On the projective geometry of conic feasibility problems

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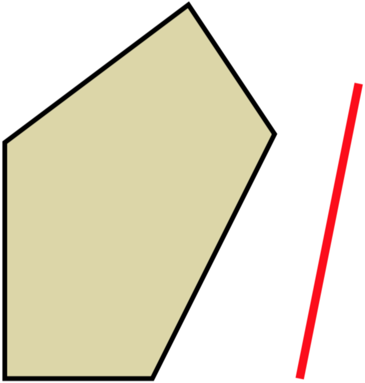

General framework

U real vector space
$\mathbf{K} \subset U$ *regular* convex cone ($=$ *closed, pointed, with interior*)
$L \subset U$ affine space

$$\mathbf{K} \cap L \quad \text{feasible set of a CP}$$

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Linear programming	$U = \mathbb{R}^n$ and $\mathbf{K} = \mathbb{R}^n_+$
Semidefinite programming	$U = \mathbb{S}^d$ and $\mathbf{K} = \mathbb{S}^d_+$
Hyperbolic programming	$U = \mathbb{R}^n$ and $\mathbf{K} = \Lambda_+(f, e)$
	(mild assumptions on f)

Feasibility types

The conic program is

	FEASIBLE $\mathbf{K} \cap L \neq \emptyset$
STRONGLY	$Int(\mathbf{K}) \cap L \neq \emptyset$
WEAKLY	$Int(\mathbf{K}) \cap L = \emptyset$

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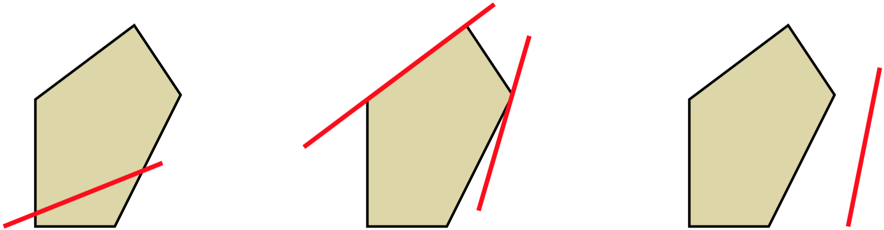
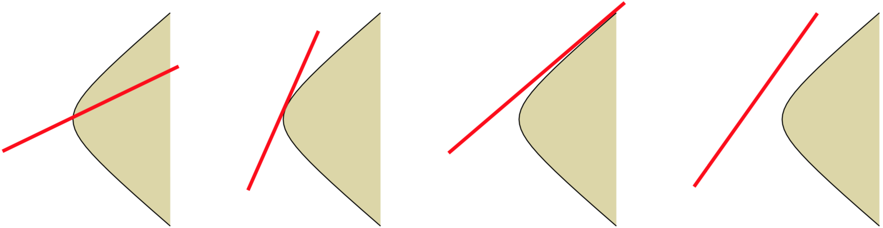
Questions:

- how to detect the feasibility type of a CP?
- if $\mathbf{K}$ is semialgebraic, algebraic certificates of infeasibility?

Linear and nonlinear CP



3 types for Linear Programming



4 types for Conic Programming

Linear programming

Consider the feasible set in a standard (primal) LP :

$$\begin{array}{ll} (L) & Ax = b \\ (\mathbf{K}) & x_i \geq 0, \;\; i = 1, \ldots, n \end{array}$$

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$$\begin{array}{ll} (\widehat{L}) & Ax = bx_0 \\ (\mathbf{K}) & \quad x_i \geq 0, \;\; i = 1, \ldots, n \\ & \quad x_0 \;\; \text{free} \end{array}$$

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This operation *lifts* $\mathbf{K} = \mathbb{R}^n_+$, which is seen as an affine slice:

$$\mathbf{K} \approx (\mathbb{R} \times \mathbb{R}^n_+) \cap \{x_0 = 1\} \quad \textit{and} \quad L \approx \widehat{L} \cap \{x_0 = 1\}$$

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Can we do the same for the general CP?

