Exact algorithms: from Semidefinite to Hyperbolic programming

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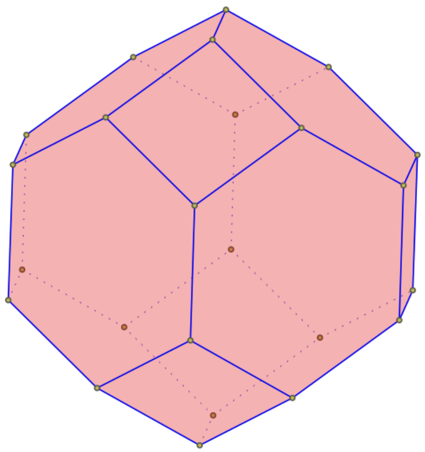
Polyhedra



Finite # linear inequalities :

$$\ell_i(x) \geq 0, \quad i = 1, \dots, m$$

One can associate a diagonal matrix :

$$A(x) = \mathrm{diag}(\ell_i(x))$$

$$\boxed{f = \ell_1(x)\ell_2(x)\cdots\ell_m(x) = \det A(x)}$$

$\mathcal{P} = \{x \in \mathbb{R}^n : \forall i, \ell_i(x) \geq 0\}$

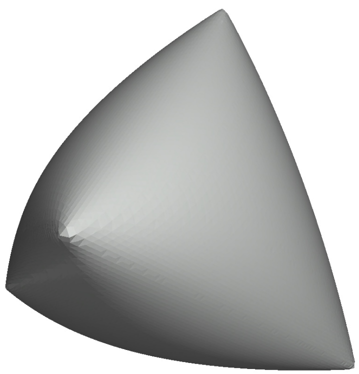
Feasibility Problem: Is $\mathcal{P} \neq \emptyset$? ($\exists\, x \in \mathbb{R}^n$ s.t. $\ell_i(x) \geq 0, \forall i$)

Linear Programming (LP): Compute inf $\ell(x)$ s.t. $x \in \mathcal{P}$

Solutions are "rational"

Combinatorics of boundary – active constraints $''\ell_i = 0''$

Spectrahedra



$\mathcal{S} = \{x \in \mathbb{R}^n : A(x) \succeq 0\}$

Start with a symmetric linear matrix :

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$$

Infinite # linear ineq. $y^T A(x) y \geq 0$

Finite # **Non**-linear inequalities :

$$\{\text{Principal Minors of } A(x)\} \quad \geq 0$$

$$\boxed{f = \det A(x)}$$

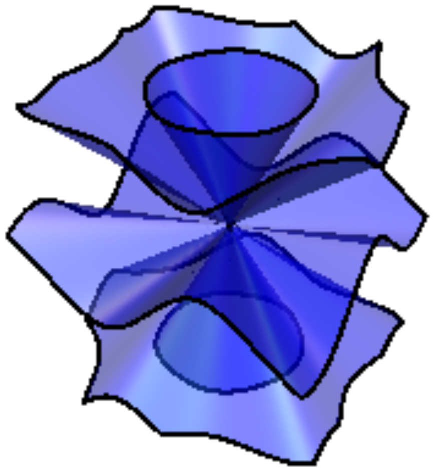
Feasibility Problem (LMI): $\exists\, x \in \mathbb{R}^n$ s.t. $A(x) \succeq 0$

Semidefinite Progr. (SDP): Compute inf $\ell(x)$ s.t. $x \in \mathcal{S}$

Irrational solutions - Algebraic degree $= \delta($**size,var,rank**$)$

Combinatorics of boundary – **co-rank** of $A(x)$

This talk: Hyperbolicity cones



$\mathcal{C}(f, e)$ hyperbolicity cone

Input data : a polynomial and a vector

$f \in \mathbb{R}[x_1, \ldots, x_n]_d, e \in \mathbb{R}^n$

Finite # **Non**-linear inequalities :

Coeff. of $t \mapsto f(te - x)$

f hyperbolic w.r. to e

Feasibility Problem? By definition $e \in \mathrm{Int}(\mathcal{C}(f, e))$!!

