Proposition Let $f \in \mathbb{R}[x]_{2d}$ be homogeneous of degree $2d$. If $f \in \Sigma_{n,2d}$ then f is a sum of squares ~~of~~ of homogeneous polynomials of deg d, that is

$$f = f_1^2 + \dots + f_r^2 \Rightarrow f_i \text{ homogeneous of degree } d.$$

Proof

Exercise.

Recall from Lecture 2:

~~Theorem~~

Definition Given $f \in \mathbb{R}[x]_d$, $x = x_1 \dots x_n$, we call

$$f^h(x_0, x) = x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

the homogenization of f. Remark that $f^h \in \mathbb{R}[x_0 x_1 \dots x_n]_d$ is homogeneous of degree d.

Example

$$f = 3x_1^2 x_2 - x_3^5 + \sqrt{2}\, x_4 x_1 x_2 + 1 \quad (d=5)$$

$$f^h = 3x_0^2 x_1 x_2 - x_3^5 + \sqrt{2}\, x_0^2 x_4 x_1 x_2 + x_0^5.$$

Exercise A polynomial $f \in P(\mathbb{R}^n)$ iff $f^h \in P(\mathbb{R}^{n+1})$. Is it still true if we replace positivity ($f \geq 0$) by strict positivity ($f > 0$)?

Theorem (Hilbert 1888, homogeneous version)

$f \in \mathbb{R}[x_0, x_1 \cdots x_n]$ homogeneous of degree $2d$. Then f is a sum of squares (of polynomials of degree d) if and only if

(1) $n = 1$ (binary forms $\mathbb{R}[x_0, x_1]$)

(2) $d = 1$ (quadratic forms $\mathbb{R}[x_0 x_1 \cdots x_n]_2$)

(3) $n = 2, d = 2$ (plane quartics $\mathbb{R}[x_0 x_1 x_2]_4$).

Recall from last lecture:

Theorem (Artin, 1927) Every nonnegative polynomial is a sum of squares of rational functions:

$$f \in P(\mathbb{R}^n) \Rightarrow f = \sum_{i=1}^{r} \left(\frac{g_i}{h_i}\right)^2 \quad g_i, h_i \in \mathbb{R}[x]$$

Remark

(a) Clearing denominators, we can re-write it:

$$hf = g$$

with $h, g \in \Sigma_{n,2D}$

(b) The degree D could be higher than the degree of f!

(c) If we fix a maximal D, then we can compute such certificate via SDP such as for SOS-certificates!

The Motzkin polynomial

<u>Example</u> $M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is positive but not SOS (exercise). However

$$(x^2+y^2)\cdot M(x,y) = y^2(1-x^2)^2 + x^2(1-y^2)^2 + x^2y^2(x^2+y^2-2)^2$$

is a sum of squares.

Hence $M(x,y) = \frac{f_1^2}{x^2+y^2} + \frac{f_2^2}{x^2+y^2} + \frac{f_3^2}{x^2+y^2} =$

$$= \frac{((x^2+y^2)f_1)^2}{(x^2+y^2)^2} + \dots$$

Sum of squares of rational functions!

<u>Proof of (c)</u>

$hf = g$ with $h, g \in \Sigma_{n,2D}$ $\iff$ $\exists\ Q_h, Q_g \succeq 0$ such that

$$\underbrace{v_D(x)^T Q_h v_D(x) \cdot f}_{} = v_D(x)^T Q_g v_D(x)$$

this one is a linear system in Q_f and Q_g.

Positivity over intervals $I \subset \mathbb{R}$

Q. What is $\mathcal{P}(I)$?

Intervals can have three forms:

$$I = (-\infty, b], \ [a, b] \ \text{or} \ [a, +\infty)$$

However, we can apply affine transformation $\varphi : \mathbb{R} \to \mathbb{R}$ (such as translations) of type

$$\tilde{f}(x) = f(b - x)$$

$$\tilde{f}(x) = f(x - a)$$

$$\tilde{f}(x) = f\left(\frac{2x - (a+b)}{b - a}\right)$$

to reduce to the cases

$$I = [0, +\infty) \ \text{and} \ I = [-1, 1].$$

Remark

$$[-1, 1] = S(g) \quad \text{with} \quad g = (1 - x^2)$$

$$\text{but also} \quad g = (1 - x, 1 + x)$$

Theorem $f \in \mathbb{R}[x]_d$, $I = [-1, 1]$. ($x = (x_1)$)

