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On the exactness of Lasserre relaxations of SPIs and POPs

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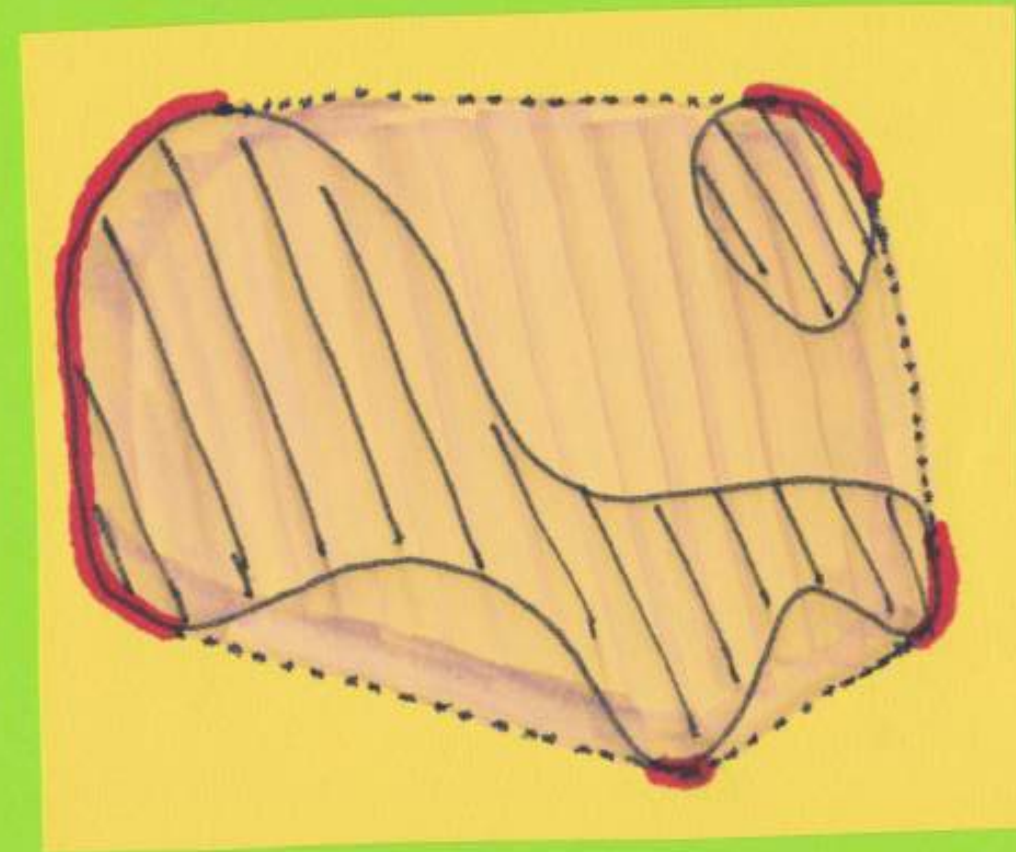

SIAM Conference on Applied Algebraic Geometry
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Reformulation and linearization technique

(RLT)

$$1 + 2x_1 + \ldots + x_1^3 x_2 - x_1^4 x_2 \geq 0$$

polynomial inequality of some degree

Reformulation and linearization technique

(RLT)

$$1 + 2x_1 + \ldots + x_1^3 x_2 - x_1^4 x_2 \geq 0$$

polynomial inequality of some degree

$$\Longleftrightarrow \forall a_1, \ldots, a_6 \in \mathbb{R} : (a_1 + a_2 x_1 + \ldots + a_6 x_2^2)^2 (1 + 2x_1 + \ldots + x_1^3 x_2 - x_1^4 x_2) \geq 0$$

infinite system of polynomial inequalities of higher degree

Reformulation and linearization technique

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$$1+2x_1+\dots+x_1^3x_2-x_1^4x_2\geq 0$$

polynomial inequality of some degree

$$\Longleftrightarrow \forall a_1,\dots,a_6\in\mathbb{R}:\ (a_1+a_2x_1+\dots+a_6x_2^2)^2(1+2x_1+\dots+x_1^3x_2-x_1^4x_2)\geq 0$$

infinite system of polynomial inequalities of higher degree

$$\Longleftrightarrow \forall a_1,\dots,a_6\in\mathbb{R}:\ \underbrace{(a_1 \dots a_6)}_{\text{row vector}}\ \underbrace{\underbrace{(1+2x_1+\dots+x_1^3x_2-x_1^4x_2)}_{\text{scalar}}\ \underbrace{\underbrace{\begin{pmatrix}1\\ x_1\\ \vdots\\ x_2^2\end{pmatrix}}_{\text{column vector}}\underbrace{(1\ x_1 \dots x_2^2)}_{\text{row vector}}}_{\text{symmetric matrix}}}_{\text{symmetric matrix}}\ \underbrace{\begin{pmatrix}a_1\\ \vdots\\ a_6\end{pmatrix}}_{\text{column vector}}\geq 0$$

Reformulation and linearization technique

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$$1+2x_1+\ldots+x_1^3x_2-x_1^4x_2\geq 0$$

polynomial inequality of some degree

$$\iff \forall a_1,\ldots,a_6\in\mathbb{R}: (a_1+a_2x_1+\ldots+a_6x_2^2)^2(1+2x_1+\ldots+x_1^3x_2-x_1^4x_2)\geq 0$$

infinite system of polynomial inequalities of higher degree

$$\iff \forall a_1,\ldots,a_6\in\mathbb{R}: \underset{\text{row vector}}{(a_1\ \ldots\ a_6)}\ \underbrace{\underset{\text{scalar}}{(1+2x_1+\ldots+x_1^3x_2-x_1^4x_2)}\ \underbrace{\underset{\text{column vector}}{\begin{pmatrix}1\\x_1\\\vdots\\x_2^2\end{pmatrix}}\underset{\text{row vector}}{(1\ x_1\ \ldots\ x_2^2)}}_{\text{symmetric matrix}}}_{\text{symmetric matrix}}\ \underset{\text{column vector}}{\begin{pmatrix}a_1\\\vdots\\a_6\end{pmatrix}}\geq 0$$

$$\iff \begin{pmatrix}1+2x_1+\ldots-x_1^4x_2 & & \cdot\\ & \cdot & \\ \cdot & & x_2^4+\ldots\end{pmatrix}\underset{\text{psd}}{\succeq} 0$$

polynomial matrix Inequality of the higher degree

Reformulation and linearization technique

(RLT)

$$1+2x_1+\dots+y_{16}-y_{17}\geq 0$$

$$\Longleftrightarrow \forall a_1,\dots,a_6\in\mathbb{R}: (\dots)^2(\dots)\geq 0$$

$$\Longleftrightarrow \forall a_1,\dots,a_6\in\mathbb{R}: \underset{\text{row vector}}{(\dots)}\;\underbrace{\underset{\text{scalar}}{(\dots)}\;\underset{\substack{\text{column}\\\text{vector}}}{\begin{pmatrix}1\\ \vdots\\ x_2^2\end{pmatrix}}\;\underbrace{\underset{\text{row vector}}{(\dots)}}_{\text{symmetric matrix}}}_{\text{symmetric matrix}}\;\underset{\substack{\text{column}\\\text{vector}}}{\begin{pmatrix}a_1\\ \vdots\\ a_6\end{pmatrix}}\geq 0$$

$$\Longleftrightarrow \begin{pmatrix}1+2x_1+\dots-y_{17} & & \cdot\\ & \cdot & \\ \cdot & & y_5+\dots\end{pmatrix}\underset{\text{psd}}{\succeq} 0$$

matrix Inequality

Reformulation and linearization technique

(RLT)

LINEARIZE

$$1+2x_1+\dots+y_{16}-y_{17}\geq 0$$

$$\Longleftrightarrow\ \forall a_1,\dots,a_6\in\mathbb{R}:\ \dots\geq 0$$

EXPAND AND LINEARIZE

$$\Longleftrightarrow\ \forall a_1,\dots,a_6\in\mathbb{R}:\ \underbrace{\dots}_{\text{row vector}}\ \underbrace{\dots}_{\text{scalar}}\ \underbrace{\underbrace{\begin{pmatrix}1\\ x_1\\ \vdots\\ x_2^2\end{pmatrix}}_{\text{column vector}}\ \underbrace{\dots}_{\text{row vector}}}_{\text{symmetric matrix}}\ \underbrace{\begin{pmatrix}a_1\\ \vdots\\ a_6\end{pmatrix}}_{\text{column vector}}\geq 0$$

symmetric matrix

LINEARIZE

$$\Longleftrightarrow\ \begin{pmatrix}1+2x_1+\dots-y_{17} & & \cdot\\ & \cdot & \\ \cdot & & y_5+\dots\end{pmatrix}\underbrace{\succeq}_{\text{psd}} 0$$

linear matrix inequality

Lasserre relaxation

$\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$

$g = (g_1, \dots, g_m) \in \mathbb{R}[\underline{X}]^m$

$$\left.\begin{array}{r} g_1(x) \geq 0 \\ \vdots \\ g_m(x) \geq 0 \end{array}\right] \text{SPI}$$

Lasserre relaxation

$\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$

$g = (g_1, \dots, g_m) \in \mathbb{R}[\underline{X}]^m$

$$\left.\begin{array}{r} 1 \geq 0 \\ g_1(x) \geq 0 \\ \vdots \\ g_m(x) \geq 0 \end{array}\right] \text{SPI} \quad \overset{\text{reformulation}}{\rightsquigarrow} \quad \left.\begin{array}{r} G_0(x) \succeq 0 \\ G_1(x) \succeq 0 \\ \vdots \\ G_m(x) \succeq 0 \end{array}\right] \text{PMI}$$

Lasserre relaxation

$\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$

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$d \in \mathbb{N}_0$ „relaxation degree"

of degree d or $d-1$

$$\left.\begin{array}{r} 1 \geq 0 \\ g_1(x) \geq 0 \\ \vdots \\ g_m(x) \geq 0 \end{array}\right] \text{SPI} \xrightarrow{\text{reformulation}} \left.\begin{array}{r} G_0(x) \succeq 0 \\ G_1(x) \succeq 0 \\ \vdots \\ G_m(x) \succeq 0 \end{array}\right] \text{PMI} \xrightarrow{\text{linearization}}$$

