Phaseless rank

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Joint work with João Gouveia (U. Coimbra)

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Phaseless rank

Given a matrix $A \in \mathbb{R}_+^{n\times m}$ we are interested in the quantity

Phaseless rank

$$\begin{aligned} \text{rank}_\theta(A) = \min_X \quad & \text{rank}\,(X) \\ \text{s.t.} \quad & |X_{ij}| = A_{ij}, \quad \forall i,j; \\ & X \in \mathbb{C}^{n\times m}. \end{aligned}$$

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$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & e^{i\theta_1} & e^{i\theta_2} & e^{i\theta_3} \\ e^{i\theta_4} & 0 & e^{i\theta_5} & e^{i\theta_6} \\ e^{i\theta_7} & e^{i\theta_8} & 0 & e^{i\theta_9} \\ e^{i\theta_{10}} & e^{i\theta_{11}} & e^{i\theta_{12}} & 0 \end{bmatrix}$$

$$\min \operatorname{rank}(X)?$$

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$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & e^{i(\theta+\pi)} & e^{i(\theta+\frac{2\pi}{3})} \\
1 & e^{i\theta} & 0 & e^{i(\theta+\frac{\pi}{3})} \\
1 & e^{i(\theta-\frac{\pi}{3})} & e^{i(\theta-\frac{2\pi}{3})} & 0
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$$\text{rank}\,(X) = 2$$

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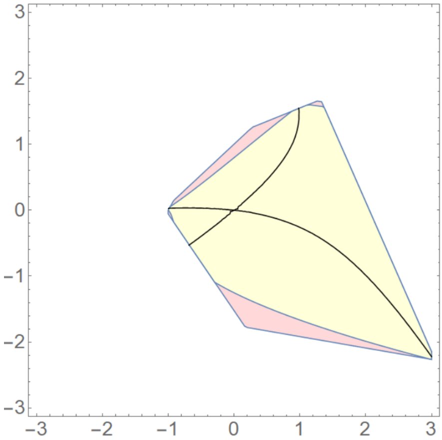
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & e^{i(\theta+\pi)} & e^{i(\theta+\frac{2\pi}{3})} \\ 1 & e^{i\theta} & 0 & e^{i(\theta+\frac{\pi}{3})} \\ 1 & e^{i(\theta-\frac{\pi}{3})} & e^{i(\theta-\frac{2\pi}{3})} & 0 \end{bmatrix}$$

$\text{rank}(A) = 4$ $\qquad$ $\text{rank}(X) = 2$

It is a problem of rank minimization under phase uncertainty.



Space of 3×3 matrices of phaseless rank at most 2 cut by a random two-dimensional affine space.

Why we care - Amoebas

Definition (Amoeba)

Given a polynomial ideal I and its set of zeros, $\mathcal{V}(I)$, we define the **amoeba** of I as

$$\mathcal{A}(I) = \{\mathrm{Log}(|z|) = (\log(|z_1|), ..., \log(|z_n|)) \mid z \in \mathcal{V}(I) \cap (\mathbb{C}^*)^n\}.$$

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These are well-known structures, studied in complex analysis and tropical algebraic geometry.

Let $P_k^{n\times m} = \{A \in \mathbb{R}_+^{n\times m} : \mathrm{rank}_\theta\,(A) \leq k\}$.

$$\mathrm{Log}\left(P_k^{n\times m}\right) = \mathcal{A}(\text{minors of order}(k+1))$$

It is an amoeba membership problem.

Why we care - Semidefinite lifts

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A **complex** semidefinite representation of size k of a d-polytope P is a description

$$P = \left\{ x \in \mathbb{R}^d \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\},$$

where A_i,B_i are $k \times k$ **complex** hermitian matrices.

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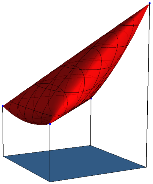

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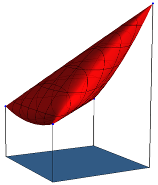

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If P has vertices $p_1, \cdots, p_v$ and facets $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, its slack matrix, S_P, is defined by $S_P(i,j) = h_i(p_j)$, for all i and j.

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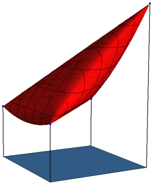

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Theorem (Gouveia-Parrilo-Thomas,2012)

The smallest size of a **complex** *semidefinite representation of P is* $\text{rank}^{\mathbb{C}}_{psd}(S_P)$*, defined as the smallest k for which there are* $U_1, \ldots, U_f, V_1, \cdots, V_v \in \mathcal{S}^k_+(\mathbb{C})$ *such that* $S_P(i,j) = \langle U_i, V_j \rangle$ *for all* i, j*.*

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The size of any complex semidefinite representation of a d-polytope P cannot be smaller than $d + 1$. If it equals $d + 1$ we call the polytope **psd$^{\mathbb{C}}$-minimal**.