1. Let $g = 1 - x^2$. Then $f \in \mathcal{P}(I)$ iff

$$f = \sigma_0 + \sigma_1 g \quad \text{with} \quad \sigma_i \in \Sigma_{n,D}$$

with $D \leq 2d$. (more precisely $\deg \sigma_0 \leq 2d$, $\deg \sigma_1 \leq 2d - 2$)

2. Let $g_1 = 1-x$, $g_2 = 1+x$. Then $f \in P(I)$ iff

$$f = \sigma_0 + \sigma_1 g_1 + \sigma_2 g_2 + \sigma_3 g_1 g_2 \qquad \sigma_i \in \Sigma_{n,D}$$

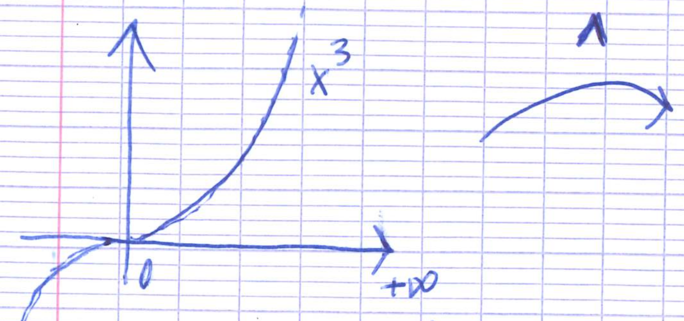
all summands here degree $\leq d$, and

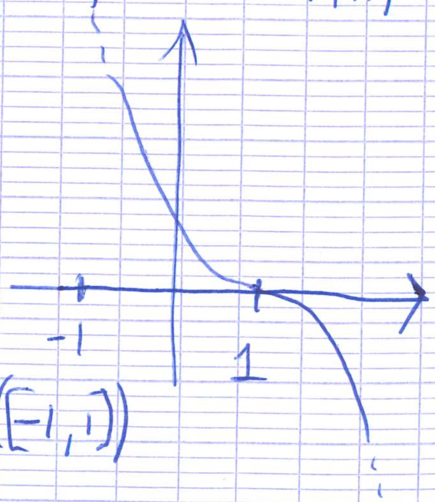
* $\sigma_1 = \sigma_2 = 0$ if d is even
* $\sigma_0 = \sigma_3 = 0$ if d is odd.

<u>Definition</u> Given $f \in \mathbb{R}[x]_d$ (univariate) we define $\hat{f} \in \mathbb{R}[x]$ as follows:

$$\hat{f}(x) = (1+x)^d f\left(\frac{1-x}{1+x}\right)$$

<u>Example</u> $f(x) = x^3$ $\qquad \hat{f}(x) = (1+x)^3 \left(\frac{1-x}{1+x}\right)^3 = (1-x)^3$





$f \in P([0,+\infty))$ $\longrightarrow$ $\hat{f} \in P([-1,1])$

Indeed, this always holds:

<u>Exercise</u> $f \in P([0,+\infty)) \iff \hat{f} \in P([-1,1])$ and $\deg \hat{f} = \deg f$

~~deg~~

Nonnegative Polynomials that are <u>not</u> SOS

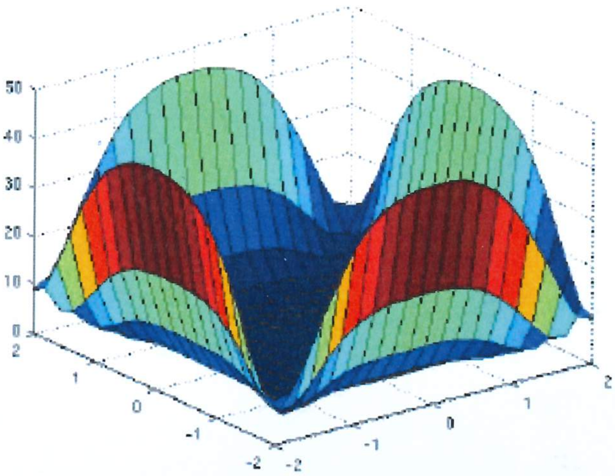


Figura 2.1: Robinson R$(x, y, 1)$

we put $z=1$

$$R(x,y,z) = x^6+y^6+z^6-\left(x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4\right)+$$
$$+3x^2y^2z^2$$

