On the projective geometry of conic feasibility problems

Conic Linear Optimization for Computer-Assisted Proofs @ MFO

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General framework

\( U \) real vector space
\( K \subset U \) regular convex cone (= closed, pointed, with interior)
\( L \subset U \) affine space

\( K \cap L \) feasible set of a CP

Linear programming
Second-order cone programming
Semidefinite programming
Hyperbolic programming

\( U = \mathbb{R}^n \) and \( K = \mathbb{R}_+^n \)
\( U = \mathbb{R}^n \) and \( K = \mathcal{L}^n \)
\( U = S^d \) and \( K = S_+^d \)
\( U = \mathbb{R}^n \) and \( K = \Lambda_+(f, e) \)
Feasibility types

The conic program is

<table>
<thead>
<tr>
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<th>FEASIBLE</th>
<th>INFEASIBLE</th>
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<tbody>
<tr>
<td>STRONGLY</td>
<td>$\text{Int}(K) \cap L \neq \emptyset$</td>
<td>$K \cap L = \emptyset$</td>
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<tr>
<td>WEAKLY</td>
<td>$\text{Int}(K) \cap L = \emptyset$</td>
<td>$d(K, L) &gt; 0$</td>
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<td>$d(K, L) = 0$</td>
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Questions we are interested in:

• how to detect feasibility / feasibility types of a CP?

• if $K$ is semialgebraic, algebraic certificates of infeasibility?
Linear and nonlinear CP

3 types for Linear Programming

4 types for Conic Programming
**Linear programming**

Consider the feasible set in a standard (primal) LP:

\[
\begin{align*}
(L) & \quad Ax = b \\
(K) & \quad x_i \geq 0, \ i = 1, \ldots, n
\end{align*}
\]

Let \(x_0\) be a new variable, and homogenize it to

\[
\begin{align*}
(\hat{L}) & \quad Ax = bx_0 \\
(K) & \quad x_i \geq 0, \ i = 1, \ldots, n \\
\quad & \quad x_0 \geq 0
\end{align*}
\]

This operation *lifts* \(K = \mathbb{R}^n_+\) to \(K = \mathbb{R}^{n+1}_+\), that is \(K\) is seen as an affine slice of a lifted cone:

\[
K \approx K \cap \{x_0 = 1\} \quad \text{and} \quad L \approx \hat{L} \cap \{x_0 = 1\}
\]

The (first) cone \(K\) is the part of the (second) cone \(\hat{K}\) that you “can see”.
Homogenization (idea)

\[ P(V) \quad \text{on the projective geometry of conic feasibility problems} \]
### Homogenization (formal construction)

**Lifted space:**
- $V$: real vector space of dimension $N$
- $K \subset V$: regular convex cone
- $L \subset V$: affine space ($\text{codim}(L) \geq 2$)
- $\hat{L}$: span of $L$ in $V$
- $K \cap \hat{L}$: homogenized CP

**Original space:**
- $U$: real v.s. = affine chart of $\mathbb{P}(V)$ where the CP is defined
- $K = K \cap U$: the cone that we see
- $K \cap L = (K \cap U) \cap L$: original CP

**Information at infinity:**
- $\text{lin}(U)$: hyperplane at infinity
- $\text{lin}(L) \subset \text{lin}(U)$: direction of $L$
- $K \cap \text{lin}(U)$: cone at infinity
Comparison of types

Theorem.

• $K \cap L$ and $(-K) \cap L$ are infeasible $\iff K \cap \hat{L} \subset \text{lin}(L)$.

• $K \cap L$ or $(-K) \cap L$ strongly feasible $\iff K \cap \hat{L}$ strongly feasible

• $K \cap \hat{L} = \{0\}$ implies\(^1\) that $K \cap L$ is strongly infeasible

\(^1\) The converse does not hold, we will need to define a more refined type of strong infeasibility.
Stability

Let $d = \dim L$. We say that $K \cap L$ is stably infeasible if there is an open neighborhood of $L$ in the Grassmannian of $d$–dimensional spaces in $\mathbb{R}^n$ s.t. $K \cap L'$ is infeasible for all $L'$ in this neighborhood.