Homogenization (idea)

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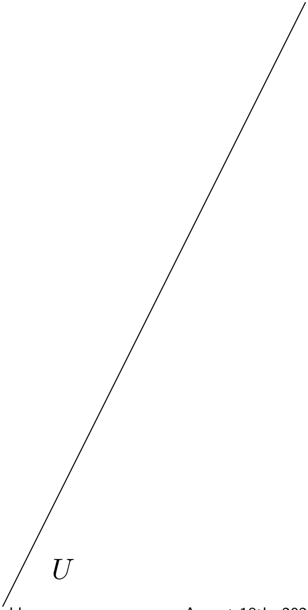

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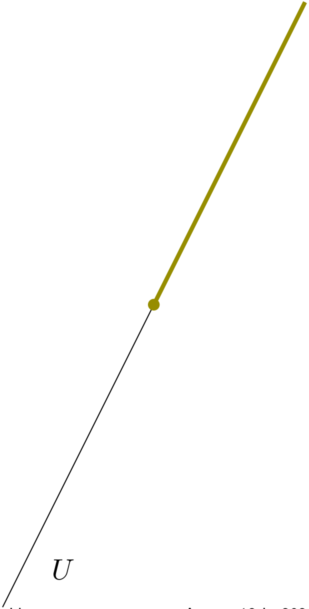

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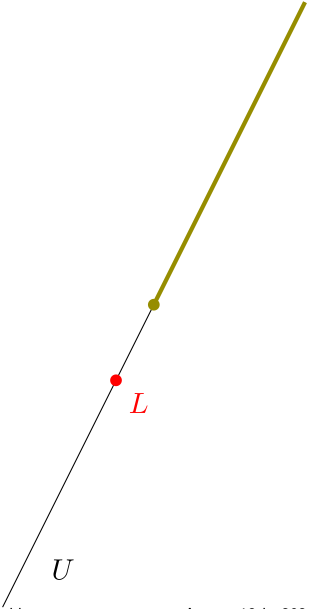

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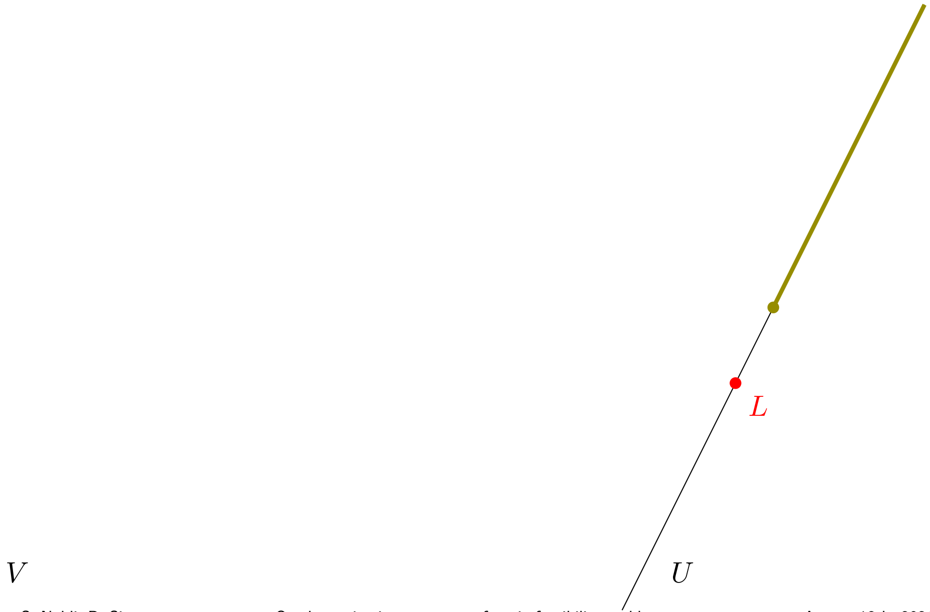

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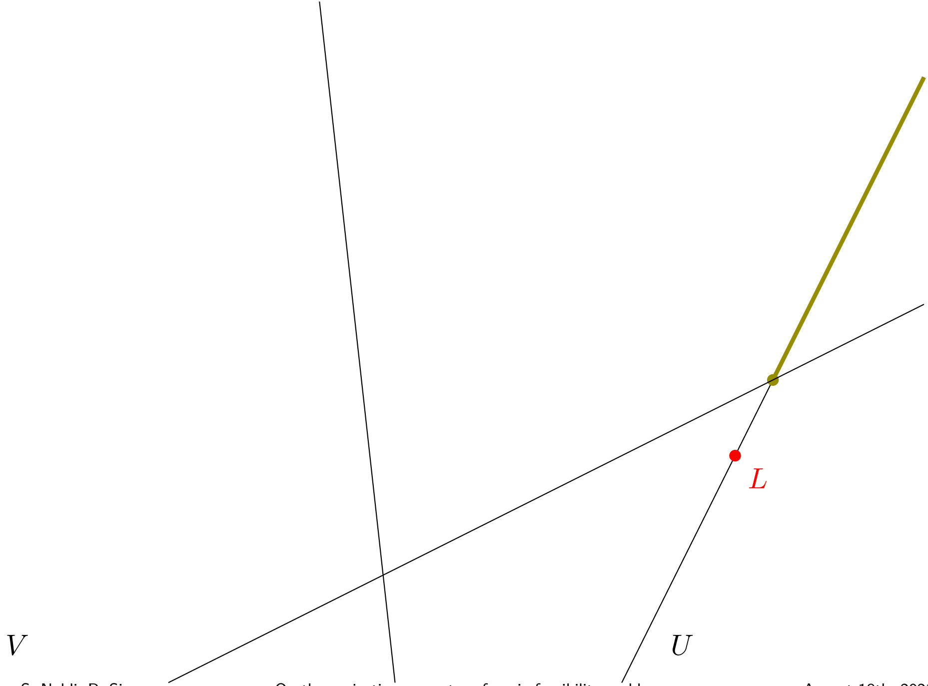