Nonnegative Polynomials that are not SOS

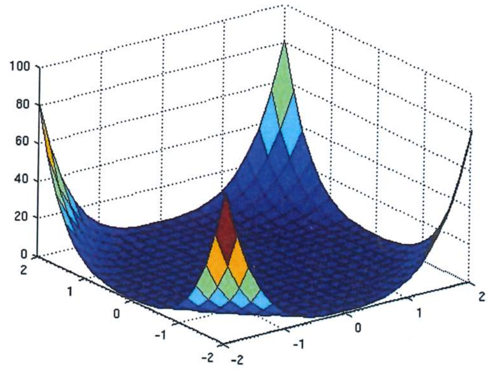


Figura 2.3: Motzkin $\mathrm{M}_3(x, y, 1)$

$z=1$

$$M_3(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

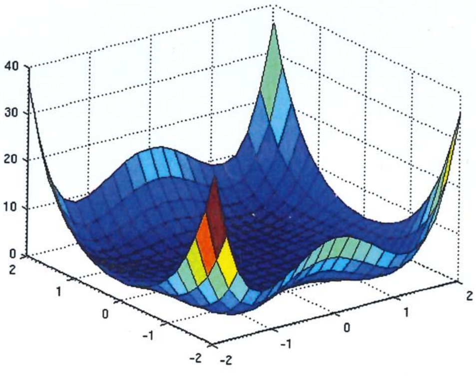


Figura 2.4: Choi-Lam $\mathrm{S}(x, y, 1)$

$z=1$

$$S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$$

(Pólya-Szegö) → **Proof** *

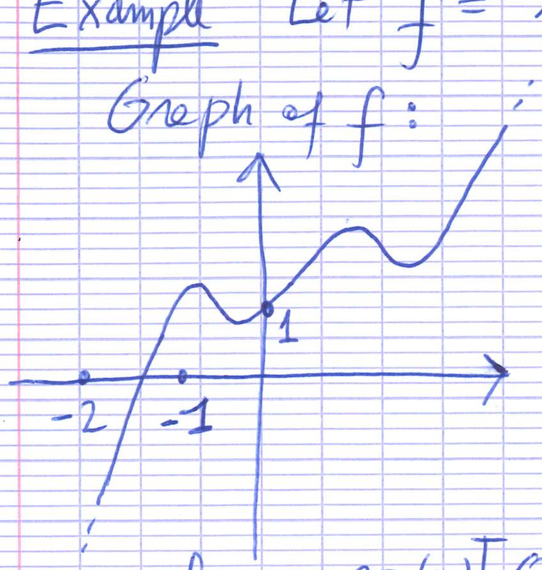
Theorem Let $f \in \mathcal{P}_d([0,+\infty))$. Then

$$f = \sigma_0 + \sigma_1 \cdot x \qquad \sigma_i \in \Sigma_{n,D}$$

and the degree of both summands is at most d. PROOF SUL FOGLIO!

Example Let $f = x^5 - 2x^3 + x + 1$. How we compute σ_0, σ_1 such that $f = \sigma_0 + \sigma_1 \cdot x$?

Graph of f:



We have that $\deg \sigma_i \leq 4$ (≤ 5, but σ_i are SOS).

$$\rightarrow f = v_2(x)^T Q_0\, v_2(x) + x \cdot v_2(x)^T Q_1\, v_2(x).$$

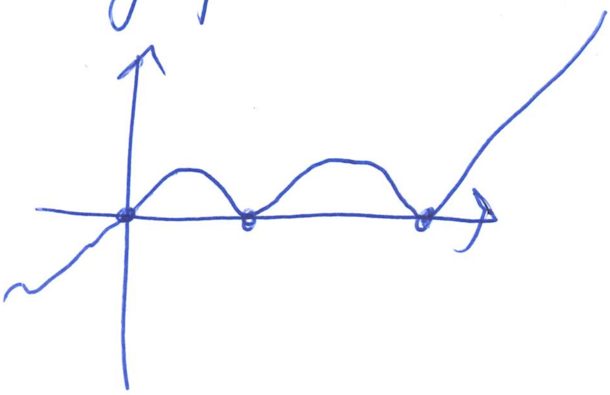
We solve this <u>linear</u> system and then we compute (with MATLAB) two matrices (solving a SDP problem):