*Theorem.*

- $K \cap L$ and $(-K) \cap L$ stably infeasible $\iff K \cap \hat{L} = \{0\}$
- Assume $(-K) \cap L = \emptyset$. Then $K \cap L$ is stably infeasible $\iff K \cap \hat{L} = \{0\}$.
- $K \cap L$ is stably infeasible $\iff \exists \ell \in \text{Int}(K^\vee)$ such that $\ell(x) < 0$ for all $x \in L$
Rationality

Assumptions:
- \( K \) is a semialgebraic set defined by inequalities with coefficients in \( \mathbb{Q} \)
- \( L \) is the solution set of linear equations with coefficients in \( \mathbb{Q} \)
- \( K \cap L = \emptyset \).

Is there a rational certificate?

Theorem. A stably infeasible program \( K \cap L \) always admits a rational infeasibility certificate.

For LP stability is not necessary by Farkas Lemma

Theorem. If \( \{ x \in \mathbb{R}^n : Ax = b \} \cap \mathbb{R}_+^n \) is infeasible, there exists \( y \in \mathbb{Q}^n \) and \( \lambda \in \mathbb{Q} \) s.t. \( H = \{ x \in \mathbb{R}^n : y^T(Ax - b) = \lambda \} \) strongly separates \( L \) and \( \mathbb{R}_+^n \).
Irrationality example in SDP

Let \( v = \{ x^2, y^2, z^2, xy, xz, yz \} \) and let \( L' \subset \mathbb{S}^6 \) be the set of \( 6 \times 6 \) symmetric matrices \( M \) satisfying

\[
v^T M v = x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z + 2x^2z^2 + xz^3 + yz^3 + z^4\]

The set \( \mathbb{S}^6_+ \cap L' \) is a 2-dimensional cone with no rational points\(^2\).

For \( L = (L')^\perp - Id_6 \), then \( \mathbb{S}^6_+ \cap L \) is strongly infeasible but has no rational certificates, since any such certificate would be a rational point in \( \mathbb{S}^6_+ \cap L' \).

A facial reduction

Theorem. **K** regular, nice\(^3\) convex cone. Let \( L \subset V \) be of codim \( \geq 2 \). If \( K \cap L = \emptyset \), there exist \( \ell_1, \ldots, \ell_k \in K^\vee \) with the following properties. Set \( F_0 = K \), \( L_1 = \hat{L} \), \( F_i = \{ x \in F_{i-1} : \ell_i(x) = 0 \} \) and \( L_i = L_{i-1} \cap \text{span}(F_{i-1}) \). Then:

\[
\begin{array}{c}
k \leq 1 + \dim(L) \\
F_i \supset F_{i+1} \\
F_i \supset K \cap L_i \supset K \cap \hat{L} \\
F_k \subset \text{lin}(L)
\end{array}
\]

One deduces \( K \cap \hat{L} \subset F_k \subset \text{lin}(L) \), a certificate that \( K \cap L = \emptyset \).

\(^3\)Pataki : A cone \( K \) is nice if \( K^\vee + F^\perp \) is closed for every face \( F \)
References

This talk is based on

“Conic programming: Infeasibility certificates and projective geometry”
(S. Naldi, R. Sinn) J. Pure Appl. Algebra 225(7), 2021

Related papers:

“Facial reduction algorithms for conic optimization problems”
(H. Waki, M. Muramatsu)

“Characterizing Bad Semid. Programs: Normal Forms and Short Proofs”
(G. Pataki) SIAM Rev. 61(4):839–859, 2019

“Bad projections of the PSD cone”
(Y. Jiang, B. Sturmfels) Collectanea Mathematica 72:261-280, 2021