LECTURE 4

We have defined two cones in $\mathbb{R}[x]$:

$$\mathcal{P}(\mathbb{R}^n) = \text{globally positive polynomials}$$
$$\Sigma_n = \text{sums of squares polynomials}$$

and truncated in degree $2d$

$$\mathcal{P}(\mathbb{R}^n)_{2d} = \mathcal{P}(\mathbb{R}^n) \cap \mathbb{R}[x]_{2d}$$
$$\Sigma_{n,2d} = \Sigma_n \cap \mathbb{R}[x]_{2d}$$

<u>Definition</u> Let $C \subset V$ be a cone. The dimension of C is the dimension of its affine hull (the smallest affine space $H \subset V$ containing C).

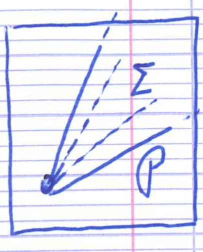
<u>Proposition</u> $\mathcal{P}(\mathbb{R}^n)$ and Σ_n (resp. $\mathcal{P}(\mathbb{R}^n)_{2d}$ and $\Sigma_{n,2d}$) are full-dimensional closed convex cones of $\mathbb{R}[x]$ (resp. of $\mathbb{R}[x]_{2d}$):

$$\dim \mathcal{P}(\mathbb{R}^n) = \dim \Sigma_n = \dim \mathbb{R}[x] = +\infty$$
$$\dim \mathcal{P}(\mathbb{R}^n)_{2d} = \dim \Sigma_{n,2d} = \dim \mathbb{R}[x]_{2d} = \binom{2d+n}{n}.$$

We suppose to work with <u>homogeneous</u> polynomials (for simplicity reasons).

The goal is to construct the dual cones $(\mathcal{P}(\mathbb{R}^n)_{2d})^*$ and $(\Sigma_{n,2d})^*$.

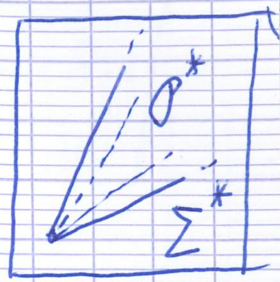
vector space of polynomials



$$\mathbb{R}[x]_{2d} \supseteq \mathcal{P}(\mathbb{R}^n)_{2d} \supseteq \Sigma_{n,2d}$$

dual ~~set~~ vector space

$$\mathbb{R}[x]_{2d}^{\vee} \supseteq (\Sigma_{n,2d})^* \supseteq (\mathcal{P}(\mathbb{R}^n)_{2d})^*$$



The same for the whole space

$$\mathbb{R}[x] \supseteq \mathcal{P}(\mathbb{R}^n) \supseteq \Sigma_n \qquad\qquad \mathbb{R}[x]^{\vee} \supseteq (\Sigma_n)^* \supseteq (\mathcal{P}(\mathbb{R}^n))^*$$

Choose the monomial basis $\langle x^\alpha \rangle_{\alpha \in \mathbb{N}^n_{2d}} = \mathbb{R}[x]_{2d}$

Then $\ell \in \mathbb{R}[x]_{2d}^{\vee}$

$\rightarrow$ $\ell : \mathbb{R}[x]_{2d} \longrightarrow \mathbb{R}$ linear functional

It is determined by its values on x^α:

$$\ell\left(\sum c_\alpha x^\alpha\right) = \sum c_\alpha \ell(x^\alpha) \qquad \alpha \in \mathbb{N}^n_{2d}$$

So if $\ell(x^\alpha) = y_\alpha \in \mathbb{R}$ $\forall \alpha$, then we have the identification of

$$\mathbb{R}[x]_{2d}^{\vee} = \mathbb{R}^{\binom{2d+n}{n}} = \{(y_\alpha)_{\alpha \in \mathbb{N}^n_{2d}} : y_\alpha \in \mathbb{R}\}$$

Analogously

$$\mathbb{R}[x]^\vee = \{ (y_\alpha)_{\alpha \in \mathbb{N}^n} : y_\alpha \in \mathbb{R} \} =: \mathbb{R}^\infty .$$

Let $S^2(\mathbb{R}[x]_d)$ be the set of all quadratic forms over $\mathbb{R}[x]_d$:

$$S^2(\mathbb{R}[x]_d) = \{ Q : \mathbb{R}[x]_d \to \mathbb{R} \ \text{bilinear and symmetric} \}$$

$$S^2(\mathbb{R}[x]_d) \cong \mathrm{Sym}_{\binom{n+d}{d}}(\mathbb{R})$$

We associate

$$\mathbb{R}[x]_{2d}^\vee \longrightarrow S^2(\mathbb{R}[x]_d)$$
$$\ell \mapsto Q_\ell$$

with

$$Q_\ell(f) = \ell(f^2) .$$

Remark What is the matrix of the quadratic form Q_ℓ? For example, consider the functional