Hyperbolic Progr. (HP): Compute inf $\ell(x)$ s.t. $x \in \mathcal{C}(f, e)$

Algebraic degree = ?(**deg,n,mult**) (“**ir**rational” solutions)

Combinatorics of boundary – **multiplicity** of x

Hyperbolic polynomials

Definition of hyperbolic polynomial

$f \in \mathbb{R}[x]_d$ is *hyperbolic w.r.t.* $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$ if

- $f(e) \neq 0$
- $\forall\, a \in \mathbb{R}^n \quad t \mapsto ch_a(t) := f(t\,e - a)$ has **only real roots**

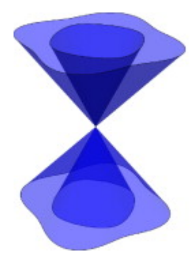
If such e exists, f is called a *hyperbolic polynomial*.

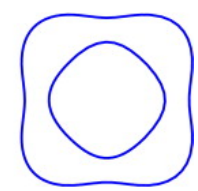
Fundamental examples:

(1) $f = x_1 \cdots x_d$
$ch_a(t) = \prod_i (te_i - a_i)$

(2) $f = \det A(x)$, $A(x)$ sym.
$ch_a(t) = \det(tI_d - A(a))$

General case (here $d = 4$) :





Brändén (2010): hyperb. polynomials without determinantal representations

Cones and Multiplicity

Hyperbolicity cone

The *hyperbolicity cone* of $f \in \mathbb{R}[x]_d$ (*w.r.t.* e) is

$$\mathcal{C}(f,e) = \{a \in \mathbb{R}^n : ch_a(t) = 0 \Rightarrow t \geq 0\}$$

Multiplicity: For $a \in \mathbb{R}^n$, we define

$$\text{mult}(a) := \text{multiplicity of 0 as root of } \text{ch}_a(t) = f(te - a)$$

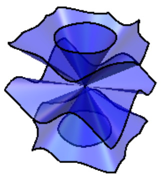
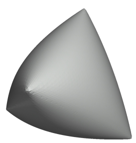
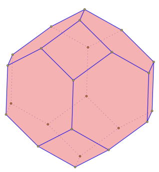
Multiplicity set: For $m \leq d$, $\Gamma_m = \{a \in \mathbb{R}^n : \text{mult}(a) \geq m\}$

Remark: The set Γ_m is **real algebraic**.

Indeed, if $ch_a(t) = t^d + g_1(a)t^{d-1} + \cdots + g_{d-1}(a)t + g_d(a)$ then

$$\Gamma_m = \{a : g_i(a) = 0, i \geq d - m + 1\}$$

Recap

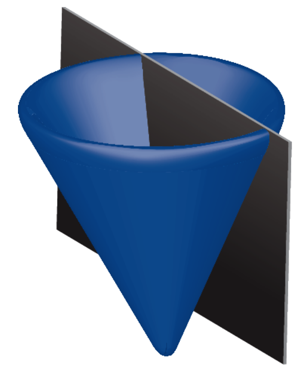
Cone	Polynomial	Optimization	Boundary
	Hyperbolic	HP	Multiplicity
	$f = \det\, A(x)$	SDP	Co-rank of $A(x)$
	$f = \prod \ell_i(x)$	LP	Active constraints

Two different approaches:

- "Interior" approach (e.g. classical interior-point methods)
- "Boundary" approach (more suitable for algebraic methods)

General goal: certification of information on the solution

Motivation in real algebraic geometry



Generalized Lax conjecture

Every hyperbolicity cone is a spectrahedron,
that is $\exists\, A_1, \ldots, A_n$ such that

$$\mathcal{C}(f, e) = \{x \in \mathbb{R}^n : A_1 x_1 + \ldots + A_n x_n \succeq 0\}$$

Helton-Vinnikov, Lewis-Parrilo-Ramana – True for $n = 3$, with the stronger result that f hyperbolic $\Rightarrow f = \det(x_1 A_1 + x_2 A_2 + x_3 A_3)$

By **Brändén**'s counterexamples, the Lax conjecture cannot be proved by proving that every hyperbolic polynomial admits a determinantal representation.

Determinantal representations : **Leykin, Netzer, Plaumann, Sinn, Sturmfels, Thom, Vinzant...** with many different techniques

Kummer (2015) – Partial results towards Lax conjecture, for hyperbolic polynomials without real singularities.

This work: Algebraic approach to HP

We prove that every hyperbolic programming problem is equivalent to linear optimization over some multiplicity locus.

