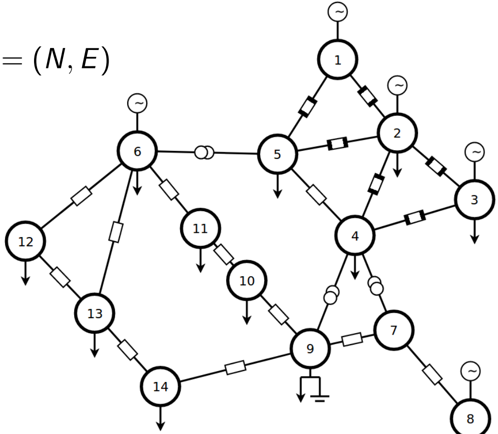
Tighter bounds for AC-OPF through rank-one convexification (?)

Tillmann Weisser
T-5/CNLS, Los Alamos National Laboratory
with Carleton Coffrin, Sidhant Misra,
Smitha Gopinath and Hassan Hijazi

SIAM AG, Bern 2019

AC-Optimal Power Flow Problem (simplified)

$G = (N, E)$



Variables:

- at each node:
  - $\mathrm{x} \in \mathbb{C}^{|N|}$ Power
  - $\mathrm{y} \in \mathbb{C}^{|N|}$ Voltage
- on each line
  - $\mathrm{z} \in \mathbb{C}^{|E|}$ power flow

AC Powerflow (Power balance):

$$z_{ij} \approx |\mathrm{y}_i|^2 - \mathrm{y}_i \mathrm{y}_j^*$$

$$\mathrm{x}_i = \sum_{(i,j) \in E} \mathrm{z}_{ij}$$

AC-Optimal Power Flow Problem (simplified)

$$\min \sum c(\mathfrak{R}(x_i)) \qquad \text{(cost)}$$

$$z_{ij} = |y_i|^2 - y_i {y_j}^*, \quad x_i = \sum_{(i,j)\in E} z_{ij} \qquad \text{(power balance)}$$

$$\underline{x}_i \leq x_i \leq \bar{x}_i \qquad \text{(generation limits)}$$

$$\underline{y}_i^2 \leq |y_i|^2 \leq \bar{y}_i^2 \qquad \text{(magnitude limits)}$$

$$|z_{ij}| \leq \bar{z}_{ij} \qquad \text{(line limits)}$$

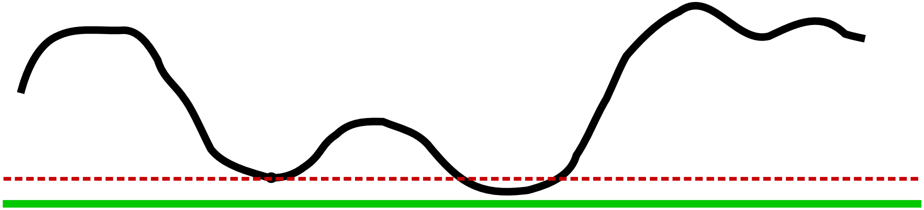
$$\underline{\theta}_{ij}\mathfrak{R}(y_i {y_j}^*) \leq \mathfrak{I}(y_i {y_j}^*) \leq \bar{\theta}_{ij}\mathfrak{R}(y_i {y_j}^*) \qquad \text{(angle differences)}$$

$\Rightarrow$ QCQP

$\Rightarrow$ N large

Practice:

- solve AC-OPF instance locally: $c(\mathrm{x}^*) = c_{\mathrm{loc}}$
- solve relaxation: c_{rel}
- if optimality gap $\frac{c_{\mathrm{loc}} - c_{\mathrm{rel}}}{c_{\mathrm{loc}}} < 1\%$ dispatch at x^*



Need **tight** relaxations!