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

7h30 $\rightarrow f = 1 + x \cdot (x^2 - 1)^2$

Proof (Polya-Szegö)

As $f \geq 0$ on $[0,\infty)$, every real positive root has even multiplicity and complex roots come in conjugated pairs



Then $f = s_0 \cdot x^m (x+a_1)\cdots(x+a_t)$

with $a_i > 0$, $s_0 \in \Sigma_{n,D}$ and $m \in \{0,1\}$.

Then since $(x+a_1)\cdots(x+a_t) = s_1 + x s_2$

with $s_i \in \Sigma$

the result follows.

positive coefficients → We can separate even powers from odd powers and get $s_1 + x s_2$.

$$(x+1)(x+2) = x^2 + 3x + 2 = (x^2+2) + x \cdot 3$$

$$(x+1)(x+2)(x+3) = x^3 + 3x^2 + 2x + 3x^2 + 9x + 6 =$$

$$= 6x^2 + 6 + x(x^2 + 11).$$

2. Let $g_1 = 1-x$, $g_2 = 1+x$. Then $f \in \mathcal{P}(I)$ iff

$$f = \sigma_0 + \sigma_1 g_1 + \sigma_2 g_2 + \sigma_3 g_1 g_2 \qquad \sigma_i \in \Sigma_{n,D}$$

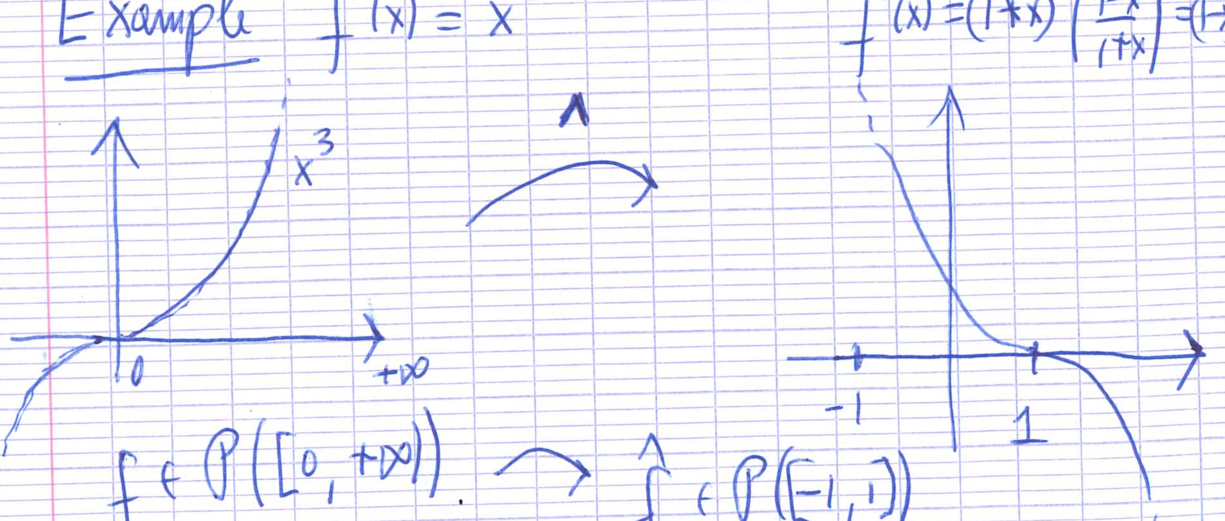
all summands have degree $\leq d$, and

* $\sigma_1 = \sigma_2 = 0$ if d is even
* $\sigma_0 = \sigma_3 = 0$ if d is odd.