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Lasserre relaxation

$\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$

$g = (g_1, \ldots, g_m) \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$

constraints — objective

$d \in \mathbb{N}_0$

„relaxation degree"

of degree d or $d-1$

minimize $f(x)$
over all $x \in \mathbb{R}^n$
s.t. $1 \geq 0$
$g_1(x) \geq 0$
$\vdots$
$g_m(x) \geq 0$

SPI — POP

reformulation ⟿

minimize $f(x)$
over all $x \in \mathbb{R}^n$
s.t. $G_0(x) \succeq 0$
$G_1(x) \succeq 0$
$\vdots$
$G_m(x) \succeq 0$

PMI

linearization ⟿

minimize $\tilde{f}(x,y)$
over all $x \in \mathbb{R}^n, y \in \mathbb{R}^{?}$
s.t. $\tilde{G}_0(x,y) \succeq 0$
$\tilde{G}_1(x,y) \succeq 0$
$\vdots$
$\tilde{G}_m(x,y) \succeq 0$

LMI — SD

Lasserre relaxation

$\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$

$\underbrace{g = (g_1, \dots, g_m)}_{\text{constraints}} \in \mathbb{R}[\underline{X}]^m, \underbrace{f \in \mathbb{R}[\underline{X}]}_{\text{objective}}$

$d \in \mathbb{N}_0$

„relaxation degree"

of degree d or $d-1$

POP (SPI):

minimize $f(x)$
over all $x \in \mathbb{R}^n$
s.t. $1 \geq 0$
$g_1(x) \geq 0$
$\vdots$
$g_m(x) \geq 0$

reformulation

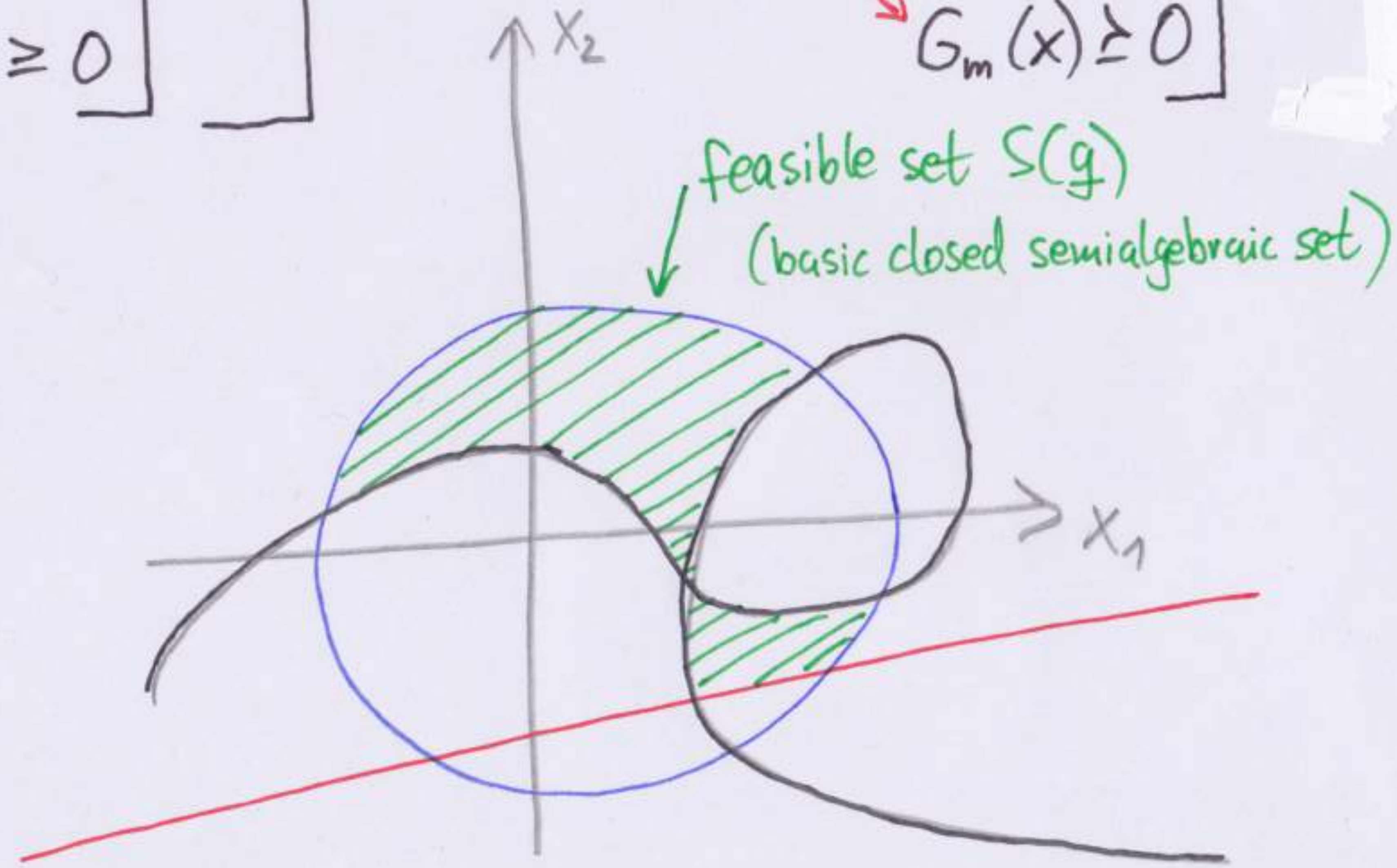
PMI:

minimize $f(x)$
over all $x \in \mathbb{R}^n$
s.t. $G_0(x) \succeq 0$
$G_1(x) \succeq 0$
$\vdots$
$G_m(x) \succeq 0$

linearization

SD (LMI):

minimize $\tilde{f}(x,y)$
over all $x \in \mathbb{R}^n, y \in \mathbb{R}^?$
s.t. $\tilde{G}_0(x,y) \succeq 0$
$\tilde{G}_1(x,y) \succeq 0$
$\vdots$
$\tilde{G}_m(x,y) \succeq 0$



Lasserre relaxation

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constraints, objective

$d \in \mathbb{N}_0$

"relaxation degree"

of degree d or $d-1$

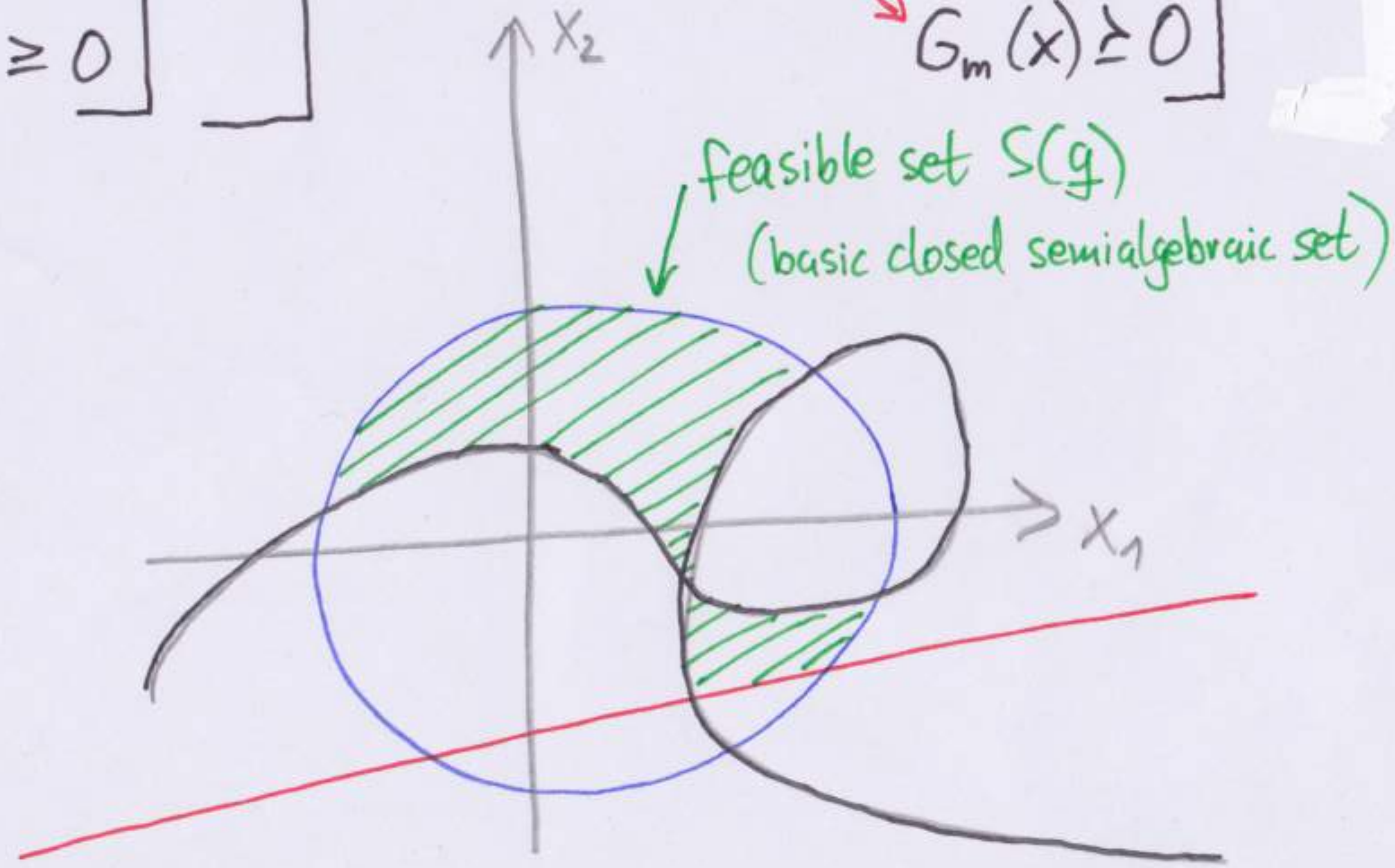
$$\left.\begin{array}{l} \text{minimize } f(x) \\ \text{over all } x \in \mathbb{R}^n \\ \text{s.t.} \left.\begin{array}{r} 1 \geq 0 \\ g_1(x) \geq 0 \\ \vdots \\ g_m(x) \geq 0 \end{array}\right] \text{SPI} \end{array}\right] \text{POP}$$