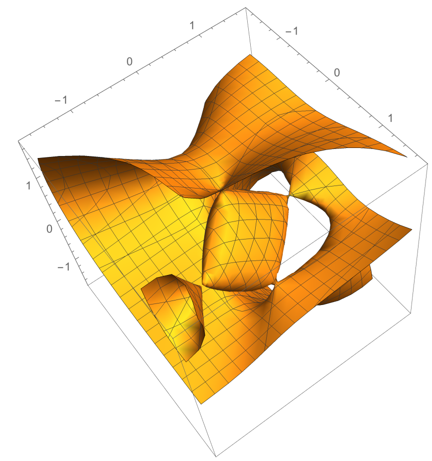
Moreover, computing $\max \mathrm{mult}(x)$ over $\mathcal{C}(f, e)$ is equivalent to the computation of witness points on connected components of the multiplicity loci (classical problem in R.A.G.)

All the reduced problems have *simply exponential complexity* with respect to the number n of variables.

Representation of the solution:

$$x_i = q_i(t)/q_0(t), \quad q(t) = 0 \ \text{ with } q_i \in \mathbb{Q}[t]$$

A test on a hyperbolic quartic



Optimizing random linear forms yields solutions of

- multiplicity 1 for 64% of the times
- multiplicity 2 for 36% of the times

One can get rational solutions (multiplicity 2) :

$$x_1 = 0 \quad x_2 = 1/2 \quad x_3 = 0$$

or irrational smooth boundary solutions (mult 1) :

$$x_1 \in [-\tfrac{29707767148026666593}{147573952589676412928}, -\tfrac{29707767148024593931}{147573952589676412928}] \approx -0.2013076605$$

$$x_2 \in [-\tfrac{18765770300641154993}{73786976294838206464}, -\tfrac{18765770300640685591}{73786976294838206464}] \approx -0.2543236116$$

$$x_3 \in [\tfrac{21153099339285995043}{1180591620717411303424}, \tfrac{661034354352767111}{36893488147419103232}] \approx 0.01791737208$$

Renegar's derivative cones

Based on the remark that

f hyperbolic w.r. to e $\Rightarrow$ $D_e f = \sum_i e_i \frac{\partial f}{\partial x_i}$ still hyperbolic

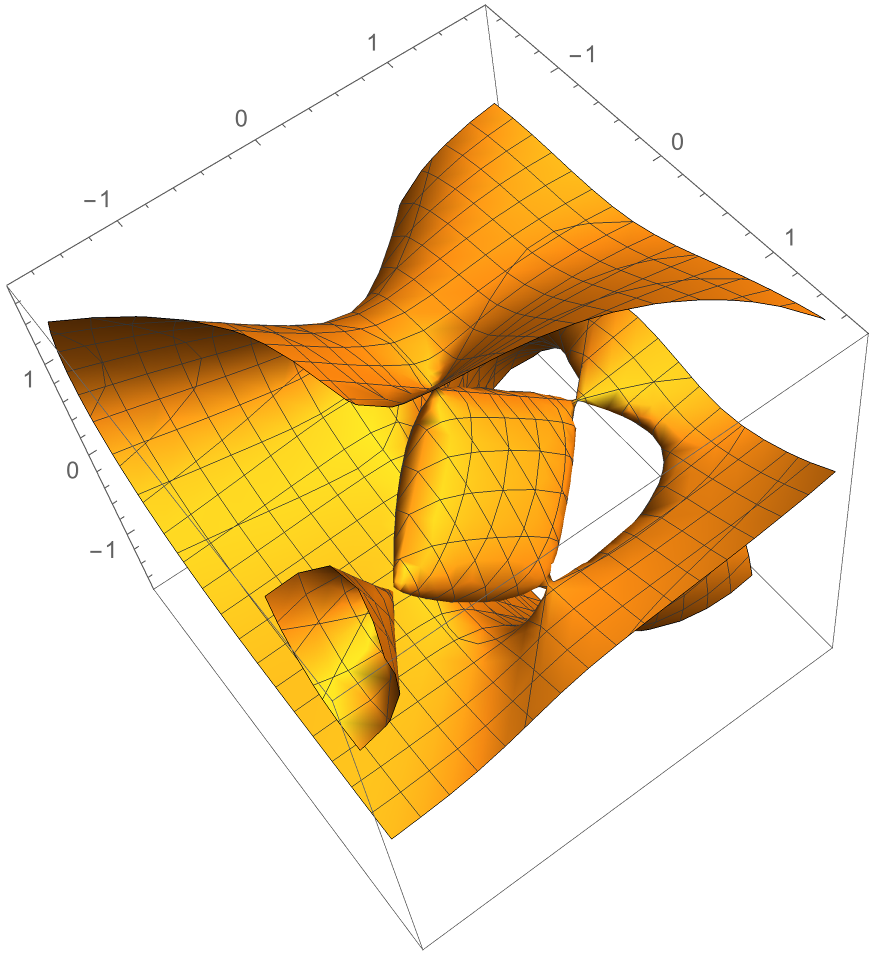
This gives a nested sequence of convex hyperbolicity cones:

$$\mathcal{C}(f, e) \subset \mathcal{C}(D_e f, e) \subset \cdots \subset \mathcal{C}(D_e^{(d-1)} f, e)$$