Definition Given $f \in \mathbb{R}[x]_d$ (univariate) we define $\hat{f} \in \mathbb{R}[x]$ as follows:

$$\hat{f}(x) = (1+x)^d f\left(\frac{1-x}{1+x}\right)$$

Example $f(x) = x^3$ $\qquad \hat{f}(x) = (1+x)^3 \left(\frac{1-x}{1+x}\right)^3 = (1-x)^3$



$f \in \mathcal{P}([0,+\infty)) \curvearrowright \hat{f} \in \mathcal{P}([-1,1])$

Indeed, this always holds:

Exercise $f \in \mathcal{P}([0,+\infty)) \iff \hat{f} \in \mathcal{P}([-1,1])$ and $\deg \hat{f} = \deg f$

(Pólya-Szegö) $\longrightarrow$ **Proof** *

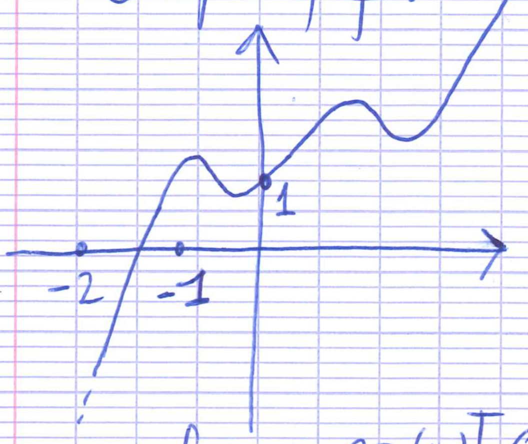
<u>Theorem</u> Let $f \in \mathcal{P}_d([0,+\infty))$. Then

$$f = \sigma_0 + \sigma_1 \cdot x \qquad \sigma_i \in \Sigma_{n,D}$$

and the degree of both summands is at most d. PROOF SUL FOGLIO!

<u>Example</u> Let $f = x^5 - 2x^3 + x + 1$. How we compute σ_0, σ_1 such that $f = \sigma_0 + \sigma_1 \cdot x$? We have that $\deg \sigma_i \leq 4$ (≤ 5, but σ_i are SOS).

Graph of f:



$$\rightarrow f = v_2(x)^T Q_0 v_2(x) + x \cdot v_2(x)^T Q_1 v_2(x).$$

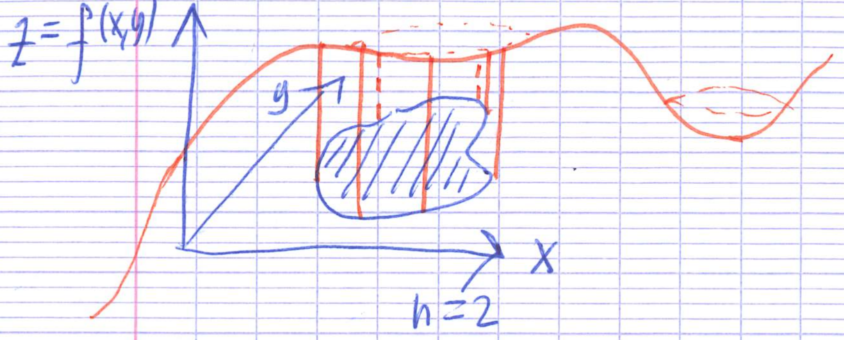
We solve this <u>linear</u> system and then we compute (with MATLAB) two matrices (solving a SDP problem):

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

7h30 $\rightarrow f = 1 + x \cdot (x^2-1)^2$

Multivariate case

We consider the more general case $f \in P(S(g))$ when $S(g) = \{x \in \mathbb{R}^n : g_i(x) \geq 0 \ \forall i = 1 \dots m\}$.



Q. What are the certificates for $f \in P(S(g))$?

Suppose $S(g) = \mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^k : x_i \geq 0 \ i = 1 \dots n\}$

$$\left(\text{for instance } g = (\underset{g_1(x)}{x_1}, x_2, \dots, \underset{g_n(x)}{x_n}) \right)$$

__Theorem__ (Polyá)

If f is homogeneous (and $f \in P(\mathbb{R}^n_{\geq 0})$,) and if $f > 0$ on $(\mathbb{R}^n_{\geq 0} \setminus \{0\})$ (strictly positive in the interior of $\mathbb{R}^n_{\geq 0}$) then for $k \gg 0$ (large enough) the coefficients of

$$(x_1 + \dots + x_n)^k f(x) \quad \text{are nonnegative.}$$

Given $g = (g_1 \cdots g_m)$ we define

For $J \subset \{1..m\} \to g_J = \prod_{i \in J} g_i$

If $J = \phi \to g_\phi = 1$.