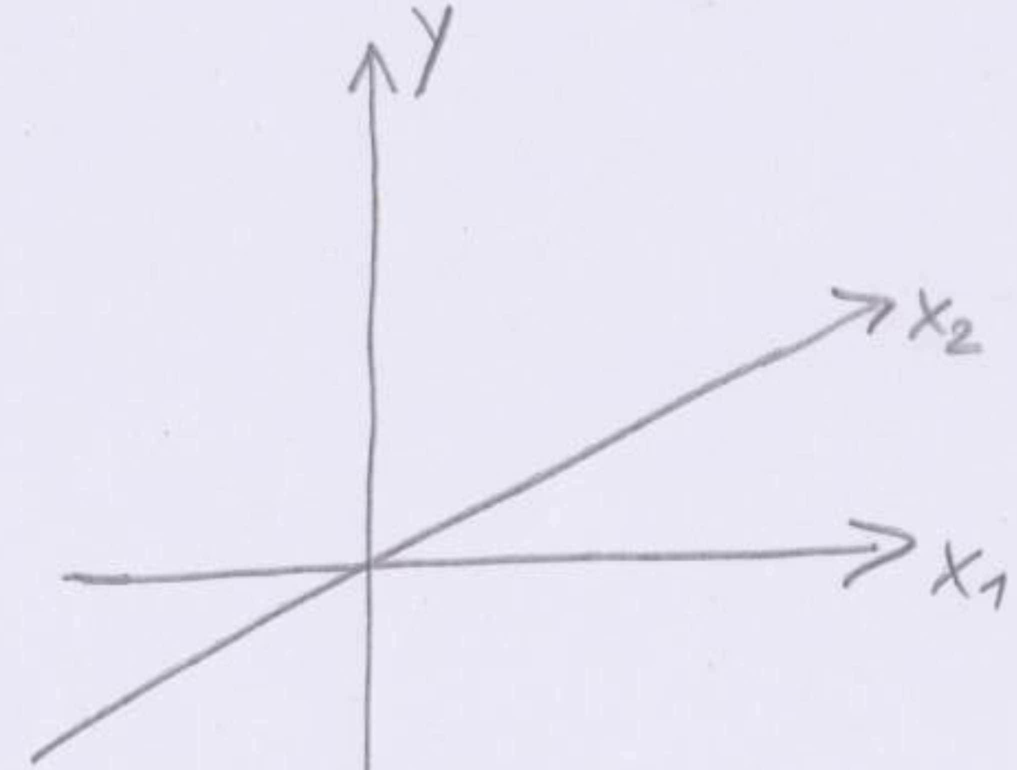
reformulation ⟿

$$\left.\begin{array}{l} \text{minimize } f(x) \\ \text{over all } x \in \mathbb{R}^n \\ \text{s.t.} \left.\begin{array}{r} G_0(x) \succeq 0 \\ G_1(x) \succeq 0 \\ \vdots \\ G_m(x) \succeq 0 \end{array}\right] \text{PMI} \end{array}\right.$$

linearization ⟿

$$\left.\begin{array}{l} \text{minimize } \tilde{f}(x,y) \\ \text{over all } x \in \mathbb{R}^n, y \in \mathbb{R}^? \\ \text{s.t.} \left.\begin{array}{r} \tilde{G}_0(x,y) \succeq 0 \\ \tilde{G}_1(x,y) \succeq 0 \\ \vdots \\ \tilde{G}_m(x,y) \succeq 0 \end{array}\right] \text{LMI} \end{array}\right] \text{SD}$$





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constraints, objective

$d \in \mathbb{N}_0$

„relaxation degree"

of degree d or $d-1$

POP:
minimize $f(x)$
over all $x \in \mathbb{R}^n$
s.t. (SPI)
$1 \geq 0$
$g_1(x) \geq 0$
$\vdots$
$g_m(x) \geq 0$

reformulation

PMI:
minimize $f(x)$
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linearization

SD
minimize $\tilde{f}(x,y)$
over all $x \in \mathbb{R}^n, y \in \mathbb{R}^?$
s.t. (LMI)
$\tilde{G}_0(x,y) \succeq 0$
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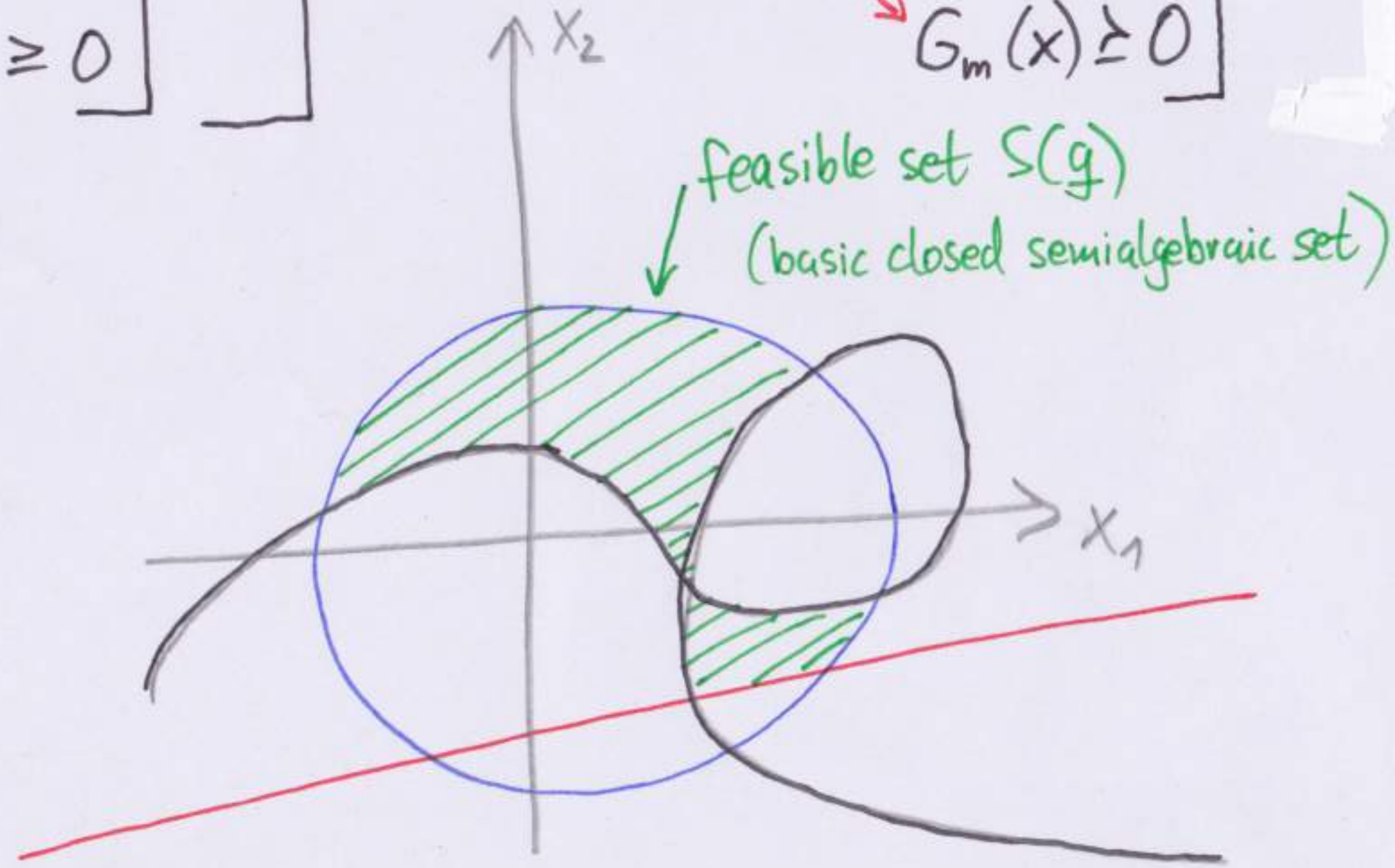