(the last one being a half-space), giving a sequence of *lower bounds* for the linear function to optimize:

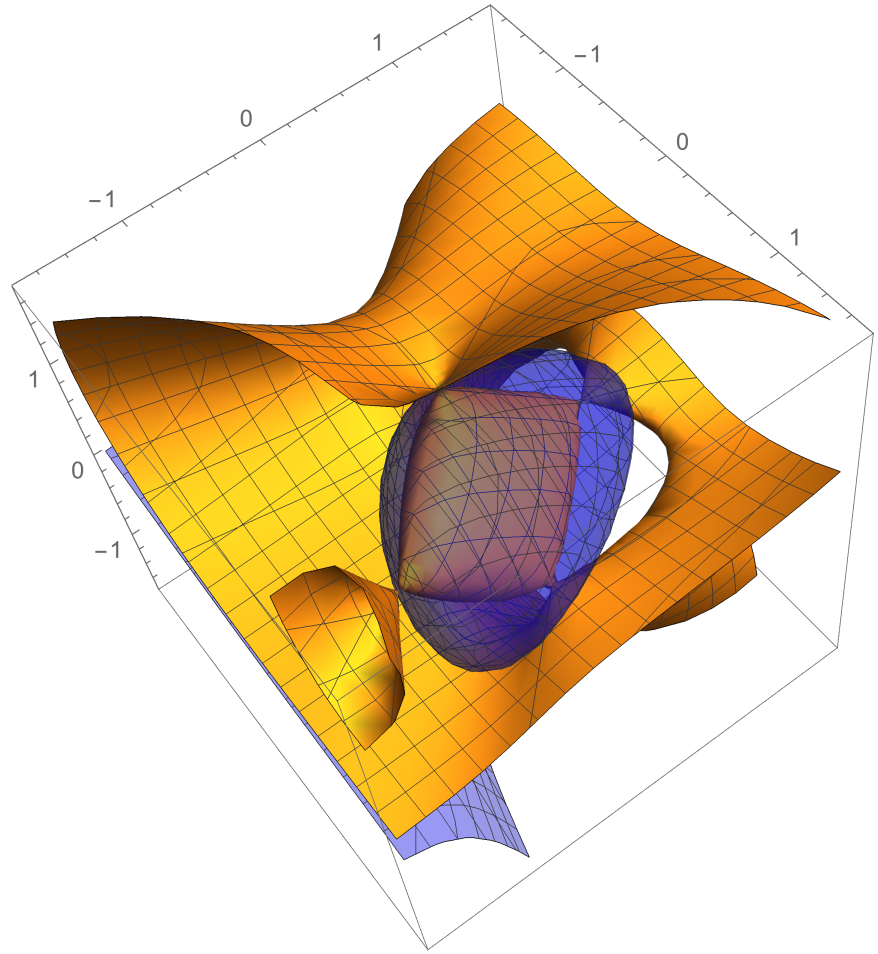
$$\inf_{\mathcal{C}(f,e)} \ell(a) \geq \inf_{\mathcal{C}(D_e f,e)} \ell(a) \geq \cdots \geq \inf_{\mathcal{C}(D_e^{(d-1)} f,e)} \ell(a)$$

The quartic



Quartic hyperbolic polynomial

The quartic



+ First derivative

Renegar's derivative cones (continued)

Why is Renegar's method useful from an effective viewpoint?