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Proposition (Gouveia-Robinson-Thomas,2013+G.-Gouveia-Silva,2017)

A d-dimensional polytope P with slack matrix $S_P \in \mathbb{R}_+^{f\times v}$ is psd$^{\mathbb{C}}$-minimal if and only if

$$\text{rank}_\theta\left(\sqrt[\circ]{S_P}\right) = d+1,$$

where $\sqrt[\circ]{S_P}$ is the entrywise nonnegative square root of S_P.

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where $\sqrt[\circ]{S_P}$ is the entrywise nonnegative square root of S_P.

In fact, more generally, we also have $\mathrm{rank}_{\mathrm{psd}}^{\mathbb{C}}(A) \leq \mathrm{rank}_\theta\left(\sqrt[\circ]{A}\right)$ for any nonnegative matrix A.

The phaseless-singular matrices

So the first interesting thing one can ask is the following:

Question

When does a matrix $A \in \mathbb{R}^{n \times m}$, with $m \geq n$ verify $\operatorname{rank}_{\theta}(A) < n$?

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$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{2} \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{2}{3} & \frac{4}{3} & 2 \\ 1 & 1 & 0 \\ \sqrt{2} & \sqrt{2} & 2 \end{bmatrix} \longrightarrow \begin{cases} \sqrt{2} & \leq 1 + 2/3; \\ \sqrt{2} & \leq 1 + 4/3; \\ 2 & \leq 2 + 0. \end{cases}$$

LP-feasibility formulation

But this can be written in a very nice form.

Lemma (Camion-Hoffman,1966)

Let $A \in \mathbb{R}_+^{n \times m}$, with $m \geq n$. Then, $\text{rank}_\theta(A) < n$ if and only if the LP-feasibility problem

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\begin{array}{rl}
\text{find} & \lambda \in \mathbb{R}^n \\
\text{s.t.} & \lambda_i A_{ij} - \sum_{k \neq i} \lambda_k A_{kj} \leq 0, \quad \forall i, j; \\
 & \sum \lambda_i = 1; \\
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has a solution.

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has a solution.

We can now leverage this result to get a number of interesting consequences.

Nonmaximal phaseless rank - rectangular case

Proposition (G.-Gouveia)

A matrix $A \in \mathbb{R}_+^{n \times m}$, with $m \geq n$, verifies $\text{rank}_\theta(A) < n$ if and only if for any $n \times n$ submatrix B we have $\text{rank}_\theta(B) < n$.

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Proof:

- $C_i = \{$ multipliers λ making column i verify the triangular inequality$\}$.
- C_i are convex and every n of them intersect, so by Helly's theorem all intersect.

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Hence, solving the rectangular case consists in solving multiple square case

$$M = \begin{bmatrix} x - y + 1 & x - y + 1 & x + 1 & 1 \\ 1 - x & -x + y + 1 & 1 - y & x + y + 1 \\ 1 - y & 1 - x & 1 & x - y + 1 \end{bmatrix}$$

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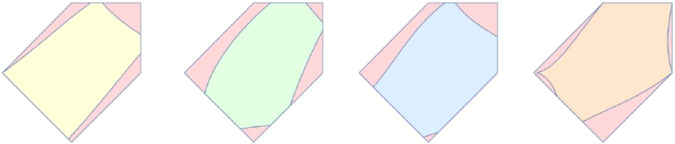

phaseless singular 3×3 submatrices

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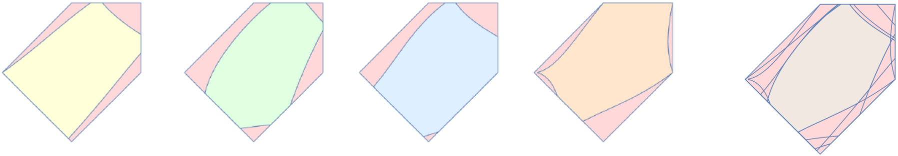

phaseless singular 3×3 submatrices

phaseless singular M

Nonmaximal phaseless rank - square case

Theorem (Camion-Hoffman,1966)

Given $A \in \mathbb{R}_+^{n \times n}$, $\operatorname{rank}_\theta(A) = n$ if and only if there exists a permutation matrix P such that $\mathcal{M}(AP)$ is a nonsingular M-matrix.