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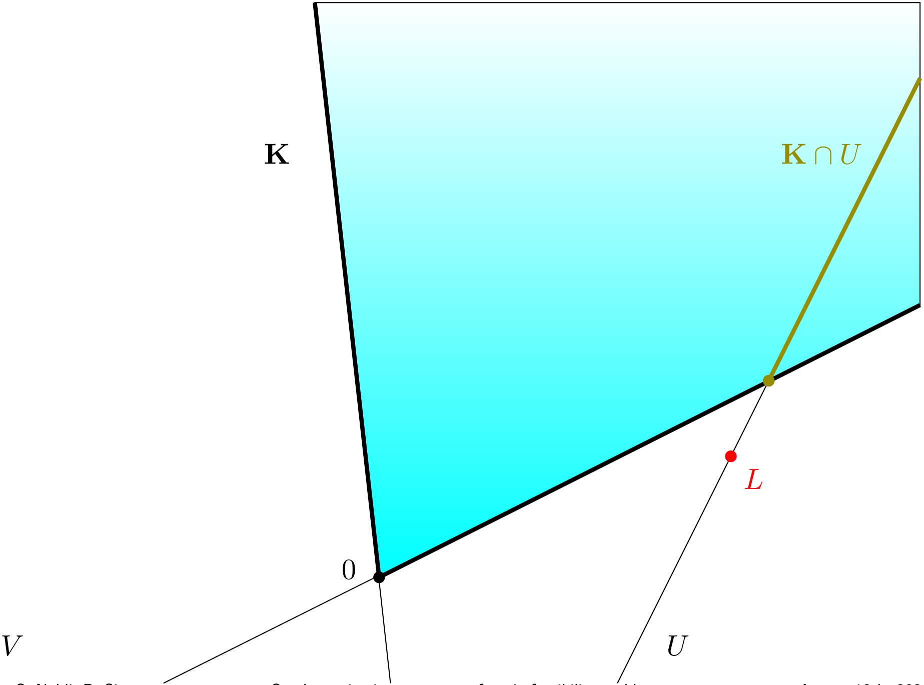

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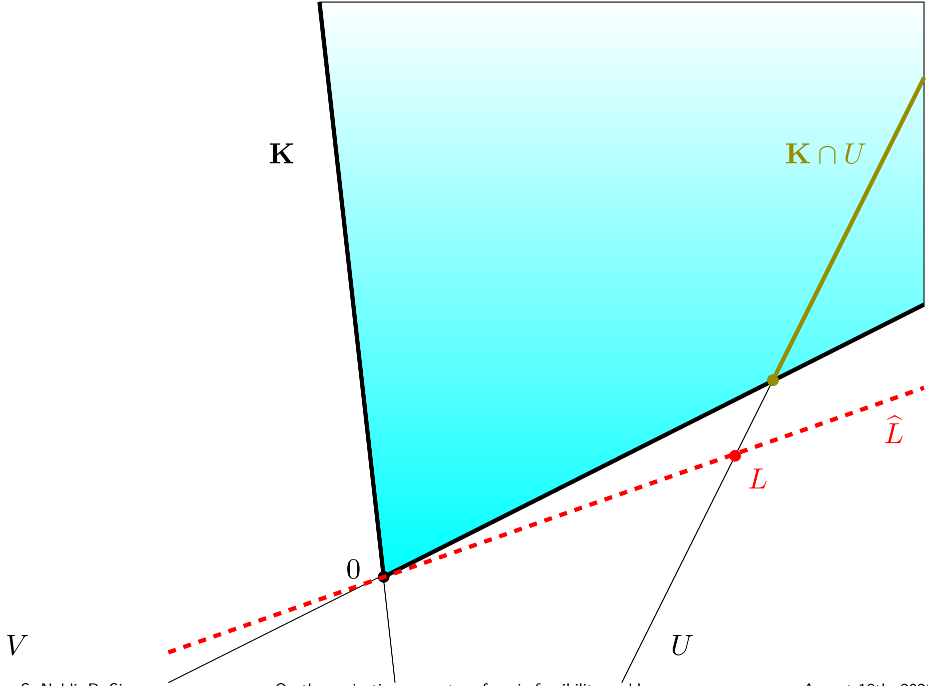