We define the PREORDERING generated by g:

$$P(\underline{g}) = \left\{ \sum_{J \subset \{1..m\}} \sigma_J g_J : \sigma_J \text{ are SOS} \right\}$$

<u>Remark</u> $P(g) \subset \mathcal{P}(S(g))$

<u>Proof with students</u>

$f = \sum_{J \subset \{1..m\}} \sigma_J g_J$ positive if g_i are all positive. $\square$

<u>Remark</u> If $S(g) = \mathbb{R}^n$, what is $P(g)$?

For instance $\mathbb{R}^n = S(1) = \{ x \in \mathbb{R}^n : 1 \geq 0 \}$.

In this case $P(1) = \Sigma_n := \bigcup_{d=0}^{\infty} \Sigma_{n,2d}$.

One of the first results about certificates of positivity over $S(g)$ is using $P(g)$:

Theorem (Stengle) Let $f \in \mathbb{R}[x]$ and let $S(g) = \{x \in \mathbb{R}^n : g_i(x) \geq 0,\ i = 1..m\}$.

(1) NICHT NEGATIVE STELLEN SATZ

$$f \in P(S(g)) \text{ iff } \exists\, l \in \mathbb{N}\ \exists\, f_1, f_2 \in P(g)$$
$$\text{s.t. } f \cdot f_1 = f^{2l} + f_2$$

(2) POSITIVSTELLEN SATZ

$$f > 0 \text{ on } S(g) \text{ iff } \exists\, f_1\, f_2 \in P(g)$$
$$\text{s.t. } f \cdot f_1 = 1 + f_2$$

(3) NULLSTELLENSATZ

$$f = 0 \text{ on } S(g) \text{ iff } \exists\, l \in \mathbb{N}, \exists\, f_1 \in P(g)$$
$$\text{such that } f^{2l} + f_1 = 0.$$

Remark Again, these certificates can be computed via SDP solves, when the degree of f_1, f_2 are fixed.

Compact semi-algebraic sets

Recall: a set $S \subset \mathbb{R}^n$ is compact (in the Euclidean topology) when it is closed and bounded ($S \subset B(0,R)$ for ~~...~~ large R).

We look at polynomials $f \in \mathcal{P}(S(g))$ for $S(g) \subsetneq \mathbb{R}^n$ compact.

Theorem (Schmüdgen's Positivstellensatz)

Let $g = (g_1 \dots g_m) \in \mathbb{R}[x]$ be such that $S(g)$ is compact. If $f > 0$ on $S(g)$ then $f \in P(g)$, that is

$$f = \sum_{J \subset \{1..m\}} \sigma_J g_J \quad \text{for some } \sigma_J \in \Sigma_n$$

Hence we have for the moment

$\Rightarrow$ $f \in P(g) \Longrightarrow f \in \mathcal{P}(S(g))$

$\Rightarrow$ $f > 0$ on $S(g) \Longrightarrow f \in P(g)$ (with $S(g)$ compact)

$\Rightarrow$ Putinar's Positivstellensatz

Definition QUADRATIC MODULE

$$Q(g) = \{ \sigma_0 + \sum_{j=1}^{m} \sigma_j g_j : \sigma_j \text{ are sos} \}$$

Assumption (Archimedianity)

There exists $u(x) \in Q(g)$ such that

$$S(u) = \{ x \in \mathbb{R}^n : u(x) \geq 0 \} \text{ is compact.}$$

If this is true, $Q(g)$ is called archimedean.

Remark

(1) $Q(g)$ archimedean $\Rightarrow$ $S(g)$ compact

(2) $S(g)$ compact, $S(g) \subset B(0, R)$ $\exists R \in \mathbb{R}_{\geq 0}$,

then we define $u(x) = R^2 - \|x\|^2 =$

$= R^2 - x_1^2 - x_2^2 - \dots - x_n^2$.

Clearly $S(u) = B(0,R)$ is compact!

So, to make $Q(g)$ archimedean, it suffices to add u in the description of $S(g)$:

$$S(g) = S(g, u) = \{ x \in \mathbb{R}^n : g_i(x) \geq 0, i=1..m, u(x) \geq 0 \}$$

$\rightarrow$ $Q(\underbrace{g, u}_{\ni u})$ is archimedean, with the new description.

