Lower Bounds for Polynomials in Exact Arithmetic

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(joint with Victor Magron and Timo de Wolff)

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Bern

The Problem

Task

Global minimisation of multivariate polynomial

Let $p \in \mathbb{R}[x_1, \ldots, x_n]$

$$\min \{p(x) : x \in \mathbb{R}^n\}$$

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want to prove non-negativity

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- Numerically compute lower bound for polynomial in polynomial time
- Symbolic Postprocessing
  Rounding to feasible rational solution
  - idea independent of future software improvements
  - even applicable for constrained optimisation

Non-Negativity

New Problem

Given sparse $p \in \mathbb{Q}[x]$. How to show $\forall x \in \mathbb{R}^n \,.\, p(x) \geq 0$?

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- notation: t summands, n variables, degree $2d$
- input: list of (exponent-vector $\times$ coefficient)
- size: $\mathcal{O}\left(t \cdot n \cdot \log d + t \cdot \max\left\{\log |c| : c \in \text{coeff}(p)\right\}\right)$

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  - semi-definite programme with linear equations
  - size of problem is $\mathcal{O}\left(\binom{n+d}{d}^2\right)$

    $\rightsquigarrow$ exponential in input size

A New Approach

Circuit Polynomial: $$q = \sum_{i=0}^{n} b_i x^{a_i} + c x^y \qquad b_i > 0$$

points $a_i \in (2\mathbb{N})^n$ span simplex $\quad y \in \mathrm{int}(\mathrm{hull}(\{a_0, \ldots, a_n\}))$

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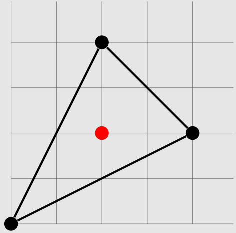
Example (Motzkin polynomial)

$$q = 1 + x^2y^4 + x^4y^2 - 3x^2y^2$$



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Theorem (Iliman, de Wolff)

Let $y = \sum \lambda_i a_i$ unique convex combination.
Define Circuit Number $\Theta = \prod \left(\frac{b_i}{\lambda_i}\right)^{\lambda_i}$

- $|c| \leq \Theta$ *or*
- $c \geq -\Theta$ *and y all even*

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Decompose into

$$p \geq_{\min} \mathrm{relax(p)} = \sum q_k + C \geq C$$

where $$q_k = \sum_{i=0}^{n} b_{ik} x^{a_{ik}} + c_k x^{y_k}$$ as above

$$c_k = -\prod \left(\frac{b_{ik}}{\lambda_{ik}} \right)^{\lambda_{ik}}$$

SONC Optimisation in Simple Case

Let $p = b_0 + b_1 \cdot x_1^6 x_2^2 + b_2 \cdot x_1^2 x_2^4 - c_1 \cdot x_1^2 x_2 - c_2 \cdot x_1^3 x_2^2$.

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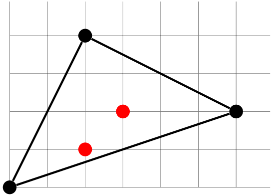

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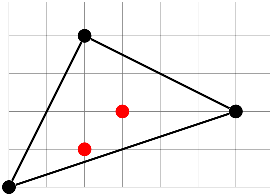

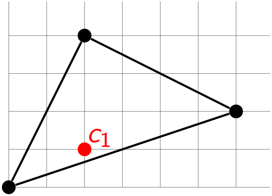

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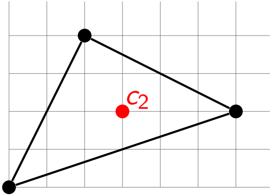

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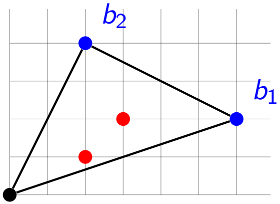