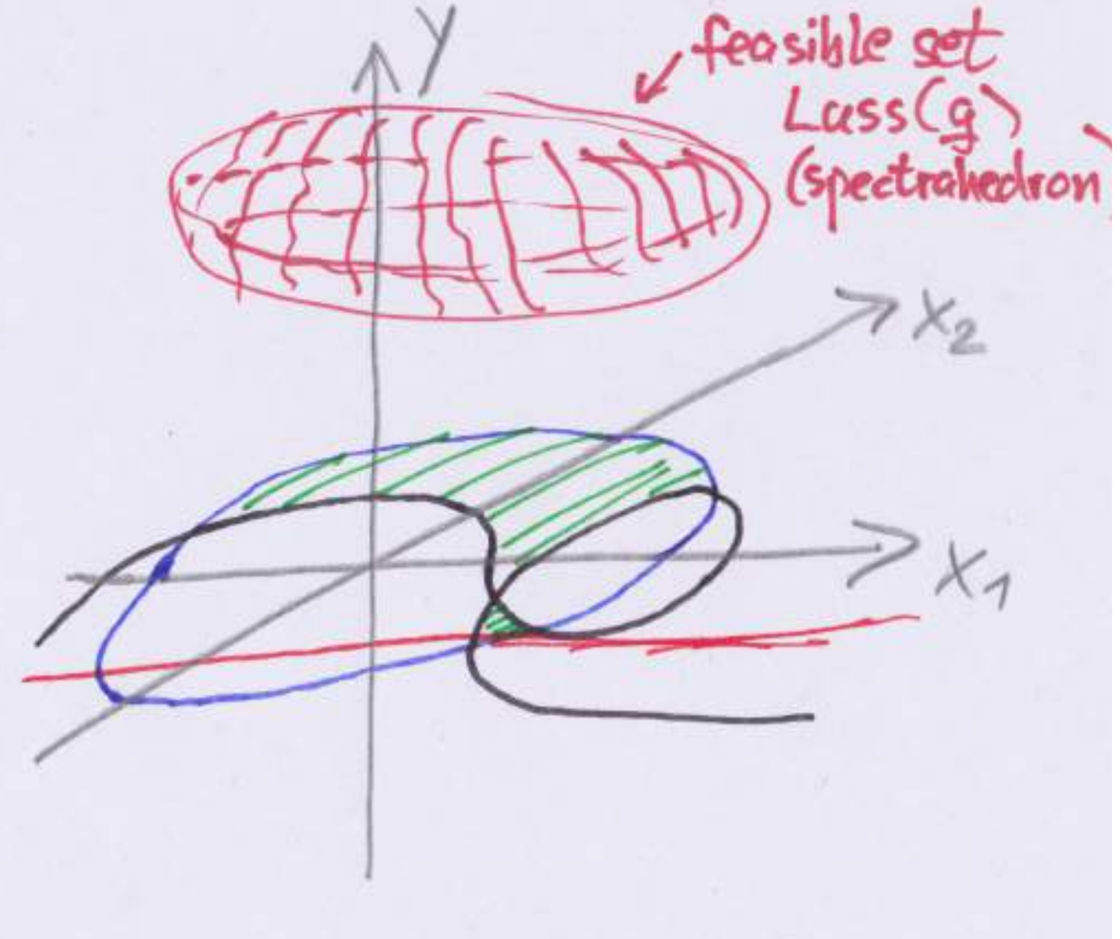

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constraints, objective

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SPI, POP

reformulation

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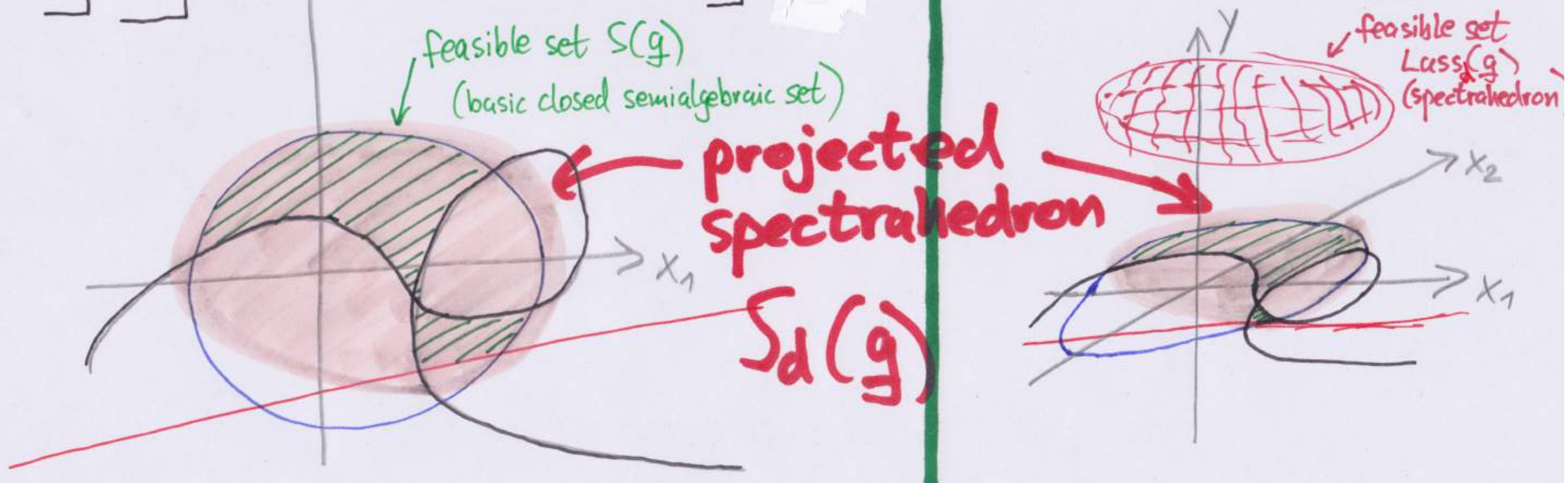
LMI, SD



Lasserre in an abstract way

$g \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$, $d \in \mathbb{N}_0$

constraints objective relaxation degree

Lasserre in an abstract way

$\underline{g} \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$, $d \in \mathbb{N}_0$

constraints objective relaxation degree

$S(\underline{g}) := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$ „basic closed semialgebraic set"

$\operatorname{opt}(f, \underline{g}) := \inf\{f(x) \mid x \in S(\underline{g})\} \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ „optimal value"

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$M_d(\underline{g}) := \left\{ \sum_{i=0}^{m} \sum_j p_{ij}^2 g_i \mid p_{ij} \in \mathbb{R}[\underline{X}], \deg(p_{ij}^2 g_i) \leq d \right\} \subseteq \mathbb{R}[\underline{X}]_d$

„truncated quadratic module"

where $g_0 := 1 \in \mathbb{R}[\underline{X}]$

$M(\underline{g}) := \bigcup_{d \in \mathbb{N}_0} M_d(\underline{g})$ „quadratic module"

Lasserre in an abstract way

$g \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$, $d \in \mathbb{N}_0$

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$M(g) := \bigcup_{d \in \mathbb{N}_0} M_d(g)$ „quadratic module"

$\mathrm{Lass}_d(g) := \{L \mid L: \mathbb{R}[\underline{X}]_d \to \mathbb{R} \text{ linear}, L(M_d(g)) \subseteq \mathbb{R}_{\geq 0}, L(1) = 1\}$ „Lasserre spectrahedron"

$S_d(g) := \{(L(X_1), \ldots, L(X_n)) \mid L \in \mathrm{Lass}_d(g)\}$ „projected Lasserre spectrahedron"

$\mathrm{opt}_d(f,g) := \inf\{L(f) \mid L \in \mathrm{Lass}_d(g)\}$ „relaxed optimal value"

Lasserre in an abstract way

$g \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$, $d \in \mathbb{N}_0$

constraints | objective | relaxation degree

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Obviously, $S(g) \subseteq \ldots \subseteq S_{d+1}(g) \subseteq S_d(g) \subseteq \ldots$ „SPIs through LMIs"
and $\mathrm{opt}(f,g) \geq \ldots \geq \mathrm{opt}_{d+1}(f,g) \geq \mathrm{opt}_d(f,g)$ „POPs through SDPs"

History

Let $\underset{\text{constraints}}{g} \in \mathbb{R}[\underline{x}]^m$, $\underset{\text{objective}}{f} \in \mathbb{R}[\underline{x}]$.

Suppose $M(g)$ is Archimedean $\left(\underset{\substack{\text{slightly} \\ \text{change } g}}{\overset{\Longrightarrow}{\Longleftarrow}} S(g) \text{ is compact} \right)$.

History

Let $g \in \mathbb{R}[\underline{X}]^m$, $f \in \mathbb{R}[\underline{X}]$.

constraints — objective

Suppose $M(g)$ is Archimedean $\left(\underset{\text{slightly change } g}{\overset{\Longrightarrow}{\Longleftarrow}} S(g) \text{ is compact} \right)$.

Lasserre 2001 based on Putinar 1993 : $\operatorname{opt}_d(f,g) \xrightarrow{d\to\infty} \operatorname{opt}(f,g)$

Lasserre 2009 based on Prestel 2001 : $S_d(\;\,g) \xrightarrow{d\to\infty}$ conv $S(\;\,g)$

Nie & S. 2007

History

Let $g \in \mathbb{R}[\underline{x}]^m$, $f \in \mathbb{R}[\underline{x}]$.
constraints objective

Suppose $M(g)$ is Archimedean ($\underset{\text{slightly change } g}{\rightleftarrows}$ $S(g)$ is compact).

Always:

Lasserre 2001 based on Putinar 1993: $\text{opt}_d(f,g) \xrightarrow{d\to\infty} \text{opt}(f,g)$

Lasserre 2009 based on Prestel 2001, Nie & S. 2007: $S_d(\ ,g) \xrightarrow{d\to\infty}$ conv $S(\ ,g)$

Under mild conditions:

Scheiderer 2005: For large d, $\text{opt}_d(f,g) = \text{opt}(f,g)$.

Helton & Nie 2009: For large d, conv $S(g)$ is a projected spectrahedron

UNDER MILD CONDITIONS

History

Let $\underline{g} \in \mathbb{R}[\underline{x}]^m$, $f \in \mathbb{R}[\underline{x}]$.

constraints — objective

Suppose $M(\underline{g})$ is Archimedean $\left(\xRightarrow{\quad} \; S(\underline{g}) \text{ is compact} \right)$.

slightly change $\underline{g}$

Always:

Lasserre 2001 based on Putinar 1993: $\operatorname{opt}_d(f, \underline{g}) \xrightarrow{d \to \infty} \operatorname{opt}(f, \underline{g})$

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Under mild conditions:

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Helton & Nie 2009: For large d, $\operatorname{conv} S(\underline{g})$ is a projected spectrahedron.

Kriel & S. 2018 SIOPT: $S(\underline{g})$ convex $\Rightarrow$ For large d, $S(\underline{g}) = S_d(\underline{g})$.

Kriel & S. FoCM: For large d, $\operatorname{conv} S(\underline{g}) = S_d(\underline{g})$.

UNDER MILD CONDITIONS

History

Let $g \in \mathbb{R}[\underline{x}]^m$, $f \in \mathbb{R}[\underline{x}]$.

constraints objective

Suppose $M(g)$ is Archimedean ($\underset{\text{slightly change } g}{\overset{\Longrightarrow}{\Longleftarrow}}$ $S(g)$ is compact).

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Linear g_i are fine.
a problem.

Kriel & S. FoCM: For large d, conv $S(g) = S_d(g)$.

History

Let $g \in \mathbb{R}[\underline{x}]^m$, $f \in \mathbb{R}[\underline{x}]$.

constraints — objective

Suppose $M(g)$ is Archimedean $\left(\underset{\text{slightly change } g}{\overset{\Longrightarrow}{\Longleftarrow}} S(g) \text{ is compact} \right)$.

Always:

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Scheiderer 2005: • For large d, $\mathrm{opt}_d(f,g) = \mathrm{opt}(f,g)$.

Helton & Nie 2009: • For large d, $\mathrm{conv}\ S(g)$ is a projected spectrahedron

Kriel & S. 2018 SIOPT: • $S(g)$ convex $\Rightarrow$ For large d, $S(g) = S_d(g)$.

Linear g_i are ~~a problem.~~ fine.

Kriel & S. FoCM: • For large d, $\mathrm{conv}\ S(g) = S_d(g)$.

proof uses real closed fields

• $\forall g: \forall$ complexity bounds: $\exists d_0: \forall f$ of bounded complexity: $\forall d \geq d_0$: $\mathrm{opt}_d(f,g) = \mathrm{opt}(f,g)$.

UNDER MILD CONDITIONS

The result for convex sets

For $h \in \mathbb{R}[\underline{X}]$ and $x \in \mathbb{R}^n$, we call h <u>strictly quasiconcave</u> at x if

$$\forall v \in \mathbb{R}^n \setminus \{0\} : \left((\nabla h)(x)^T v = 0 \implies v^T ((\operatorname{Hess} h)(x)) v < 0 \right).$$

„Hessian negative definite on tangent space"

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Rough intuition: mountain hike

mountain: subgraph of h
$\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid y \leq h(x)\}$

your ground position: x

Can a bird flying a straight line at constant altitude crash into you?

contour map

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We call $h \in \mathbb{R}[\underline{X}]$ <u>$\underline{g}$-sos-concave</u> if there exists a certain sums-of-squares certificate for $\operatorname{Hess} h \preceq 0$ on $S(\underline{g})$.

