On Taylor polynomials of rational functions

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(joint with A. Conca, G. Ottaviani and B. Sturmfels)
Introduction

\[
\frac{1 + x + x^2}{1 - x - x^2} = 1 + 2x + 4x^2 + 6x^3 + 10x^4 + 16x^5 + O(x^6)
\]

Indeed:

\[
(1 - x - x^2)(1 + 2x + 4x^2 + 6x^3 + 10x^4 + 16x^5) = 1 + x + x^2 - 26x^6 - 16x^7.
\]

\[
\frac{1 + p_1x + p_2x^2}{1 + q_1x + q_2x^2} = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + O(x^6)
\]

where \( c_1, c_2, c_3, c_4, c_5 \) are (polynomial) functions of \( p_1, p_2, q_1, q_2 \).

4 parameters, 5 target coefficients \( \Rightarrow \) 1 relation ?
Introduction

From the equality modulo $x^6$ one gets

\[c_1 = p_1 - q_1\]
\[c_2 = p_1 q_1 - q_1^2 - p_2 + q_2\]
\[c_3 = p_1 q_1^2 - q_1^3 - p_1 q_2 - p_2 q_1 + 2 q_1 q_2\]
\[c_4 = p_1 q_1^3 - q_1^4 - 2p_1 q_1 q_2 - p_2 q_1^2 + 3 q_1^2 q_2 + p_2 q_2 - q_2^2\]
\[c_5 = p_1 q_1^4 - q_1^5 - 3p_1 q_1^2 q_2 - p_2 q_1^3 + 4 q_1^3 q_2 + p_1 q_2^2 + 2 p_2 q_1 q_2 - 3 q_1 q_2^2\]

eliminating the parameters, one gets the \textit{cubic} equation:

\[c_1 c_3 c_5 - c_1 c_4^2 - c_2^2 c_5 + 2 c_2 c_3 c_4 - c_3^3 = 0\]

\[
\begin{vmatrix}
c_5 & c_4 & c_3 \\
c_4 & c_3 & c_2 \\
c_3 & c_2 & c_1 \\
\end{vmatrix} = c_1 c_3 c_5 - c_1 c_4^2 - c_2^2 c_5 + 2 c_2 c_3 c_4 - c_3^3 = 0
\]
Taylor varieties

Fix $n$ variables $x = (x_1, \ldots, x_n)$, and degrees $(d, e)$ and $m$:

$$C(x) \ni \frac{P(x)}{Q(x)} = \sum_{|\gamma| \leq m} c_\gamma x^\gamma + \langle x_1, \ldots, x_n \rangle^{m+1}$$

with $Q(0, \ldots, 0) = 1$.

The (Zariski) closure in $\mathbb{P}^{(n+m)-1}$ of the set of all Taylor polynomials of degree $\leq m$ of rational functions of degree $\leq (d, e)$ in $n$ variables is called the **Taylor variety** and denoted $T^n_{d,e,m}$.

**Main questions:** dimension, defining equations, hypersurfaces...
Main motivation: Padé Approximation

Central question of Approximation Theory and Computer Algebra.

Given a function $f$, known through its Taylor approximation up to some order $m$, find two polynomials $p, q$ of degree $d, e$, such that

$$\frac{p}{q} = f \mod x^{m+1}$$

More generally given $f_1, \ldots, f_s$, find $(p_1, \ldots, p_s)$ such that

$$p_1f_1 + \cdots + p_sf_s = 0 \mod I \quad \leadsto \text{syzygy modules}$$

The geometry of Taylor varieties is related to

- **existence** of solutions: is there such a rational function approximation?
- **uniqueness/identifiability**: how many rational function approximations?
Dimension

The Taylor variety $\mathcal{T}_{d,e,m}^n \subset \mathbb{P}^{(n+m)-1}$ is irreducible, as (closure of the) image of the following morphism:

$$
\psi : \mathbb{C}^{(n+d)} \times \mathbb{C}^{(n+e)} \to \mathbb{C}^{(n+m)}
$$

$$(P, Q) \mapsto T := \sum_{|\gamma| \leq m} c_\gamma x^\gamma$$

Its dimension is bounded by the expected dimension:

$$\dim(\mathcal{T}_{d,e,m}^n) \leq \exp \dim(\mathcal{T}_{d,e,m}^n) := \min \left\{ \binom{n+m}{m} - 1, \binom{n+d}{d} + \binom{n+e}{e} - 2 \right\}$$

$\downarrow$

this inequality can be strict when $n \geq 2$. 

