ for all $x \in \mathbb{R}^n_{\geq 0}$ and $\log(f)$ is concave on $\mathbb{R}^n_{>0}$.

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For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let $D_v = \sum_{i=1}^n v_i \frac{\partial f}{\partial z_i}$.

$f \in \mathbb{R}[z_1, \ldots, z_n]$ is completely log-concave (CLC) on $\mathbb{R}^n_{>0}$ if for all $k \in \mathbb{N}$, $v_1, \ldots, v_k \in \mathbb{R}^n_{\geq 0}$,

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Equivalent Def: CLC = strongly log-concave = Lorentzian
(Gurvits) (Brändén-Huh)

Example: stable polynomials

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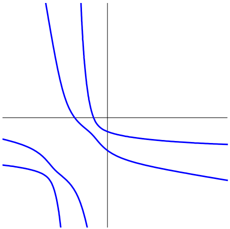

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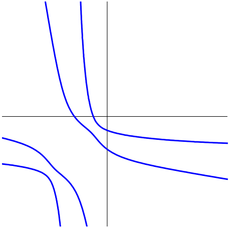

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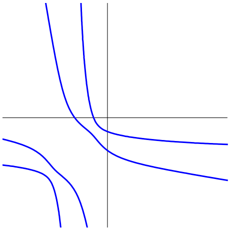

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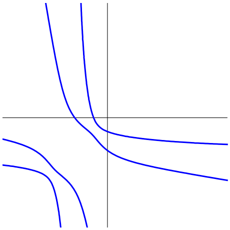

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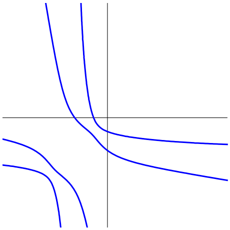

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Example: $e_k(z_1, \ldots, z_n) = c \cdot (D_{(1,\ldots,1)})^{n-k} \prod_{i=1}^n z_i$

Stable polynomials & supports

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$$\det\left(\sum_{i=1}^n z_i v_i v_i^T\right) = \sum_{I \in \binom{[n]}{d}} \det(v_i : i \in I)^2 \prod_{i \in I} z_i$$

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Brändén: Fano matroid $\neq$ support of a stable polynomial f

Matroids

A matroid on ground set $[n] = \{1, \ldots, n\}$ is a nonempty collection $\mathcal{B}$ of subsets of $[n]$ ("bases") for which

$$\mathcal{P}_{\mathcal{B}} \;=\; \mathrm{conv}\{\mathbf{1}_B \;:\; B \in \mathcal{B}\} \;\subset\; [0,1]^n$$

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Examples:

- linear independence of vectors $v_1, \ldots, v_n \in \mathbb{R}^d$
- cyclic independence of n edges in a graph

Characterization of CLC

Theorem (Anari, Liu, Oveis Gharan, V.): For $f \in \mathbb{R}_{\geq 0}[z_1, \dots, z_n]_d$, f is completely log-concave

$$\Leftrightarrow \begin{cases} \mathrm{supp}(f) = P \cap \mathbb{Z}^n \text{ where } P \text{ is M-convex, and} \\ \text{for all } |\alpha| = d-2, \text{ quadratic } \partial^\alpha f = z^T Q_\alpha z \text{ with } \lambda_2(Q_\alpha) \leq 0 \end{cases}$$

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Cor. (Gurvits/ALOV) For $f = \sum_{k=0}^{d} c_k y^{d-k} z^k$,

$$f \text{ is CLC } \Leftrightarrow \begin{cases} \{k : c_k \neq 0\} \text{ has no gaps, and} \\ \left(\frac{c_k}{\binom{d}{k}}\right)^2 \geq \frac{c_{k-1}}{\binom{d}{k-1}} \cdot \frac{c_{k+1}}{\binom{d}{k+1}} \text{ for all } k \end{cases}$$

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Theorem (Anari, Liu, Oveis Gharan, V.): For any matroid with bases $\mathcal{B}$ and independent sets $\mathcal{I}$,

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Cor: $$\left(\frac{\mathcal{I}_k}{\binom{n}{k}} \right)^2 \geq \frac{\mathcal{I}_{k-1}}{\binom{n}{k-1}} \cdot \frac{\mathcal{I}_{k+1}}{\binom{n}{k+1}}$$ (Mason's conjecture)

Other results

Let $\mathcal{B}$ = bases of a matroid with rank r.

Anari, Oveis Gharan, V: The solution to the concave program

$$\tau \quad = \quad \max_{p \in \mathcal{P}_{\mathcal{B}}} \quad \sum_{i=1}^{n} p_i \log \tfrac{1}{p_i} + (1 - p_i) \log \tfrac{1}{1-p_i}$$

can be computed in polynomial time and $\beta = e^{\tau}$ satisfies

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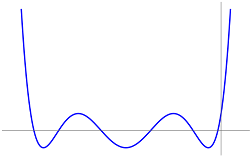
Anari, Liu, Oveis Gharan, V: There is a Markov chain on $\mathcal{B}$ with uniform stationary distribution that mixes quickly:

$$\min\{t \in \mathbb{N} : ||P^t(B, \cdot) - \pi||_1 \le \epsilon\} \le r^2 \log(n/\epsilon)$$

where P = transition matrix.

Sum up: completely log-concave polynomials

- log-concavity of polynomial as functions
  $\Rightarrow$ log-concavity of coefficients
- many matroid polynomials are completely log-concave
- much of the theory of stable polynomials extends to CLC



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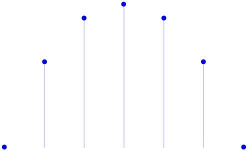

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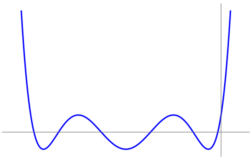

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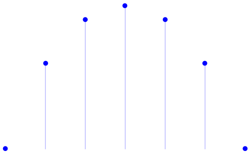

Thanks!

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