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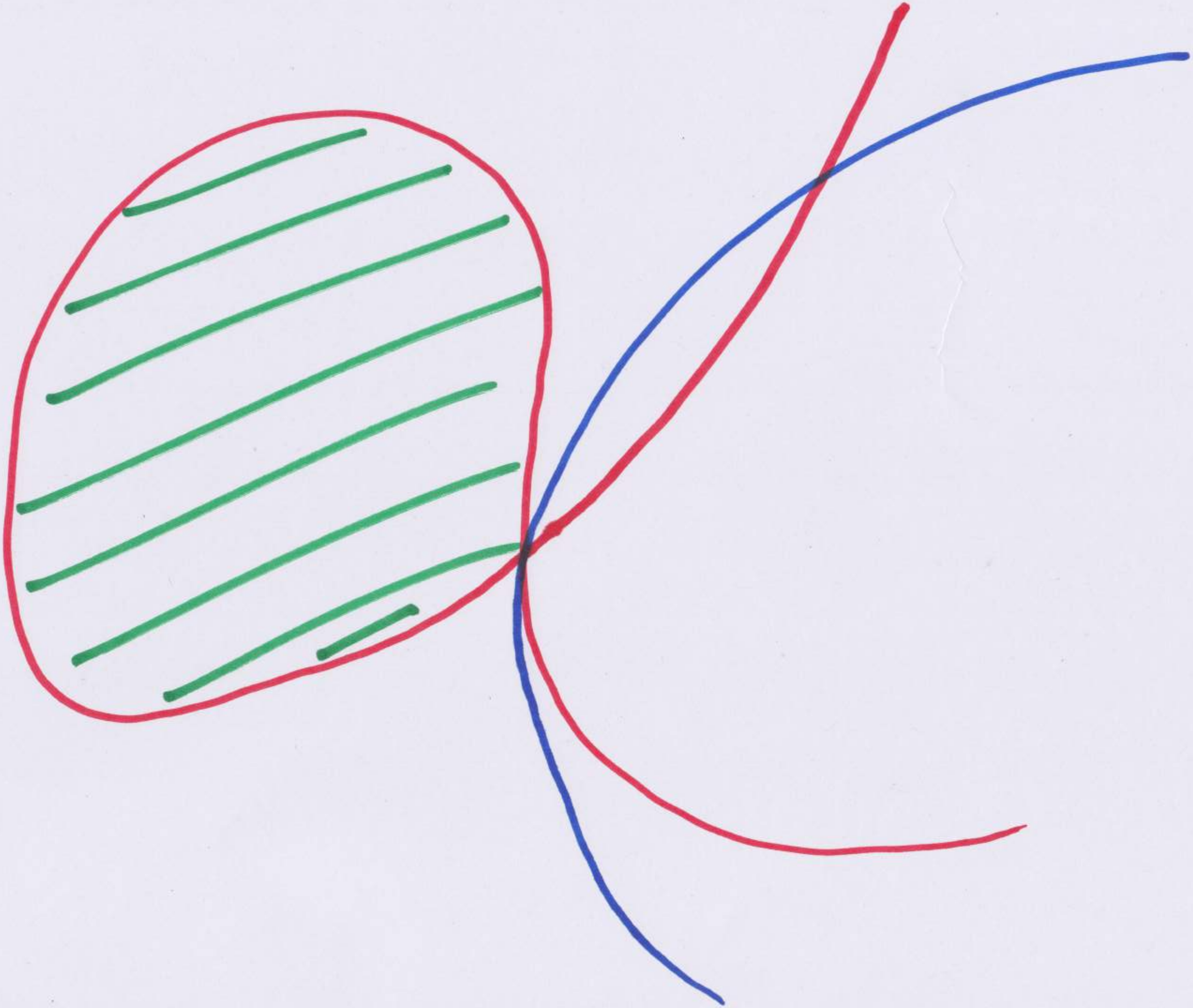
Linear polynomials are trivially $\underline{g}$-sos-concave!

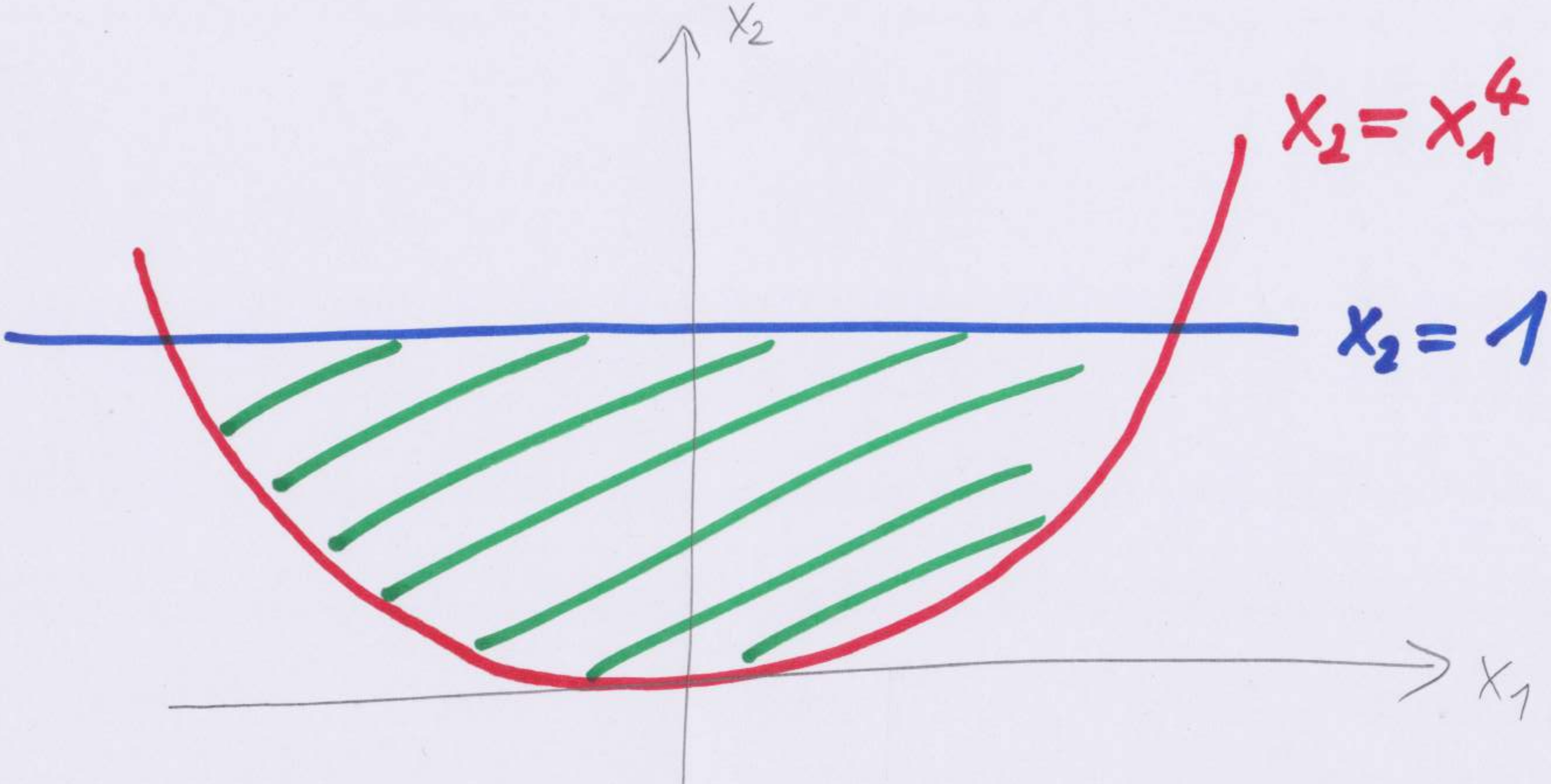
<u>Thm (2018 SIOPT)</u> Let $\underline{g} \in \mathbb{R}[\underline{X}]^m$ such that $M(\underline{g})$ is Archimedean.

Suppose $S(\underline{g})$ is convex with nonempty interior.

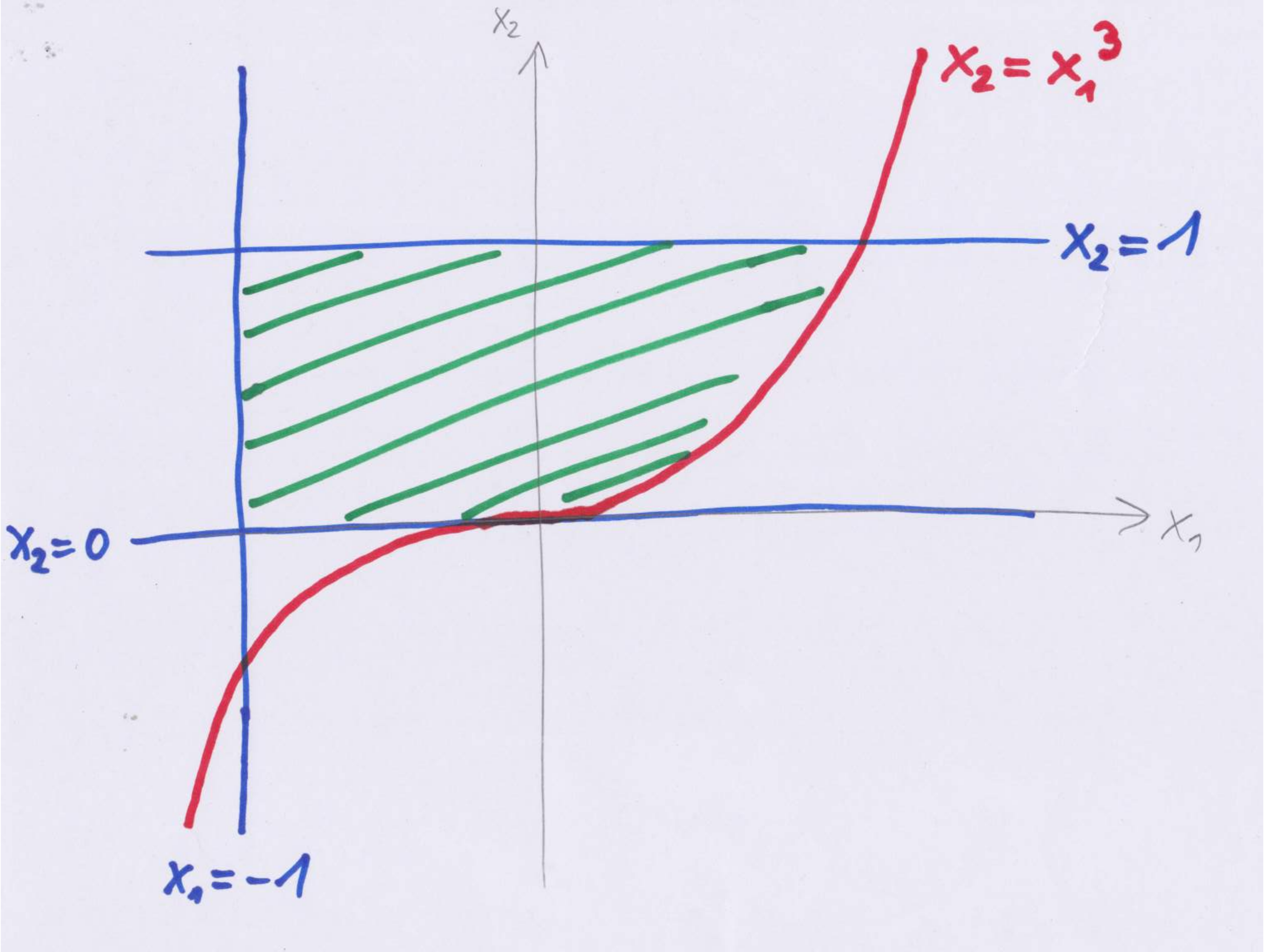
Suppose that each g_i is strictly quasiconcave on $S(\underline{g}) \cap Z(g_i)$ or $\underline{g}$-sos-concave (for example linear).