$\ell \in \mathbb{R}[x_1, x_2]_4^\vee$
$$\ell : \mathbb{R}[x_1 x_2]_4 \to \mathbb{R}$$
$$f \mapsto f(1,2)$$

evaluation in $(1,2)$ is linear! $(f_1 + f_2)(1,2) = f_1(1,2) + f_2(1,2)$

Monomial basis

$$\langle x_1^2, x_1 x_2, x_2^2 \rangle_{\mathbb{R}} = \mathbb{R}[x_1, x_2]_2$$

represented by the matrix

$$Q_\ell \sim \begin{array}{c} \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{ccc} x_1^2 & x_1x_2 & x_2^2 \end{array} \\ \begin{bmatrix} \ell(x_1^4) & \ell(x_1^3x_2) & \ell(x_1^2x_2^2) \\ \ell(x_1^3x_2) & \ell(x_1^2x_2^2) & \ell(x_1x_2^3) \\ \ell(x_1^2x_2^2) & \ell(x_1x_2^3) & \ell(x_2^4) \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$$

Hankel Matrix

<u>Proposition</u> Let $\ell \in \mathbb{R}[x]_{2d}^\vee$. Then $\ell \in \Sigma_{n,2d}^*$ if and only if Q_ℓ is positive semidefinite.

<u>Proof</u>

$\boxed{\Rightarrow}$ ℓ is positive on sums of squares, in particular $\forall f \in \mathbb{R}[x]_d$

$$0 \leq \ell(f^2) = Q_\ell(f) \Rightarrow Q_\ell \text{ is psd.}$$

$\boxed{\Leftarrow}$ $\forall f \in \mathbb{R}[x]_d \quad Q_\ell(f) = \ell(f^2) \geq 0.$

$\rightarrow \ell(\sum f_i^2) = \sum \ell(f_i^2) \geq 0$ (sum of positive numbers is positive)

$\square$

We conclude that

$$\Sigma_{n,2d}^* = \{ \ell \in \mathbb{R}[x]_{2d}^\vee : Q_\ell \succeq 0 \}$$

It is the feasible set of a SDP program (a Spectrahedron)

Show Lasserre's book.

What about $\mathcal{P}(\mathbb{R}^n)_{2d}^{*}$?

We want functionals $\ell : \mathbb{R}[x]_{2d} \to \mathbb{R}$ that are "positive over positive polynomials".

For instance, point evaluations:

<u>Definition</u> We denote by ℓ_v, for a given $v \in \mathbb{R}^n$, the <u>evaluation</u> on v:

$$\ell_v : \mathbb{R}[x]_{2d} \longrightarrow \mathbb{R}$$
$$f \longmapsto f(v).$$

<u>Exercise</u> Prove that $\ell_v \in \mathbb{R}[x]_{2d}^{\vee}$ $\forall v$.

We define

$$K_{n,2d} = \text{conical hull}\left(\{ \ell_v : v \in \mathbb{R}^n \}\right), \qquad \ell_v : \mathbb{R}[x]_{2d} \to \mathbb{R}$$

conical hull

<u>Theorem</u> $\mathcal{P}(\mathbb{R}^n)_{2d}^{*} = K_{n,2d}$.

<u>Proof</u>

(1) Prove by exercise that $K_{n,2d}$ is a closed convex cone (exercise). We know the same for $\mathcal{P}(\mathbb{R}^n)_{2d}$

(2) Deduce that $K_{n,2d}^{**} = K_{n,2d}$, $\mathcal{P}(\mathbb{R}^n)^{**} = \mathcal{P}(\mathbb{R}^n)$

(3) Hence it is sufficient to prove that

$W = \mathbb{R}[x]_{2d}$ biduality $W \rightsquigarrow W^\vee \rightsquigarrow (W^\vee)^\vee \approx W$

$p \mapsto \; \longmapsto \varphi_p \qquad \varphi_p(\ell) = \ell(p)$

$p \mapsto \varphi_p$ injective $\Rightarrow$ isom.

$$K^*_{n,2d} = \mathcal{P}(\mathbb{R}^n)_{2d}.$$ Indeed, this is true:

$$K^*_{n,2d} = \{ p \in \mathbb{R}[x]_{2d} : \varphi_p(\ell) \geq 0 \ \forall \ell \in K_{n,2d} \} =$$

$(\mathbb{R}[x]_{2d})^{\vee\vee} \simeq \mathbb{R}[x]_{2d}$

canonical isomorphism

$$= \{ p \in \mathbb{R}[x]_{2d} : \varphi_p(\ell_v) \geq 0 \ \forall v \in \mathbb{R}^n \} =$$

because $K_{n,2d}$ is the conical hull of (ℓ_v)

$$= \{ p \in \mathbb{R}[x]_{2d} : \ell_v(p) \geq 0 \ \forall v \} =$$

$$= \{ p \in \mathbb{R}[x]_{2d} : p(v) \geq 0 \ \forall v \in \mathbb{R}^n \} = \mathcal{P}(\mathbb{R}^n)_{2d}$$

$\square$

We deduce that $\mathcal{P}(\mathbb{R}^n)^*_{2d}$ is the conical hull of the point evaluations.