W-Space Formulation

Replace $\mathrm{y}\mathrm{y}^* = W$:

$$
\begin{aligned}
& \min \sum c(\Re(\mathrm{x}_i)) && \text{(cost)} \\
& \mathrm{z}_{ij} = W_{ii} - W_{ij}, \quad \mathrm{x}_i = \sum_{(i,j)\in E} \mathrm{z}_{ij} && \text{(power balance)} \\
& \underline{\mathrm{x}}_i \le \mathrm{x}_i \le \bar{\mathrm{x}}_\mathrm{i} && \text{(generation limits)} \\
& \underline{\mathrm{y}}_i{}^2 \le W_{ii} \le \bar{\mathrm{y}}_\mathrm{i}{}^2 && \text{(magnitude limits)} \\
& |\mathrm{z}_{ij}| \le \bar{\mathrm{z}}_\mathrm{ij} && \text{(line limits)} \\
& \underline{\theta}_{ij}\Re(W_{ij}) \le \Im(W_{ij}) \le \bar{\theta}_{ij}\Re(W_{ij}) && \text{(angle differences)} \\
& W \succeq 0 \\
& \mathrm{rank}(W) = 1
\end{aligned}
$$

W-Space: SDP relaxation

$$\min \sum c(\Re(\mathrm{x}_i)) \qquad \text{(cost)}$$

$$\mathrm{z}_{ij} = W_{ii} - W_{ij}, \quad \mathrm{x}_i = \sum_{(i,j)\in E} \mathrm{z}_{ij} \qquad \text{(power balance)}$$

$$\underline{\mathrm{x}}_i \leq \mathrm{x}_i \leq \bar{\mathrm{x}}_\mathrm{i} \qquad \text{(generation limits)}$$

$$\underline{\mathrm{y}}_i{}^2 \leq W_{ii} \leq \bar{\mathrm{y}}_\mathrm{i}{}^2 \qquad \text{(magnitude limits)}$$

$$|\mathrm{z}_{ij}| \leq \bar{\mathrm{z}}_\mathrm{ij} \qquad \text{(line limits)}$$

$$\underline{\theta}_{ij}\Re(W_{ij}) \leq \Im(W_{ij}) \leq \bar{\theta}_{ij}\Re(W_{ij}) \qquad \text{(angle differences)}$$

$$W \succeq 0$$

$$\cancel{\mathrm{rank}(W) = 1} \qquad \text{(drop rank-1 constraint)}$$

$\Rightarrow$ SDP relaxation often not tight enough.

Link Lasserre-Hierarchy (Putinar Positivstellensatz)

- $\mathrm{y}\mathrm{y}^* = W$ can be understood as lifting in the space of moments:

$$\mathrm{y}_i\mathrm{y}_j{}^* = \int \mathrm{y}_i\mathrm{y}_j^* \, d\mu$$

  for some unknown measure μ.
- W corresponds to the moment matrix of μ.

$\Rightarrow$ SDP relaxation is first order Lasserre Relaxation for quadratic problems.

Higher order relaxation would provide better bounds
BUT too expensive to compute.

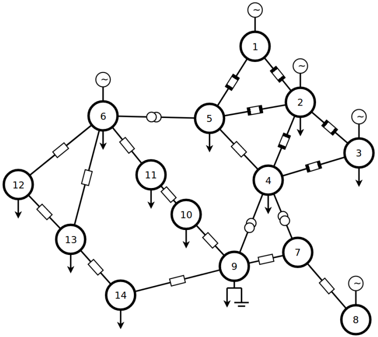
For large N (first order) SDP relaxation is too expensive.

$\Rightarrow$ relax even further:

- $W \succeq 0 \Leftrightarrow$ principal k-minors≥ 0, $\forall k \leq N$.

$\rightarrow$ consider only principal k-minors up to $k \leq n << N$ (convex NLP, smaller SDPs)

$\rightarrow$ consider only principal 2-minors (SOC)



Reformulation Linearization Technique Relaxation (first Order)

Assuming real variables in the subsequent.

Introduce redundant constraints before lifting:

$$(\bar{y}_i - y_i)(\bar{y}_j - y_j) \geq 0$$

McCormick envelope:

$$\begin{aligned}
W_{ij} &\geq \underline{y}_i y_j + \underline{y}_j y_i - \underline{y}_i \underline{y}_j \\
W_{ij} &\geq \bar{y}_i y_j + \bar{y}_j y_i - \bar{y}_i \bar{y}_j \\
W_{ij} &\leq \underline{y}_i y_j + \bar{y}_j y_i - \underline{y}_i \bar{y}_j \\
W_{ij} &\leq \bar{y}_i y_j + \underline{y}_j y_i - \bar{y}_i \underline{y}_j
\end{aligned}$$

$\Rightarrow$ Linear constraints instead of SDP constraint $W \succeq 0$.