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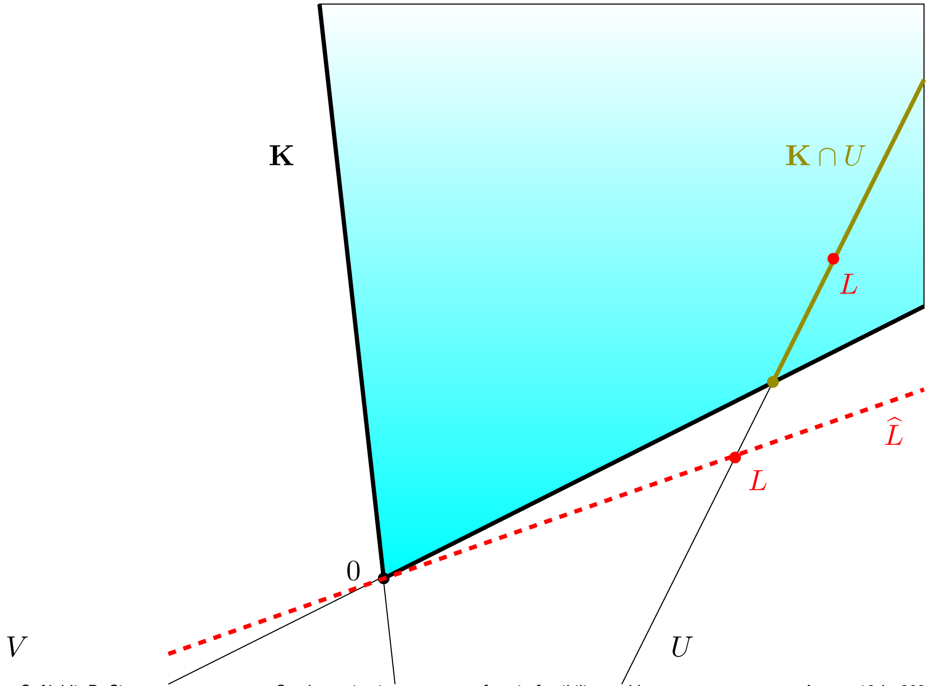

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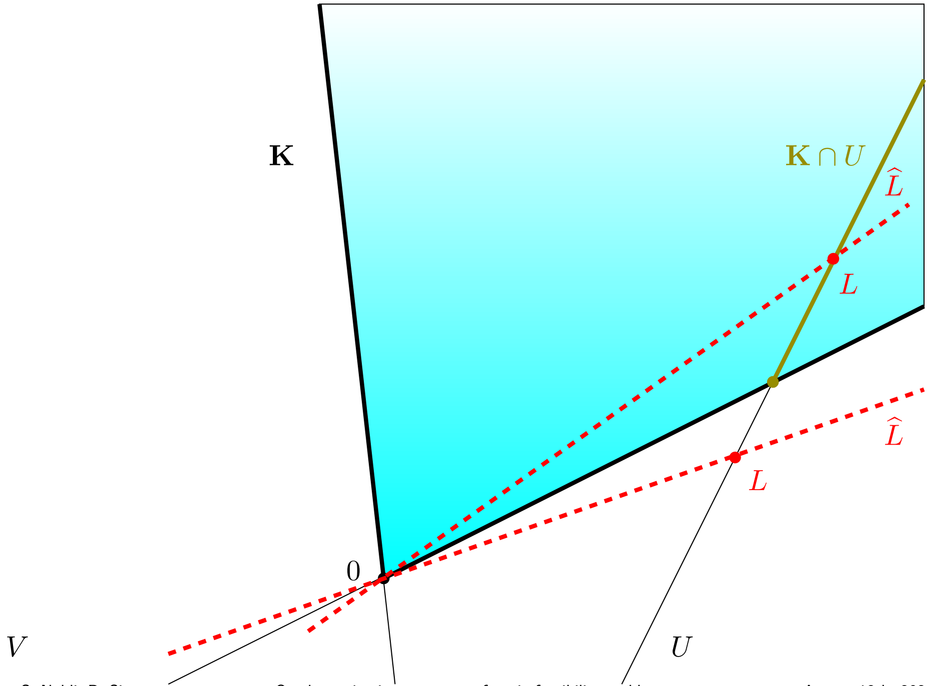