Moment Problem

Moments of measures will be the key to understand what is $\mathcal{P}(\mathbb{R}^n)^*_{2d}$ and to solve our original problem, that is:

$$(SA_0) \quad f^* = \operatorname{Inf} \ f(x)$$

$$\text{s.t. } x \in S(g) = \{ y \in \mathbb{R}^n : g_i(y) \geq 0 \ \forall i = 1..m \}$$

Definition

A <u>Borel measure</u> on $\mathbb{R}^n$ is a function

$\mu : \mathbb{R}^n \to \mathbb{R}$ such that

(•) $\mu(x) \geq 0 \quad \forall x \in \mathbb{R}^n$ (nonnegative)

(•) If $\{E_k\}_{k \in \mathbb{N}}$ is a family of pairwise disjoint sets on which μ is defined, then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$ (countable additivity on disjoint sets).

A <u>σ-algebra</u> on a set X is a family F of subsets of X, such that

(•) $X \in F$

(•) $Y \in F \to X \setminus Y \in F$

(•) $\{Y_k\}_{k \in \mathbb{N}} \subset F \to \bigcup_{k=1}^{\infty} Y_k \in F$.

The ~~sets~~ elements of F are called measurable sets. If X is a <u>topological space</u>, the smallest σ-algebra containing the open sets of X is called the <u>Borel algebra</u> $\mathbb{B}$, and every $Y \in \mathbb{B}$ is called a Borel set.

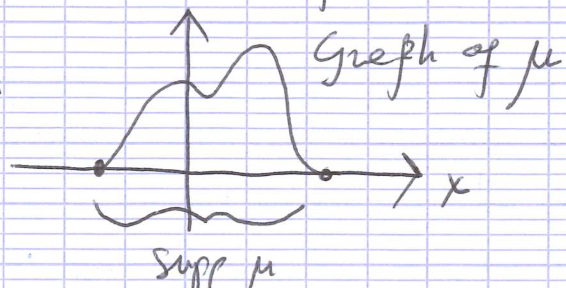
10h30

The <u>support</u> of a measure μ is the complement of the largest open set where μ vanishes:

$$\operatorname{Supp}(\mu) = \mathbb{R}^n \setminus \bigcup_{\substack{A \subset \mathbb{R}^n \\ \text{open set}}} \{A : \mu(A) = 0\}.$$

$\searrow$ closed set!

<u>Example</u>:



<u>Example</u> (Gaussian measure)

Let λ be the Lebesgue measure on $\mathbb{R}^n$. Then the <u>standard Gaussian measure</u> is

$$E \subset \mathbb{R}^n \text{ set}: \quad \mu(E) = \frac{1}{\sqrt[n]{2\pi}} \int_E e^{-\frac{1}{2}\sum x_i^2}\, d\lambda$$

is a Borel measure on $\mathbb{R}^n$, with $\mu(\mathbb{R}^n) = 1$ (probability measure).

$\mu(E)$ probability of the event E. $\qquad \operatorname{Supp}\mu = \mathbb{R}^n$

<u>Example</u> (Dirac's delta)

For $\bar{x}^* \in \mathbb{R}^n$, we denote the Dirac measure with support $\bar{x}^*$ the measure $\delta_{\bar{x}}$ such that

$\forall E \subset \mathbb{R}^n$

$$\delta_{\bar{x}}(E) = \begin{cases} 0 & \text{if } \bar{x} \notin E \\ 1 & \text{if } \bar{x} \in E \end{cases}.$$

For instance $n=1$, $x=2$ then

$$\delta_2((0,1)) = 0, \quad \delta_2((0,3)) = 1, \quad \delta_2(\{5\}) = 0$$
$$\delta_2(\{2,3\}) = 1.$$

Let μ be a measure with $\operatorname{Supp}(\mu) \subset S(g)$ for $S(g)$ a given semialgebraic set.