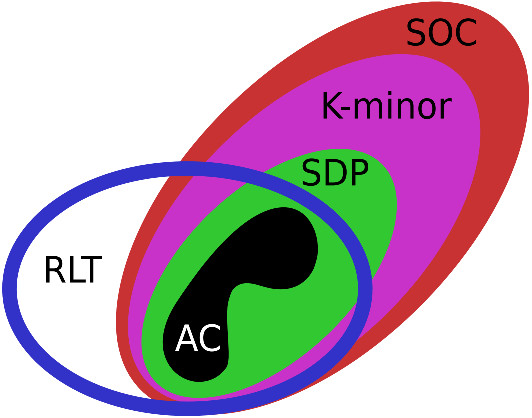
Link LP-based Hierarchy (Krivine Positvstellensatz)

- Again understanding $\mathrm{yy}^* = W$ as moment lift for some unknown measure μ.
- Dual of RLT relaxation corresponds to searching for a Krivine type certificate

$$\sum_{|\alpha|\leq 2} \prod \lambda_\alpha g^\alpha.$$

- higher order relaxations become ill-conditioned.

- SDP dominates k-minor and SOC but computational much more expensive
- SDP and RLT do **not** dominate each other
  $\Rightarrow$ combine SDP+RLT to get better relaxation



(dual side: BSOS[1])

[1] Tillmann Weisser, Jean B. Lasserre, and Kim-Chuan Toh. "Sparse-BSOS: a bounded degree SOS hierarchy for large scale polynomial optimization with sparsity". In: *Mathematical Programming Computation* 10 (2018), pp. 1–32.

Redundant Constraints to Strengthen Relaxations

Redundant information in the original formulation can be valuable in the relaxation.

- use additional physical knowledge (Smitha, Hassan)
- employ other liftings (T., Carleton)
- use mathematical description of QCQP (T., Hassan, Sidhant)

Redundant Constraints from Rank-1 Condition

Remember:

$$W = yy^* \quad \Leftrightarrow \quad W \succeq 0 \text{ and } \operatorname{rank}(W) = 1$$

Proposition:[2]

Matrix is psd and rank-1 $\Leftrightarrow$ all diagonal elements ≥ 0 and all 2-minors $= 0$.

- $\frac{1}{2}\binom{n}{2}^2$ 2-minors to consider $\rightarrow$ $\binom{n}{2}^2$ inequalities
- half of the inequalities non-convex
- McCormick (on non-convex part)

[2] Burak Kocuk, Santanu Dey, and Xu Sun. "Matrix Minor Reformulation and SOCP-based Spatial Branch-and-Cut Method for the AC Optimal Power Flow Problem". In: *Mathematical Programming Computation* 10 (2018), pp. 557–596.

Classification Convexified Rank-1 Constraints

From bounds $\underline{y}_i, \bar{y}_i$ on y_i derive bounds $\underline{W}_{ij}, \bar{W}_{ij}$ on $y_i {y_j}^*$.

Closer look to convexified constraints (in original space)

- Type A: (Products of) original bound constraints, e.g.,

$$\bar{W}_{ij} - y_i {y_j}^* \geq 0, \qquad (\bar{y}_i - y_i)(\bar{y}_j - {y_j}^*) \geq 0$$

- Type B: Sums of products of Type A of total degree 2, e.g,

$$(y_i {y_i}^* - \bar{W}_{ii})(\bar{y}_j - {y_j}^*) + (y_i {y_j}^* - \bar{W}_{ij})({y_i}^* - \underline{y}_i) \geq 0$$

Higher degree cancels out! – Higher order information in lower order relaxation.

Summary: Strengthen Relaxations in the W-Space

Summary:

- AC-OPF is a QCQP
- only first relaxation can be computed
- can we use higher order information in lower order relaxation?

Status:

- implementing/testing different ideas on power grid test-cases

Thank You!

Contact: tweisser@lanl.gov