So, compactness of $S(g)$ and archimedianity of $Q(g)$ is equivalent, up to changing description of $S(g)$.

<u>Theorem</u> (Putinar's Positivstellensatz)

Suppose that $Q(g)$ is archimedean. If $f > 0$ on $S(g)$, then $f \in Q(g)$ that is

$$f = \sigma_0 + \sum_{j=1}^{m} \sigma_i g_i \qquad \sigma_i \in SOS.$$

In terms of "complexity": # of σ_i

- Schmüdgen $f = \sum \sigma_J g_J$ EXPONENTIAL in m
- Putinar $f = \sigma_0 + \sum \sigma_i g_i$ LINEAR in m.

<u>Degree bound for Putinar</u>

Defined $\|f\|_0 = \max_{\alpha} |f_\alpha| \frac{\alpha_1! \dots \alpha_n!}{|\alpha|!}$

$$f^* = \operatorname{Inf}_{x \in S(g)} f(x)$$

$$\rightarrow \deg(\sigma_j g_j) \leq c \cdot \exp\left(\left(d^2 n^d \frac{\|f\|_0}{f^*}\right)^c\right)$$

(doubly exponential in d
simply exponential in n)

(with degree bounds)

<u>Remark</u> Whether $f \in Q(g)$ or not, can be tested via semidefinite programming.

As in the "global" case, we define matrices for the quadratic forms

$g_0 = 1$

$$g_j(x) \cdot v_{d-d_j}(x)\, v_{d-d_j}(x)^T = \sum_{\alpha \in \mathbb{N}^n_{2d}} C^j_\alpha x^\alpha$$

Here $j = 0 .. m$

d = degree ~~~~ bound for $\sigma_i g_i$

$d_j = \left\lceil \frac{\deg g_j}{2} \right\rceil$ $\rightsquigarrow$ $\lceil 1.5 \rceil = 2$

$i = 1..m$ $\quad \lceil 2 \rceil = 2$

$f = \sum_\alpha f_\alpha x^\alpha$

$$f = \sum_{i=0}^m \sigma_i g_i \qquad d_0 = \deg f$$

We need to compute symm. matrices $X_0 .. X_m$ such that

This is a SDP feasibility problem $\Leftarrow$

$$\begin{cases} X_j \succcurlyeq 0, & j = 0 \cdots m \\ f_\alpha = \sum_{j=0}^m \langle C^j_\alpha, X_j \rangle & \forall \alpha \in \mathbb{N}^n_{d_0} \\ 0 = \sum_{j=0}^m \langle C^j_\alpha, X_j \rangle & \forall \alpha \in \mathbb{N}^n_{2d},\ |\alpha| > d_0 \end{cases}$$

Suppose we have a solution $(X_0, \ldots, X_m)$:

$$f(x) = \sum_{\alpha \in \mathbb{N}^n_{d_0}} f_\alpha x^\alpha = \sum_{j=0}^{m} \left\langle \left(\sum_\alpha c^j_\alpha x^\alpha\right), X_j \right\rangle =$$

$$= \sum_{j=0}^{m} g_j(x) \underbrace{\left\langle v_{d-d_j}(x) v_{d-d_j}(x)^T, X_j \right\rangle} =$$

$$= \sum_{j=0}^{m} g_j(x) \left(\text{SOS poly} \right)$$

Indeed, suppose have the spectral decomposition

$$X_j = \sum_{t=1}^{s_j} q_{jt} q_{jt}^T \succeq 0$$

$$\rightarrow \left\langle v_{d-d_j}(x) v_{d-d_j}(x)^T, \sum_{t=1}^{s_j} q_{jt} q_{jt}^T \right\rangle =$$

$$= \sum_{t=1}^{s_j} \underbrace{\left\langle v_{d-d_j}(x) v_{d-d_j}(x)^T, q_{jt} q_{jt}^T \right\rangle}_{\text{Trace}(\cdots)} =$$

$$= \sum_{t=1}^{s_j} \text{Trace}\left(v_{d-d_j}(x)^T q_{jt} \cdot q_{jt}^T \cdot v_{d-d_j}(x) \right) =$$

9h00

$$= \sum_t \left(q_{jt}^T v_{d-d_j}(x) \right)^2 \in \Sigma_{n, d-d_j} \; !$$