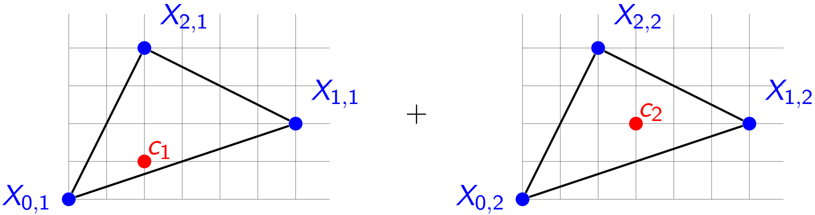

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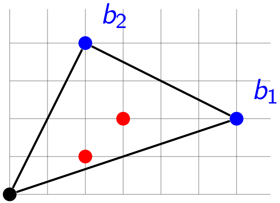
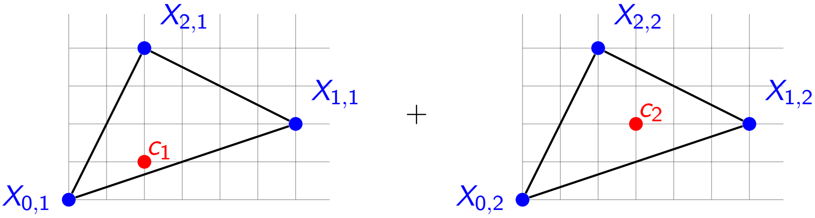


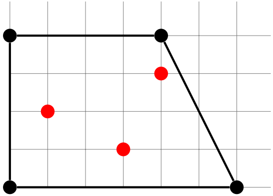
$$\min X_{0,1} + X_{0,2}$$
$$b_1 \geq X_{1,1} + X_{1,2}$$
$$b_2 \geq X_{2,1} + X_{2,2}$$

each non-negative



Decompose into Circuits

- write red as convex combination of black $\rightsquigarrow$ solve LPs
  $\rightsquigarrow$ covering of red points (no triangulation)



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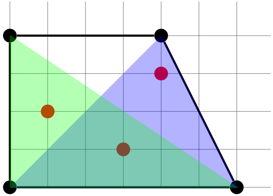
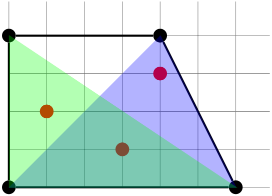
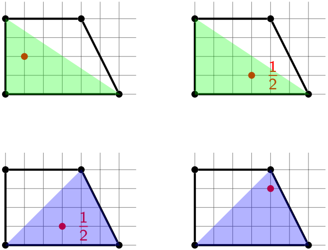


Decompose into Circuits

- write red as convex combination of black $\rightsquigarrow$ solve LPs
  $\rightsquigarrow$ covering of red points (no triangulation)
- distribute negative coefficients





Optimisation Programme

$$\text{relax}(p) = \sum_{i=0}^{m} b_i x^{a_i} - \sum_{k=1}^{r} c_k x^{y_k} \qquad b_i, c_k > 0$$

compute circuits with $\overline{c}_k x^{\overline{y}_k}$ as negative term
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$$\min \sum_{k=1}^{\overline{r}} X_{0k} \quad \text{s.t.} \quad b_i \geq \sum_{k=1}^{\overline{r}} X_{ik} \qquad (i = 1, \ldots, m)$$

$$\bar{c}_k = \prod_{i \in \text{circuit}_k} \left(\frac{X_{ik}}{\lambda_{ik}} \right)^{\lambda_{ik}} \qquad (k = 1, \ldots, \overline{r})$$

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Observation: Size independent of degree!

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$\Rightarrow$ can refine any assignment to feasible solution

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2. Numerically solve the GP
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Similar idea works for SAGE

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software POEM: lower bound of polynomial and minimisers, in polynomial time; improved bounds in FPT, bounds in exact arithmetic

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- handle case $\lambda_{k,0} = 0$
- constrained polynomial optimisation (long term)

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