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**Univariate case** \( (n = 1) \)

Fix \( d, e \in \mathbb{N} \), and let \( T = 1 + c_1 x + c_2 x^2 + \cdots + c_m x^m \in \mathbb{C}[x] \).

We define the \((m - d) \times (e + 1)\) *univariate* Padé Matrix:

\[
M_T = \begin{bmatrix}
  c_m & c_{m-1} & \cdots & c_{m-e} \\
  c_{m-1} & c_{m-2} & \cdots & c_{m-e-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{d+2} & c_{d+1} & \cdots & c_{d-e+2} \\
  c_{d+1} & c_d & \cdots & c_{d-e+1}
\end{bmatrix}
\]

- The Taylor variety \( \mathcal{T}_{d,e,m}^1 \subset \mathbb{P}^m \) has the expected dimension \( \min\{d + e, m\} \).
- If \( d + e \geq m \), then \( \mathcal{T}_{d,e,m}^1 = \mathbb{P}^m \) (in words, every degree-\( m \) polynomial is a Taylor polynomial of a rational function of degree \( \leq (d, e) \)).
- If \( d + e < m \), its defining ideal is the (prime) ideal of \( e + 1 \) minors of \( M_T \), and it equals some *secant variety* of the *rational normal (moment) curve* of degree \( m \).
Multivariate Padé Matrix

Define $M_{d+1,m} := \text{monomials}^1$ in $x$ of degree between $d + 1$ and $m$.

The Padé Matrix $M_T$ constructed before is the matrix of the linear map

$$
\mathbb{C}[x]_{\leq e} \rightarrow \mathbb{C}[x]_{\leq e + m} \rightarrow \mathbb{C}\{M_{d+1,m}\},
Q \mapsto QT \mapsto QT \text{ restricted to } M_{d+1,m}.
$$

For instance for $(n, d, e, m) = (2, 1, 1, 3)$ the linear map is

$$
Q = \begin{bmatrix}
1 \\
q_{01} \\
q_{10}
\end{bmatrix}
\mapsto
\begin{bmatrix}
c_{30} & 0 & c_{20} \\
c_{21} & c_{20} & c_{11} \\
c_{12} & c_{11} & c_{02} \\
c_{03} & c_{02} & 0 \\
c_{20} & 0 & c_{10} \\
c_{11} & c_{10} & c_{01} \\
c_{02} & c_{01} & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
q_{01} \\
q_{10}
\end{bmatrix}
\leftarrow \text{coeff. of } x^3 \text{ of } P := QT
\leftarrow \text{coeff. of } x^2y
\leftarrow \text{coeff. of } xy^2
\leftarrow \text{coeff. of } y^3
\leftarrow \text{coeff. of } x^2
\leftarrow \text{coeff. of } xy
\leftarrow \text{coeff. of } y^2
$$

1These coefficients of $QT$ must vanish if $T$ is the Taylor polynomial of some $P/Q$. 7/11 A. Conca, S. Naldi, G. Ottaviani, B. Sturmfels On Taylor polynomials of rational functions July 2023
Example \((n, d, e, m) = (2, 1, 1, 3)\) (continued)

The variety \(T_{1,1,3}^2 \subset \mathbb{P}^9\) contains ternary cubics that are (order 3) Taylor polynomials of rational functions of degree \((1, 1)\):

\[
\frac{1 + p_{10}x + p_{01}y}{1 + q_{10}x + q_{01}y} = 1 + c_{10}x + c_{01}y + \cdots + c_{12}xy^2 + c_{03}y^3 + \langle x, y \rangle^4
\]

The ideal \(I_3(M_T)\) of maximal minors of \(M_T\) has the expected codimension 5, but it is not prime: it has two components, one of which is the (prime) ideal of \(T_{1,1,3}^2\).