Then $S(\underline{g}) = S_d(\underline{g})$ for large d.

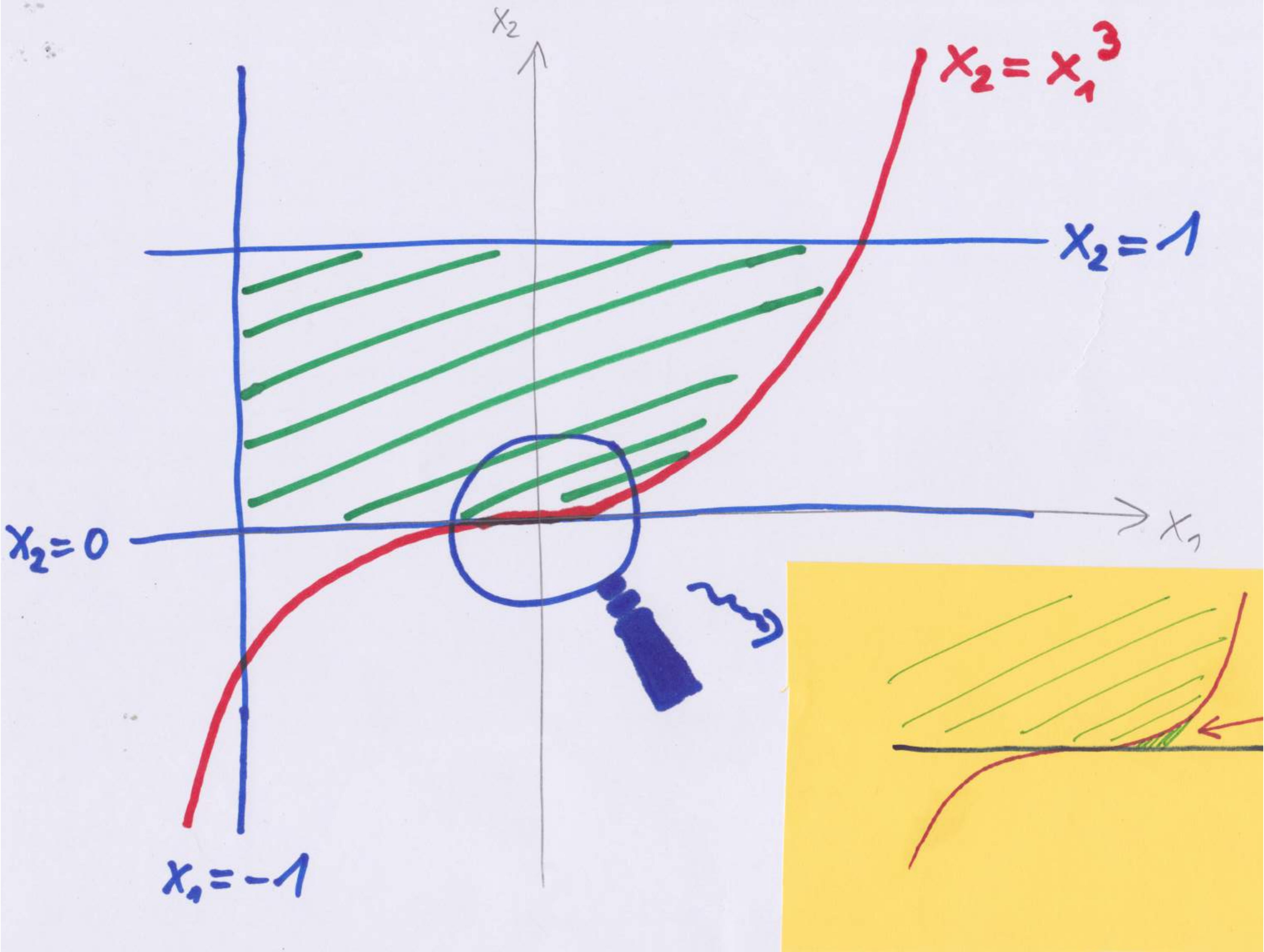




$$\mathrm{Hess}\left(\bar{x}_2 - \bar{x}_1^4\right) = \begin{pmatrix} -12\bar{x}_1^2 & 0 \\ 0 & 0 \end{pmatrix}$$



x_2
$x_2 = x_1^3$
$x_2 = 1$
$x_2 = 0$
x_1
$x_1 = -1$



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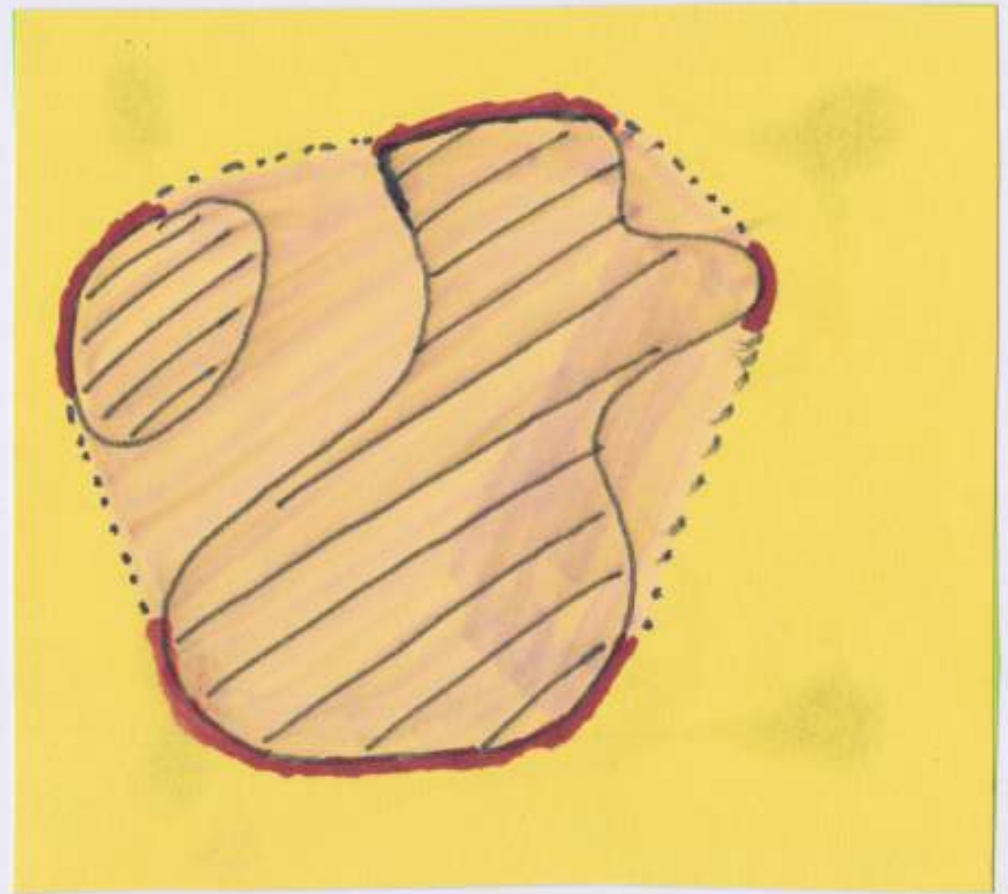
The result for not necessarily convex sets

For $S \subseteq \mathbb{R}^n$, we call

$$\text{convbd}\, S := S \cap \partial \operatorname{conv} S$$

the <u>convex boundary</u> of S.

We say that <u>S has nonempty interior near its convex boundary</u> if $\text{convbd}\, S \subseteq \overline{S^\circ}$.