We fix the monomial basis $\langle x^\alpha \rangle_{\alpha \in \mathbb{N}^n}$ of $\mathbb{R}[x]$. We define the <u>moments</u> of μ as the (infinite sequence)

$$y_\alpha = \int_{S(g)} x^\alpha \, d\mu$$

$$\boxed{\begin{array}{c}\text{Measure with}\\ \text{support in } S(g)\\ \mu\end{array}} \quad \begin{array}{c}\longrightarrow\\ \xleftarrow{?}\end{array} \quad \boxed{\begin{array}{c}\text{Moments of}\\ \text{measures}\\ (y_\alpha)_{\alpha \in \mathbb{N}^n}\end{array}}$$

<u>Moment Problem</u> Given a sequence $(y_\alpha)_{\alpha \in \mathbb{N}^n}$ of real numbers, find a measure μ such that (y_α) is the sequence of its moments

on a given sa-set $S(g)$.

<u>Definition</u> We say that $(y_\alpha)_{\alpha\in\mathbb{N}^n}$ <u>has a representing measure</u> if there exists μ such that $y_\alpha = \int_{S(g)} x^\alpha d\mu \quad \forall \alpha \in \mathbb{N}^n$.

<u>Truncated Moment Problem</u> We substitute $\mathbb{R}[x]_{2d}$ to $\mathbb{R}[x]$ and $\mathbb{N}^n_d$ to $\mathbb{N}^n$, and we have finite sequences.

<u>Special cases of moment problems</u> (MP)

* Hamburger ~~[illegible]~~ MP : $S(g) = \mathbb{R}$
* Stieltjes MP : $S(g) = \mathbb{R}_{\geq 0}$, $y_\alpha \geq 0$.

<u>Q.</u> Why MP is related to SAO ?

<u>Definition</u> For $y = (y_\alpha)_{\alpha\in\mathbb{N}^n} \subset \mathbb{R}$, the <u>Riesz functional</u> $\mathcal{L}_y : \mathbb{R}[x] \to \mathbb{R}$ is defined by

$$\text{For } f = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha \mapsto \mathcal{L}_y(f) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha y_\alpha .$$

$\mathcal{L}$ "linearizes" polynomials.

Theorem (Riesz-Haviland)

Let $y = (y_\alpha)_{\alpha \in \mathbb{N}^n}$, $S \subset \mathbb{R}^n$ be closed. There exists a Borel measure μ on S such that

$$y_\alpha = \int_S x^\alpha \, dx \qquad \forall \alpha \in \mathbb{N}^n$$

if and only if $\mathcal{L}_y(f) \geq 0$ for all

$f \in \mathcal{P}(S)$.

Hence we have ~~another~~ a characterization of the dual cone $(\mathcal{P}(\mathbb{R}^n))^*$ of $\mathcal{P}(\mathbb{R}^n)$:

$$\mathcal{P}(\mathbb{R}^n)^* = \left\{ y = (y_\alpha) \in \mathbb{R}^\infty : \begin{array}{l} \exists \mu \text{ such that} \\ y_\alpha = \int x^\alpha \, d\mu \ \forall \alpha \end{array} \right\}$$

the same for $\mathcal{P}(S)^*$ ($\operatorname{supp}(\mu) \subset S$).

A second link with SAO comes by looking at the problem:

$$\text{(GMP)} \qquad \begin{array}{rl} m^* = & \inf \displaystyle\int_{S(g)} f \, d\mu \\ \text{s.t.} & \operatorname{supp} \mu \subset S(g) \\ & \displaystyle\int_{S(g)} d\mu = 1 \end{array}$$

We recall the SAO

$$(SAO) \quad f^* = \operatorname{Inf} f(x) \quad \text{s.t. } x \in S(g).$$

<u>Theorem</u> With the notation above, $f^* = m^*$.

<u>Proof</u>

(1) $f^* = -\infty$, then ~~we choose~~ for any M such that we have an $\bar{x} \in S(g)$ $f(\bar{x}) \leq M$, ~~thus for a given $\bar{x} \in S(g)$~~. Then $\delta_{\bar{x}}$ is an admissible measure and

$$\int_{S(g)} f \, d\delta_{\bar{x}} = f(\bar{x}) \leq M$$

$\rightarrow m^* = -\infty$.

(2) $f^* > -\infty$, $f^* \in \mathbb{R}$. Since $f(x) \geq f^*$ $\forall x \in S(g)$, then $\int_{S(g)} f \, d\mu \geq \int_{S(g)} f^* d\mu =$

$= f^* \int_{S(g)} d\mu = f^*$ $\quad \forall \mu$ such that $\int_{S(g)} d\mu = 1$.

Hence $m^* \geq f^*$.

Viceversa, for $\bar{x} \in S(g)$, we consider $\delta_{\bar{x}}$, which is a feasible measure:

$m^* \leq \int_{S(g)} f \, d\delta_{\bar{x}} = f(\bar{x})$ $\quad$ passing to Inf we have $m^* \leq f^*$ $\square$