**Conclusion:** For \(n \geq 2\), taking maximal minors of the Padé Matrix is not sufficient to get the equations of \(T_{d,e,m}^n\).
Defective cases

The variety $\mathcal{T}^3_{2,2,3} \subset \mathbb{P}^{19}$ has codimension 2 (expected to be a hypersurface)

$$\det \begin{bmatrix}
c_{300} & 0 & 0 & c_{200} & 0 & 0 & 0 & 0 & 0 & c_{100} \\
c_{210} & 0 & c_{200} & c_{110} & 0 & 0 & 0 & 0 & c_{100} & c_{010} \\
c_{201} & c_{200} & 0 & c_{101} & 0 & 0 & c_{100} & 0 & c_{010} & c_{001} \\
c_{120} & 0 & c_{110} & c_{020} & 0 & 0 & c_{100} & 0 & c_{010} & c_{001} \\
c_{111} & c_{110} & c_{101} & c_{011} & 0 & c_{100} & 0 & c_{010} & c_{001} & 0 \\
c_{102} & c_{101} & 0 & c_{002} & c_{100} & 0 & 0 & c_{001} & 0 & 0 \\
c_{030} & 0 & c_{020} & 0 & 0 & 0 & 0 & c_{010} & 0 & 0 \\
c_{021} & c_{020} & c_{011} & 0 & 0 & c_{010} & 0 & c_{001} & 0 & 0 \\
c_{012} & c_{011} & c_{002} & 0 & c_{010} & c_{001} & 0 & 0 & 0 & 0 \\
c_{003} & c_{002} & 0 & 0 & c_{001} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \equiv 0$$

The variety $\mathcal{T}^3_{8,5,9} \subset \mathbb{P}^{219}$ is a hypersurface (expected to fill the whole $\mathbb{P}^{219}$)

Indeed, the maximal minors of the Padé Matrix have one common factor.

Conjecture

- For $n = 2$, all Taylor varieties $\mathcal{T}^2_{d,e,m}$ are non-defective.
- For $n \geq 3$ there are only finitely-many defective cases.
Hessians

The variety $\mathcal{T}_{1,1,2}^2$ is known as the Perazzo\textsuperscript{2} cubic surface. It has \textit{vanishing Hessian}: the determinant of Hessian matrix is identically zero.

$$M_T = \begin{bmatrix} c_{20} & c_{10} & 0 \\ c_{11} & c_{01} & c_{10} \\ c_{02} & 0 & c_{01} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 2c_{20} & -c_{11} & 0 & -c_{10} & 2c_{01} \\ -c_{11} & 2c_{02} & 2c_{10} & -c_{01} & 0 \\ 0 & 2c_{10} & 0 & 0 & 0 \\ -c_{10} & -c_{01} & 0 & 0 & 0 \\ 2c_{01} & 0 & 0 & 0 & 0 \end{bmatrix}$$

thus $\det(H) \equiv 0$.

Let $n \geq 2$. If $\mathcal{T}_{d,e,m}^n$ is a hypersurface, then it has vanishing Hessian.

**Conjecture.**

$\mathcal{T}_{d,e,d+e+1}^1$ has \textit{zero Gaussian curvature}, for every $d \geq 1$, $e \geq 2$, that is, its Hessian determinant is a multiple of the defining polynomial of $\mathcal{T}_{d,e,d+e+1}^1$.

\textsuperscript{2}U. Perazzo, \textit{Sulle varietà cubiche la cui hessiana svinisce identicamente}. G. Mat. Battaglini 38 (1900), 337-354
Taylor varieties are defined as set of Taylor polynomials of rational functions
Classical well-known varieties for $n = 1$ (secants to rational normal curve)
Interesting phenomena for $n \geq 2$ : defectivity, vanishing Hessians...

Available on arXiv/2304.00712