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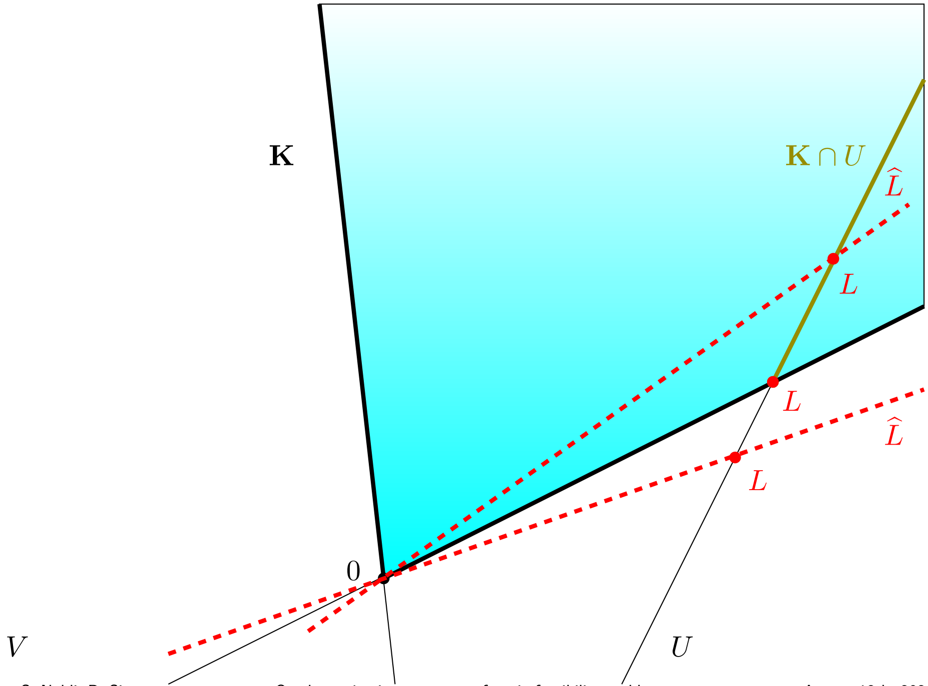

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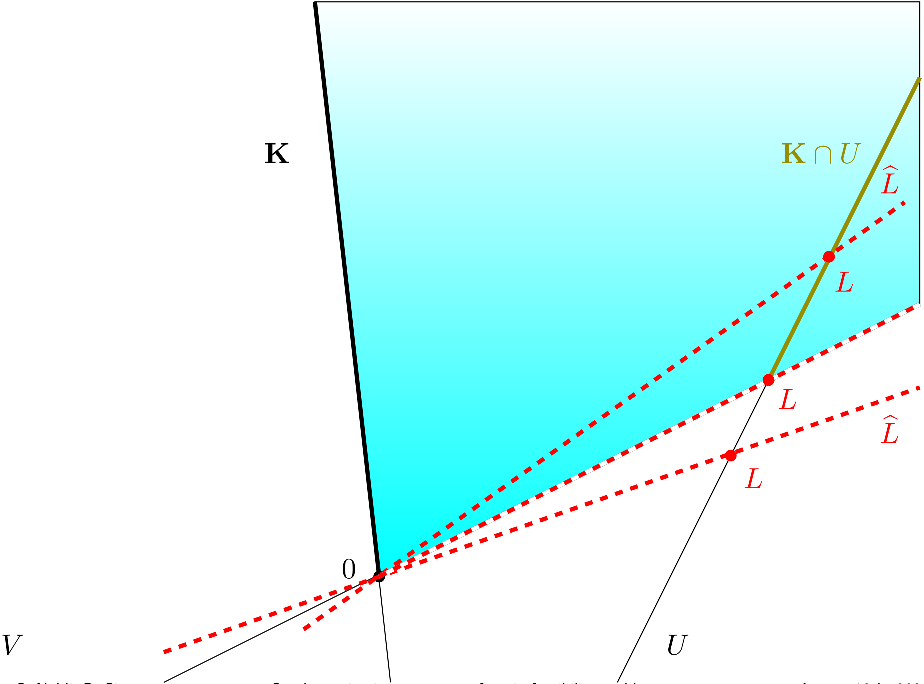