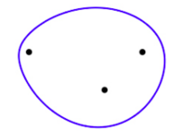
- At each step of the relaxation, the **degree** of the polynomial **decreses** by 1
- The **multiplicity** decreases: the solution becomes "smoother" at each step
- One of the $\mathcal{C}(D_e^{(j)} f, e)$ could be a section of the PSD cone (solution set of a LMI), in which case a lower bound can be computed by solving a *single SDP*:

  Sanyal (2013) : spectrahedral repr. for derivative cones of polyhedra

  Saunderson (2017) : spectrahedral repr. for first derivative of PSD cone

Planar 3−ellipse

Here I minimize a linear function $\ell(x)$ over the 3-ellipse



$$\mathcal{C}(f,e) = \{P \in \mathbb{R}^2 : \mathrm{d}(P,P_1) + \mathrm{d}(P,P_2) + \mathrm{d}(P,P_3) \leq c\}$$

The boundary is given by $f = 9x^8 - 72x^7z + 36x^6y^2 - 96x^6yz - \ldots$

Deriv.	x^*	Mult.	$\ell(x^*)$	deg$(q(t))$	Alg.Deg. of x^*
0	$(0.750, 0.000, 0.250)$	2	5.500000	56	1
1	$(0.759, -0.018, 0.258)$	1	5.499158	42	30
2	$(0.797, -0.051, 0.250)$	1	5.456196	30	26
3	$(0.862, -0.116, 0.254)$	1	5.392044	20	20
4	$(0.981, -0.254, 0.273)$	1	5.292250	12	12
5	$(1.336, -0.762, 0.426)$	1	5.090555	6	6

Still OK for 4−ellipse, becoming hard for $\geq$ 5−ellipse

Summary

Hyperbolic polynomials (resp. HP) is a rich class of real polynomials, defining highly-structured optimization problems.

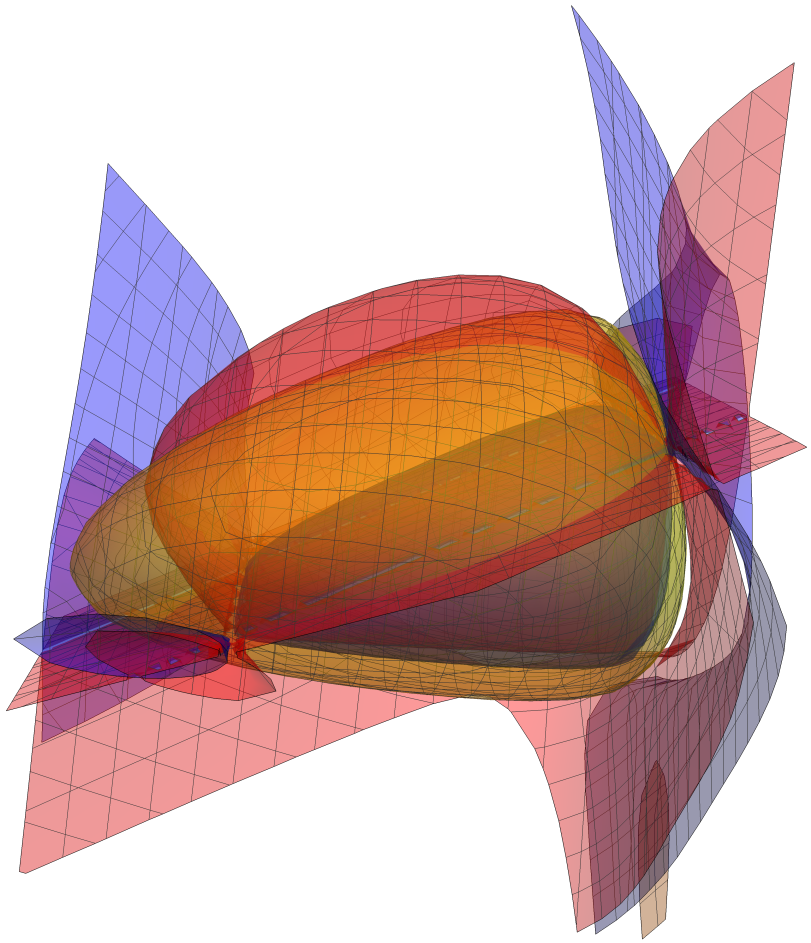
It is important to develop an **effective** approach