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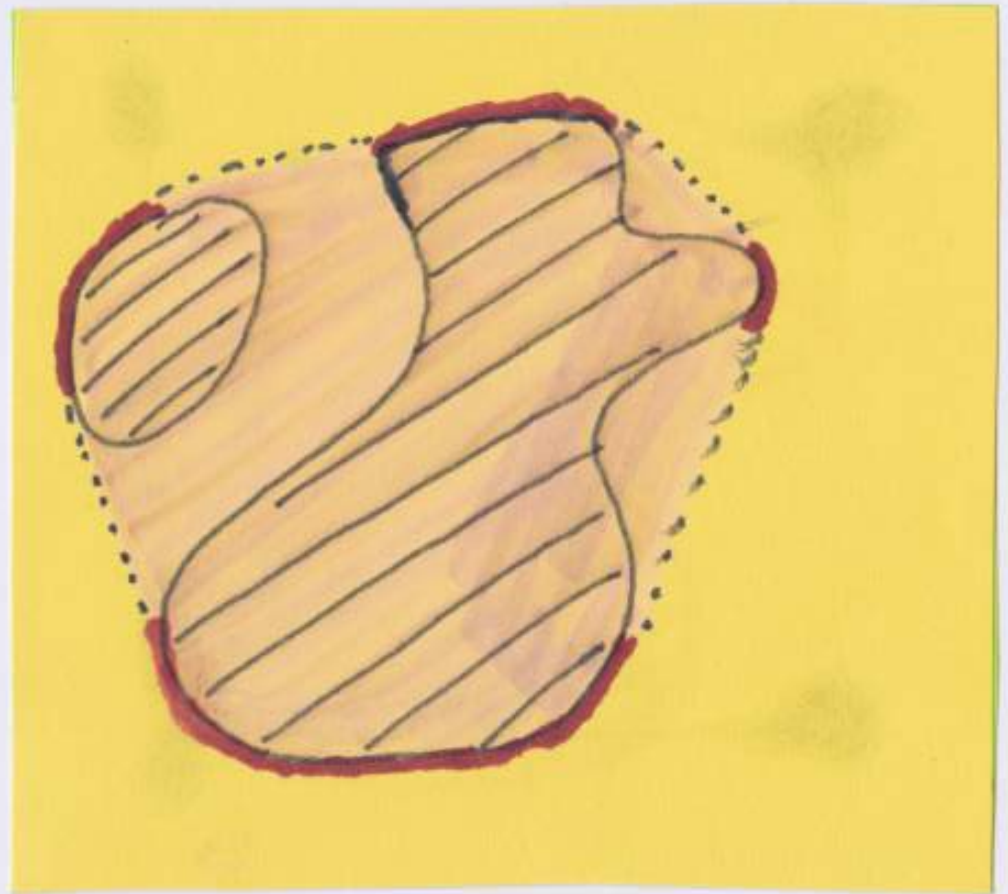
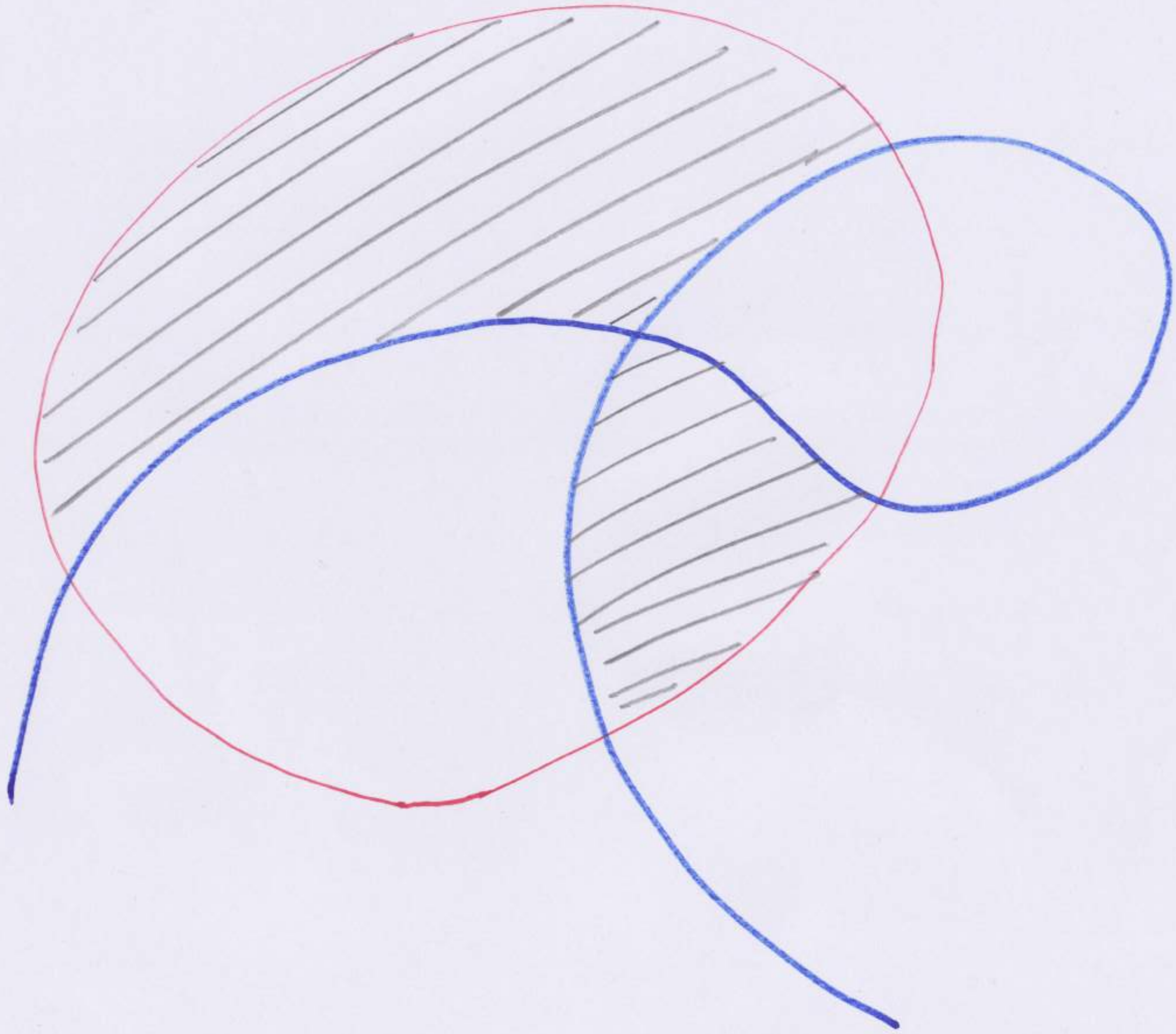


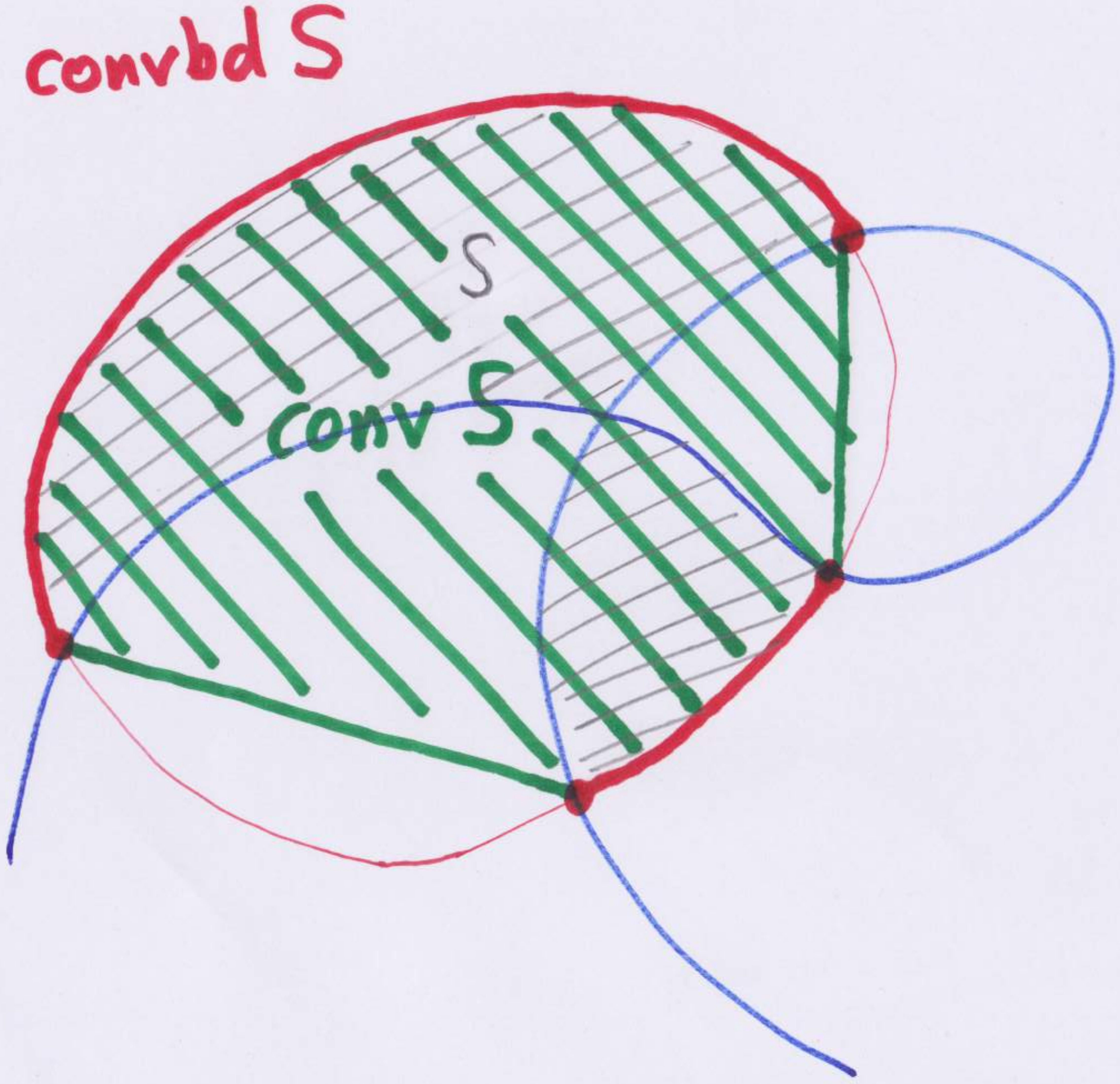
<u>Thm (FoCM)</u> Let $\underline{g} \in \mathbb{R}[\underline{X}]^m$ such that $M(\underline{g})$ is Archimedean.