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Let $X \in \mathbb{C}^{n\times n}$.

$$\mathcal{M}(X)_{ij} = \begin{cases} |X_{ij}|, & i = j \\ -|X_{ij}|, & i \neq j. \end{cases}$$

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Proposition/Definition

Let $A \in \mathbb{R}^{n\times n}$ have nonpositive off-diagonal entries. Then the following are equivalent:

- i A is a nonsingular M-matrix.
- ii All its eigenvalues have positive real part.
- iii All its leading principal minors are positive.
- iv All its leading principal minors of size at least 3 are positive.

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Corollary (G.-Gouveia)

Let $A \in \mathbb{R}_+^{n\times n}$, with $n = 3, 4$. Then, $\text{rank}_\theta(A) < n$ if and only if $\det(\mathcal{M}(AP)) \leq 0$ for all permutation matrices.

Thus, $P_2^{3\times 3}$ and $P_3^{4\times 4}$ are basic semialgebraic sets (and, consequently, $P_2^{3\times n}$ and $P_3^{4\times n}$)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}_+^{3\times 3}$.

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$$\det \begin{bmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \leq 0$$

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Bounds on phaseless rank

Question: does $\text{rank}_\theta(A) < k$ if and only if all $k \times k$ submatrices of A have nonmaximal phaseless rank?

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Proposition (Levinger,1972)

Let $A = mI + J$, where $0 < m < n - 2$, and J is the $n \times n$ all-ones matrix. All $(m+2) \times (m+2)$ submatrices have nonmaximal phaseless rank but $\text{rank}_\theta(A) > m + 1$.

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Purbhoo,2008

We have the inclusion $\mathcal{A}(f) \subseteq \text{Nlop}(f)$, where

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In general, it is not a sufficient condition.

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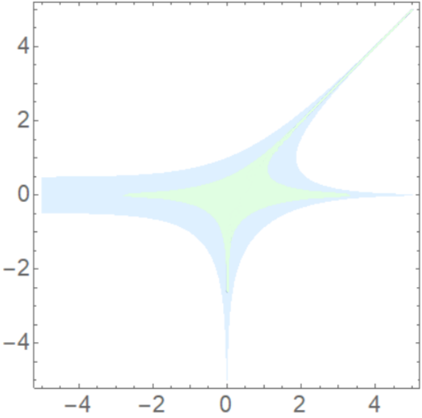
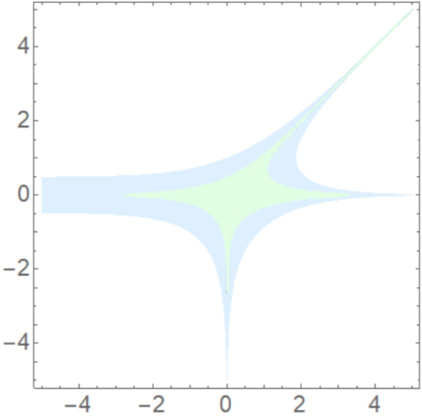


Amoeba membership

Let $f(x, y) = 1 + x + y + xy + y^2 \in \mathbb{C}[x, y]$.



$$\mathcal{A}(f) \subset \mathrm{Nlop}(f)$$

Amoeba membership

Theorem (Purbhoo,2008)

$$\mathcal{A}(I) = \bigcap_{f \in I} \mathcal{A}(f) = \bigcap_{f \in I} \mathrm{Nlop}(f)$$

This is remarkable but of little use, since it is an infinite intersection.

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Given I, a proper ideal of $\mathbb{C}[z]$, $\{f_1, \ldots, f_s\}$ is an amoeba basis if it is a set of generators of I which is minimal w.r.t the property

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In general, amoeba bases may not exist.

Our results revisited - another perspective

Proposition (G.-Gouveia)

A matrix $A \in \mathbb{R}_+^{n \times m}$ with $m \geq n$ verifies $\text{rank}_\theta(A) < n$ if and only if for any $n \times n$ submatrix B we have $\text{rank}_\theta(B) < n$.

Corollary (G.-Gouveia)

The maximal minors form an amoeba basis for the determinantal ideal they generate.

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Corollary (G.-Gouveia)

Let $A \in \mathbb{R}_+^{3 \times 3}$. Then, $\text{rank}_\theta(A) < 3$ if and only if $\det(\mathcal{M}(AP)) \leq 0$ for all permutation matrices.

Corollary (G.-Gouveia)

The amoeba of the 3×3 determinant is completely characterized by the nonlopsidedness of the determinant.

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Proposition (G.-Gouveia)

Let $A \in \mathbb{R}^{n\times m}_+$, $n \leq m$. If every column of all $k \times k$ submatrices of $\sqrt[\circ]{A}$ is not lopsided, $k \leq n$, then

$$\text{rank}^{\mathbb{C}}_{\text{psd}}(A) \leq n - \left\lfloor \frac{n-1}{k-1} \right\rfloor.$$

Thank You