to hyperbolic polynomials, independent on their determinantal representability

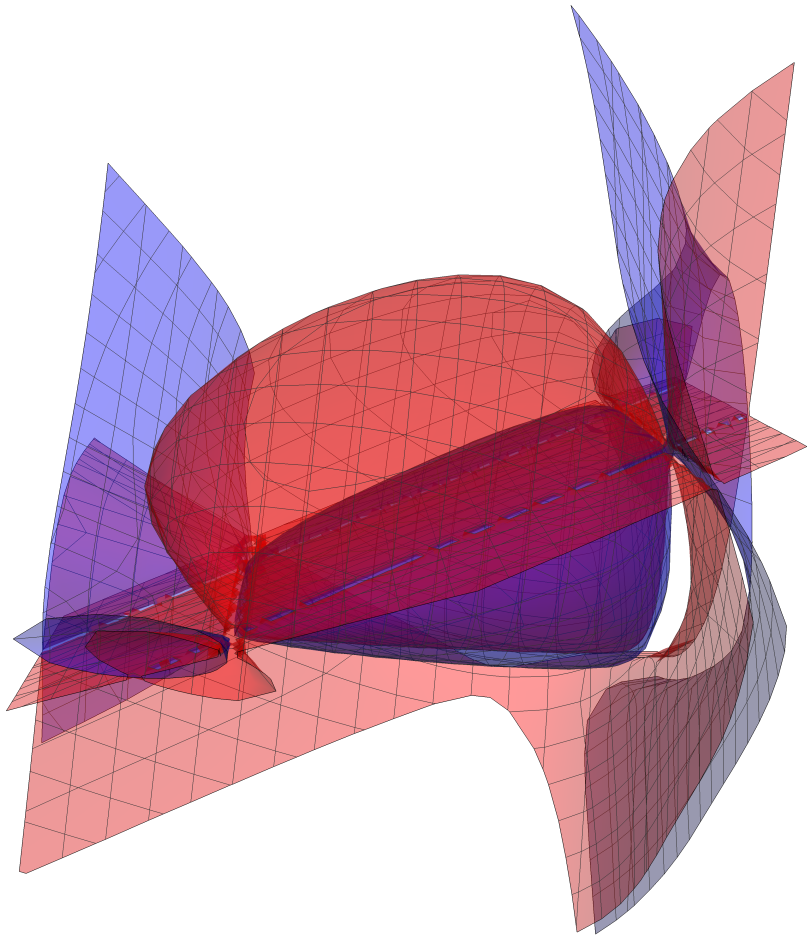
to hyperbolicity cones, independent on Lax conjecture

Algebraic alternatives to interior-point methods for polynomial optimization

Complexity of HP: polynomial in the number of variables with fixed degree? True for generic SDP



Thanks!



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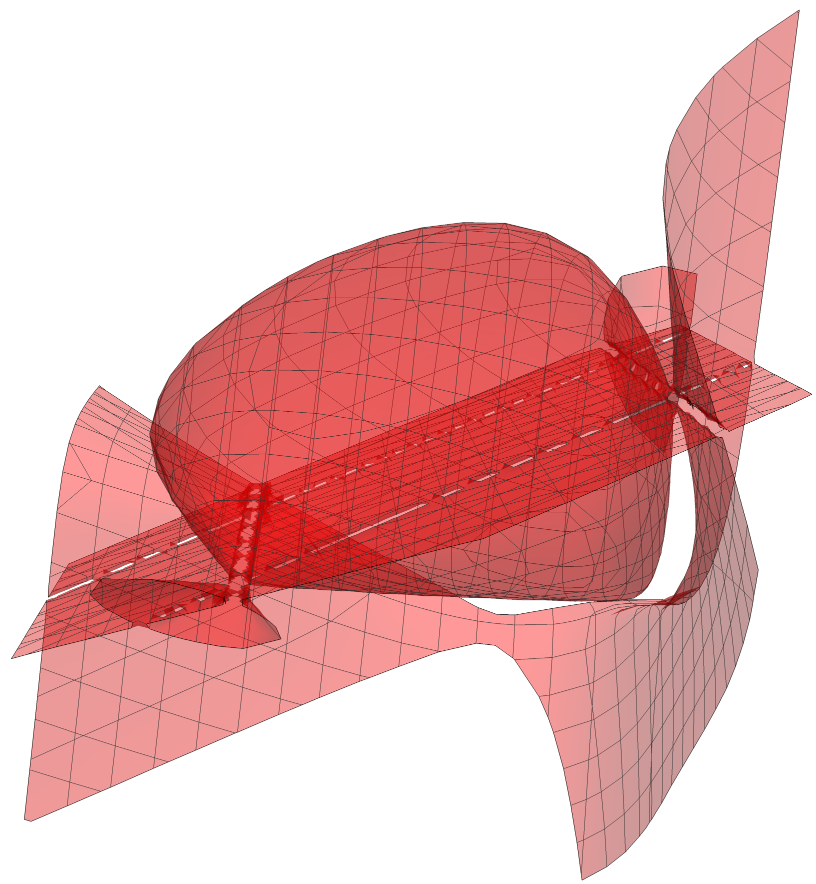

Thanks!