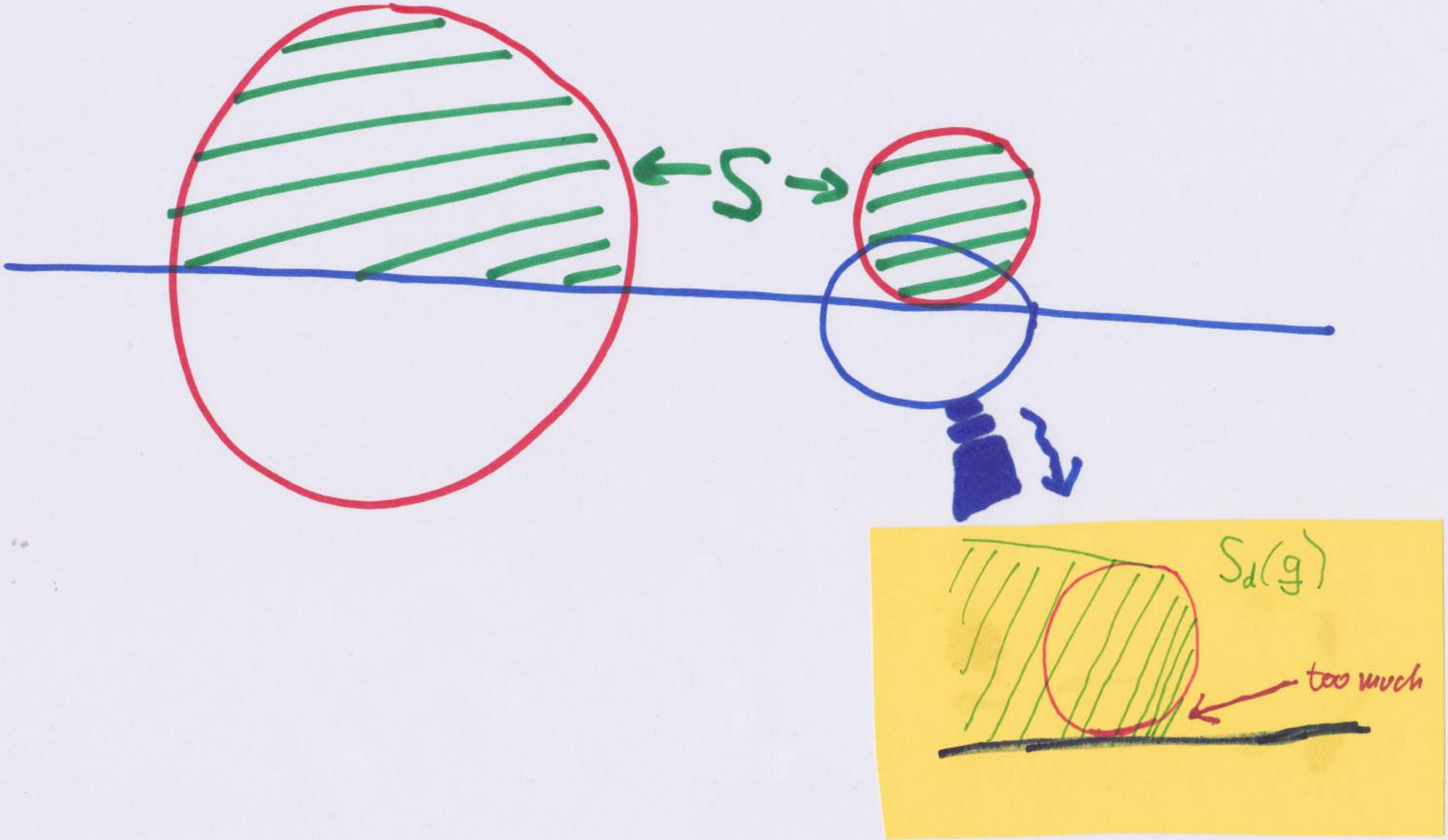
Suppose that $S(\underline{g})$ has nonempty interior near its convex boundary and each g_i is strictly quasiconcave on $(\operatorname{convbd} S) \cap Z(g_i)$.

Then $S(\underline{g}) = \boxed{\operatorname{conv} S(\underline{g}) = S_d(\overline{\underline{g}}) \text{ for large } d.}$

Proof uses real closed fields!




convbd S
S
conv S


S
$S_d(g)$
too much

The result for optimization

For $x \in \mathbb{R}^n$, $\quad u_x := (X_1 - x_1)^2 + \ldots + (X_n - x_n)^2 \in \mathbb{R}[\underline{X}]$.

Thm (FoCM) Let $n, m \in \mathbb{N}_0$ and $\underline{g} \in \mathbb{R}[\underline{X}]^m$ such that $M(\underline{g})$ is Archimedean.

Suppose $S(\underline{g}) \neq \emptyset$. Moreover, let $k \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\varepsilon > 0$.

Then there exists $d \in \mathbb{N}_0$ such that for all $f \in \mathbb{R}[\underline{X}]_N$ with all coefficients in $[-N, N]$, we have:

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If $x_1, \ldots, x_k$ are the global minimizers of f on $S(\underline{g})$, if the balls of radius ε around the x_i are pairwise disjoint and contained in $S(\underline{g})$ and if we have

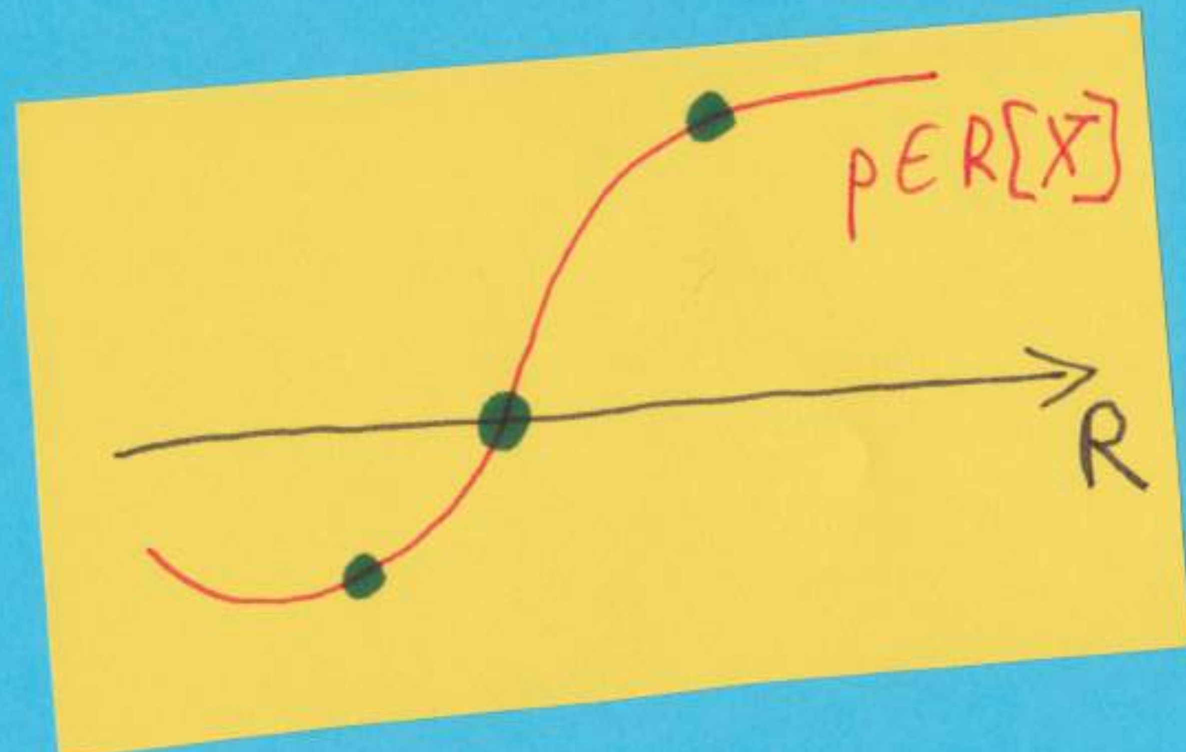
$$f \geq \mathrm{opt}(f, \underline{g}) + \varepsilon u \text{ on } S(\underline{g})$$

where $u := u_{x_1} \cdots u_{x_k} \in \mathbb{R}[\underline{X}]$, then $f - \mathrm{opt}(f, \underline{g}) \in M_d(\underline{g})$

and consequently $\mathrm{opt}(f, \underline{g}) = \mathrm{opt}_d(f, \underline{g})$.

Real closed fields

A field R is real closed if $\boxed{a \leq b :\Longleftrightarrow \exists c \in R : a + c^2 = b}$ $(a, b \in R)$ defines a linear order on the set R with respect to which the intermediate value theorem for polynomials holds.



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From now on, let R be a real closed extension field of $\mathbb{R}$.

$$\mathcal{O}_R := \{ a \in \mathbf{R} \mid \exists N \in \mathbb{N} : -N \leq a \leq N \}$$

„finite elements"

subring of R

$$\mathfrak{m}_R := \{ a \in R \mid \forall N \in \mathbb{N} : -\tfrac{1}{N} \leq a \leq \tfrac{1}{N} \}$$

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For each $a \in \mathcal{O}_R$ there is exactly one $\mathrm{st}(a) \in \mathbb{R}$ such that „standard part" $a - \mathrm{st}(a) \in \mathfrak{m}_R$.

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„standard part"

For $x \in \mathcal{O}^n$, we set $I_x := (\bar{X}_1 - x_1, \ldots, \bar{X}_n - x_n) = \{f \in \mathcal{O}[\bar{X}] \mid f(x) = 0\}$

so that $I_x^2 = \{f \in \mathcal{O}[\bar{X}] \mid f(x) = 0, \nabla f(x) = 0\}$.

A key Lemma

Let M be an Archimedean quadratic module of $\mathcal{O}[\underline{X}]$ and set

$$S := \{x \in \mathbb{R}^n \mid \forall p \in M: \ \mathrm{st}(p(x)) \geq 0\}.$$

Moreover, suppose $k \in \mathbb{N}_0$ and let $x_1, \dots, x_k \in \mathcal{O}^n$ have pairwise distinct standard parts. Let

$$f \in \bigcap_{i=1}^{k} I_{x_i}^2$$

such that

$$\mathrm{st}(f(x)) > 0$$

for all $x \in S \setminus \{\mathrm{st}(x_1), \dots, \mathrm{st}(x_n)\}$ and

$$\mathrm{st}(v^T (\mathrm{Hess}\, f)(x_i)\, v) > 0$$

for all $i \in \{1, \dots, k\}$ and $v \in \mathbb{R}^n \setminus \{0\}$. **Then $f \in M$.**

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Proof idea. Assume $f \notin M$. Separate f from the cone $M \cap I$ in the real vector space $I := I_{x_1}^2 \cdots I_{x_k}^2 \underset{\text{remainder}}{\overset{\text{Chinese}}{=}} I_{x_1}^2 \cap \dots \cap I_{x_k}^2$.

Choose extremal separating functional (real valued!) . . .

$\mathcal{O}_R^n$

line $\ell(x) = 0$

$\ell \in \mathcal{O}_R[\underline{X}]$

Lagrange multipliers from $\mathcal{O}_{\geq 0} \subseteq \mathbb{R}_{\geq 0}$