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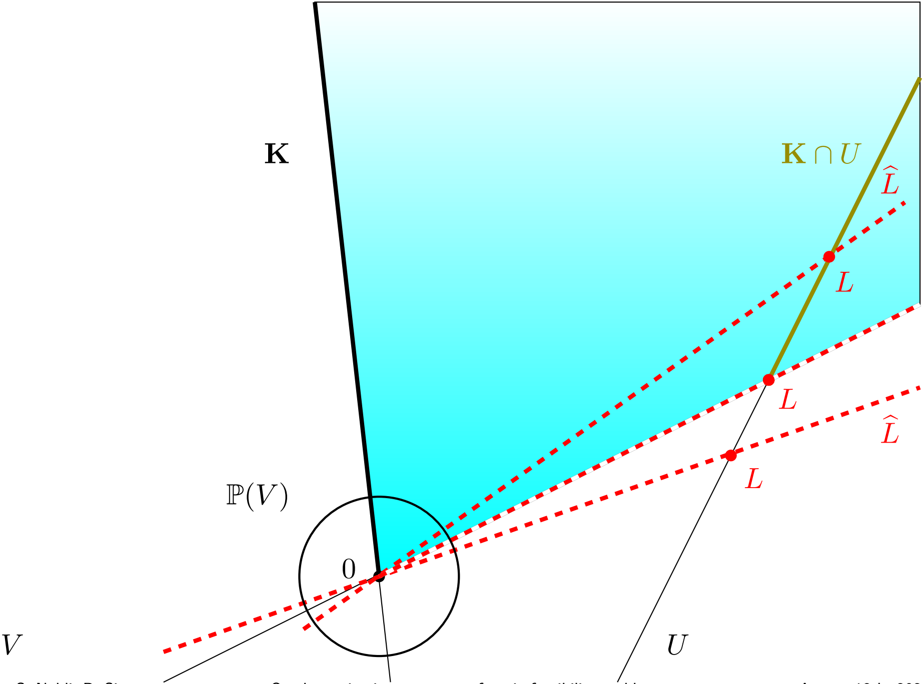

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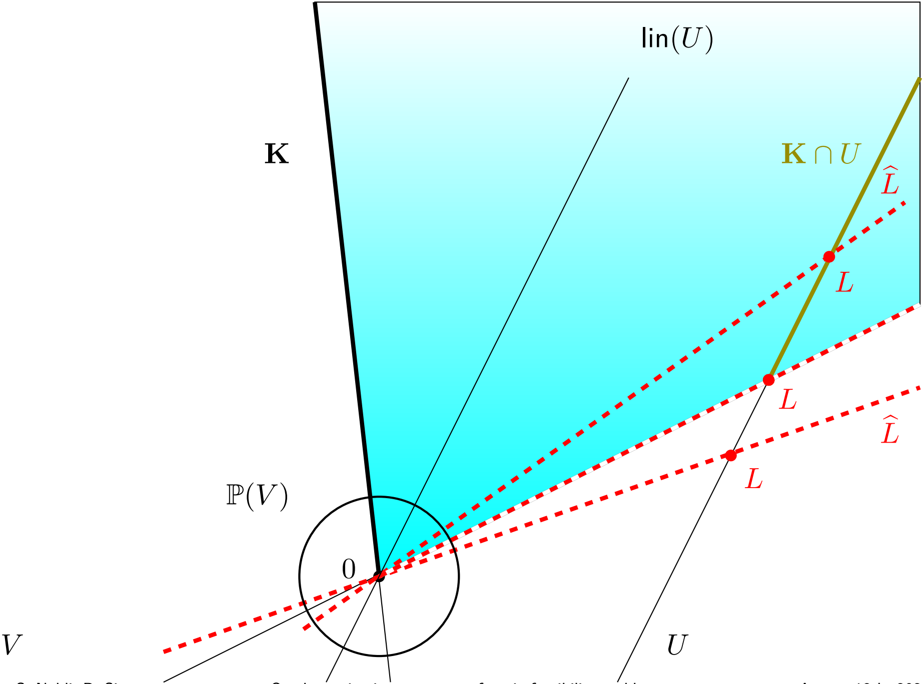

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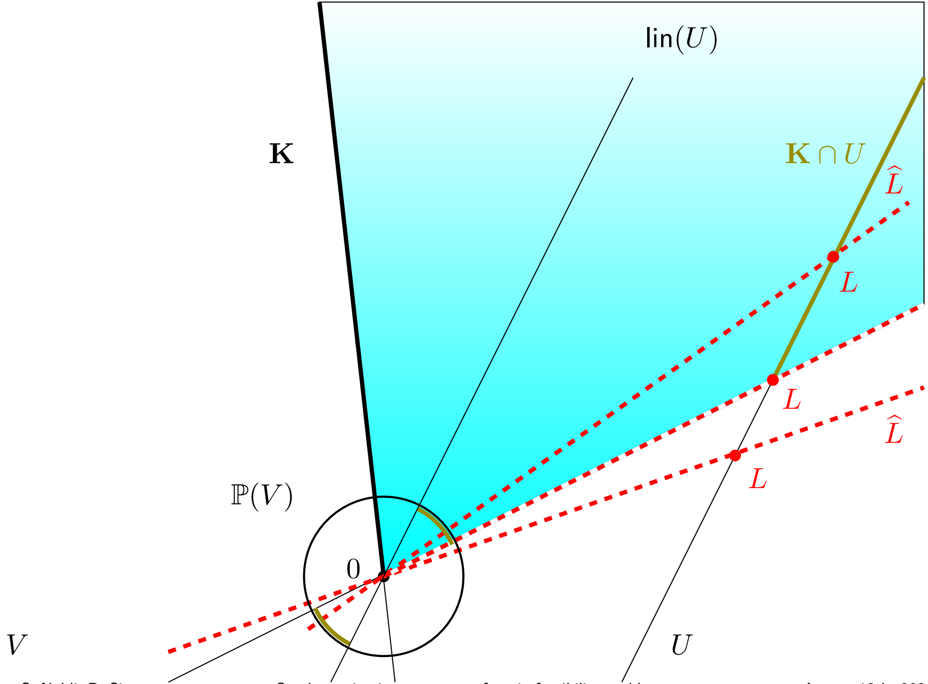

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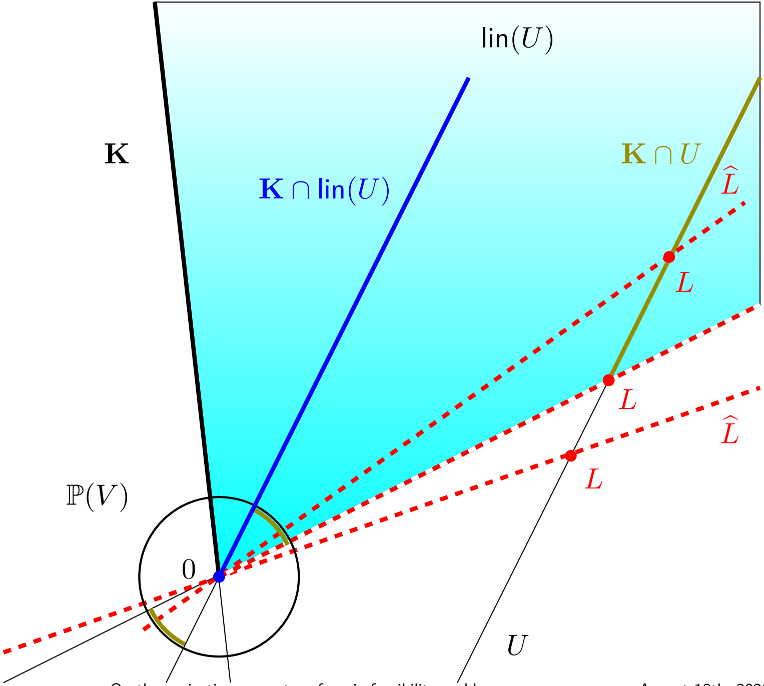

Homogenization (formal construction)

Lifted space:

V	*real vector space of dimension* N
$\mathbf{K} \subset V$	*regular convex cone*
$L \subset V$	*affine space (codim*$(L) \geq 2$*)*
$\widehat{L}$	*span of* L *in* V
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Original space:

U real v.s. $=$ affine chart of $\mathbb{P}(V)$	*where the CP is defined*
$\mathbf{K} \cap U$	*the cone that we see*
$\mathbf{K} \cap L = (\mathbf{K} \cap U) \cap L$	*original CP*

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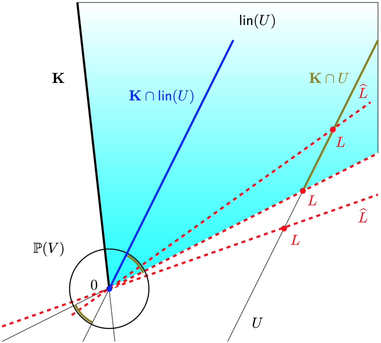
Information at infinity:

$\text{lin}(U)$	*hyperplane at infinity*
$\text{lin}(L) \subset \text{lin}(U)$	*direction of* L
$\mathbf{K} \cap \text{lin}(U)$	*cone at infinity*

Comparison of types

Theorem.

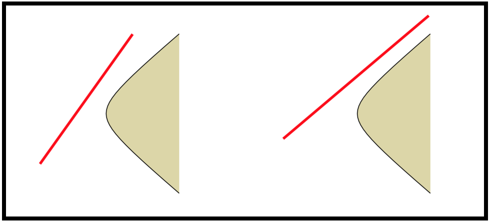
- $\mathbf{K} \cap L$ and $(-\mathbf{K}) \cap L$ are infeasible $\Leftrightarrow$ $\mathbf{K} \cap \widehat{L} \subset \mathrm{lin}(L)$.
- $\mathbf{K} \cap L$ or $(-\mathbf{K}) \cap L$ strongly feasible $\Leftrightarrow$ $\mathbf{K} \cap \widehat{L}$ strongly feasible
- $\mathbf{K} \cap \widehat{L} = \{0\}$ implies[1] that $\mathbf{K} \cap L$ is strongly infeasible



[1] The converse does not hold, we will need to define a more refined type of strong infeasibility.

Stability

Let $d = \dim L$. We say that $\mathbf{K} \cap L$ is *stably infeasible* if there is an open neighborhood of L in the Grassmannian of $d-$dimensional spaces in $\mathbb{R}^n$ s.t. $\mathbf{K} \cap L'$ is infeasible for all L' in this neighborhood.



stable vs *unstable*

Theorem.

- $\mathbf{K} \cap L$ and $(-\mathbf{K}) \cap L$ stably infeasible $\Leftrightarrow \mathbf{K} \cap \widehat{L} = \{0\}$
- Assume $(-\mathbf{K}) \cap L = \emptyset$. Then $\mathbf{K} \cap L$ is stably infeasible $\Leftrightarrow \mathbf{K} \cap \widehat{L} = \{0\}$.
- $\mathbf{K} \cap L$ is stably infeasible $\Leftrightarrow \exists \ell \in Int(\mathbf{K}^\vee)$ such that $\ell(x) < 0$ for all $x \in L$

Rationality

Assumptions:

- $\mathbf{K}$ is a semialgebraic set defined by inequalities with coefficients in $\mathbb{Q}$
- L is the solution set of linear equations with coefficients in $\mathbb{Q}$
- $\mathbf{K} \cap L = \emptyset$.

Is there a **rational** certificate ?

Theorem. A stably infeasible program $\mathbf{K} \cap L$ always admits a rational infeasibility certificate.

For LP stability is not necessary by Farkas Lemma

Theorem. If $\{x \in \mathbb{R}^n \colon Ax = b\} \cap \mathbb{R}^n_+$ is infeasible, there exists $y \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}$ s.t. $H = \{x \in \mathbb{R}^n \colon y^T(Ax - b) = \lambda\}$ strongly separates L and $\mathbb{R}^n_+$.

Irrationality example in SDP

Let $v = \{x^2, y^2, z^2, xy, xz, yz\}$ and let $L' \subset \mathbb{S}^6$ be the set of 6×6 symmetric matrices M satisfying

$$v^T M v = x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z + 2x^2z^2 + xz^3 + yz^3 + z^4$$

The set $\mathbb{S}^6_+ \cap L'$ is a 2-dimensional cone with *no rational points*[2].

For $L = (L')^\perp - Id_6$, then $\mathbb{S}^6_+ \cap L$ is strongly infeasible but has no rational certificates, since any such certificate would be a rational point in $\mathbb{S}^6_+ \cap L'$.

[2] (C. Scheiderer) "Sums of squares of polynomials with rational coefficients" J. European Math. Soc. 18, 1495-1513 (2016).

A facial reduction

Theorem. $\mathbf{K}$ regular, nice[3] convex cone. Let $L \subset V$ be of codim ≥ 2. If $\mathbf{K} \cap L = \emptyset$, there exist $\ell_1, \dots, \ell_k \in \mathbf{K}^\vee$ with the following properties. Set $F_0 = \mathbf{K}$, $L_1 = \widehat{L}$, $F_i = \{x \in F_{i-1} \colon \ell_i(x) = 0\}$ and $L_i = L_{i-1} \cap span(F_{i-1})$. Then:

$$\boxed{\begin{array}{l} k \leq 1 + \dim(L) \\ F_i \supset F_{i+1} \\ F_i \supset \mathbf{K} \cap L_i \supset \mathbf{K} \cap \widehat{L} \\ F_k \subset \mathrm{lin}(L) \end{array}}$$

One deduces $\mathbf{K} \cap \widehat{L} \subset F_k \subset \mathrm{lin}(L)$, a proof that $\mathbf{K} \cap L = \emptyset$.

[3] Pataki : A cone $\mathbf{K}$ is nice if $\mathbf{K}^\vee + F^\perp$ is closed for every face F

References

This talk is based on

"Conic programming: Infeasibility certificates and projective geometry"
(S. Naldi, R. Sinn) J. Pure Appl. Algebra 225(7), 2021

Related papers:

"Characterizing Bad Semid. Programs: Normal Forms and Short Proofs"
(G. Pataki) SIAM Rev. 61(4):839–859, 2019

"Bad projections of the PSD cone"
(Y. Jiang, B. Sturmfels) Collectanea Mathematica 72:261-280, 2021