Spectrahedral representations of plane hyperbolic curves

*BIRS Workshop on Real Polynomials: Counting and Stability*

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Introduction

Convex semialgebraic set: \( \Lambda_+(f, e) = \{ a \in \mathbb{R}^n : f(a) \geq 0, \ldots \} \)
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Second curve: \( g \in \mathbb{R}[x]_{\text{deg } f - 1} \)
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Hyperbolic curve: $V(f) = \{a \in \mathbb{R}^n : f(a) = 0\}$

Second curve: $g \in \mathbb{R}[x]_{\deg f - 1}$. Extra-factor: $\ell_1 \cdot \ell_2 \cdots \ell_s$
Hyperbolic polynomials

A homogeneous polynomial \( f \in \mathbb{R}[x]_d \) is called \textit{hyperbolic in direction} \( e \) if

- \( f(e) \neq 0 \)
- The characteristic polynomial \( f(te - a) \) is real rooted, for every \( a \in \mathbb{R}^n \).

It is \textit{hyperbolic} whenever such a direction \( e \) exists.

Examples:

- \( f = \ell_1 \ell_2 \cdots \ell_d \), with \( \ell_i \in \mathbb{R}[x]_1 \), is hyperbolic
- \( f = \text{det}(X) \), with \( X = (x_{ij}) \) symmetric, is hyperbolic in direction \( I \)
- More generally, \( f = \text{det}(x_1 A_1 + \cdots + x_n A_n) \), with \( e_1 A_1 + \cdots + x_n A_n \succ 0 \), is hyperbolic in direction \( e = (e_1, \ldots, e_n) \)
- More generally, \( f^k = \text{det}(x_1 A_1 + \cdots + x_n A_n) \), with \( e_1 A_1 + \cdots + x_n A_n \succ 0 \), for some \( k \in \mathbb{N} \), implies \( f \) hyperbolic in direction \( e = (e_1, \ldots, e_n) \)

called “\textit{determinantal}” and “\textit{weakly determinantal}” in previous talks.
Hyperbolicity cone

Let \( f \in \mathbb{R}[x]_d \) be hyperbolic in direction \( e \). The set

\[
\Lambda_+(f, e) = \{a \in \mathbb{R}^n : ch_a(t) = 0 \Rightarrow t \geq 0\}
\]

is called the hyperbolicity cone in direction \( e \) of \( f \).

Equivalent definition: the connected component of \( \mathbb{R}^n \setminus V(f) \) containing \( e \).

There are “many” hyperbolicity cones:

- Only one pair, if \( f \) is irreducible \[\text{[Kummer, 2018]}\]
- Bound of \( 2 \sum_{k=0}^{n-1} \binom{d-1}{k} \) for large \( d \), or of \( 2^d \) for large \( n \), attained for products of linear forms \[\text{[Theobald et al., 2018]}\]
Hyperbolicity cone
## Optimization viewpoint

<table>
<thead>
<tr>
<th>Feasible set</th>
<th>name</th>
<th>Optimization</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Hyperbolicity Cone" /></td>
<td>Hyperbolicity Cone</td>
<td>HP</td>
<td>Hyperbolic polynomial</td>
</tr>
<tr>
<td><img src="image" alt="Spectrahedron" /></td>
<td>Spectrahedron</td>
<td>SDP</td>
<td>$f = \det A(x)$</td>
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<tr>
<td><img src="image" alt="Polyhedron" /></td>
<td>Polyhedron</td>
<td>LP</td>
<td>$f = \prod \ell_i(x)$</td>
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</table>
A set $S \subset \mathbb{R}^n$ has a spectrahedral representation if there are real symmetric matrices $A_1, \ldots, A_n$ such that

$$S = \{ x \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succeq 0 \}$$

If $f = \det(X)$, then $\Lambda_+(f, I) = S_+^d$ (psd matrices), and more generally if $f^k = \det(x_1 A_1 + \cdots + x_n A_n)$, with $e_1 A_1 + \cdots + e_n A_n \succeq 0$, then

$$\Lambda_+(f^k, e) = \Lambda_+(f, e) = \{ x \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succeq 0 \}$$

If $f$ is hyperbolic in direction $e$, is $\Lambda_+(f, e)$ spectrahedral?
Generalized Lax conjecture

Let $f \in \mathbb{R}[x]_d$ be hyperbolic in direction $e$. Then

$$\Lambda_+(f, e) = \{ x \in \mathbb{R}^n : x_1A_1 + \cdots + x_nA_n \succeq 0 \}$$

for some real symmetric matrices $A_1, \ldots, A_n$.

or, equivalently:

Let $f \in \mathbb{R}[x]_d$ be hyperbolic in direction $e$. Then there is $g \in \mathbb{R}[x]_c$ s.t.

- $fg = \text{det}(x_1A_1 + \cdots + x_nA_n)$
- $\Lambda_+(f, e) \subset \Lambda_+(g, e)$

If such a $g$ exists, then

$$\Lambda_+(f, e) = \Lambda_+(f, e) \cap \Lambda_+(g, e) = \Lambda_+(fg, e)$$
Our construction

\[ \Lambda_+(f, e) = \text{green region} \]

\[ f = \text{blue curve} \]

\[ g = \ell_1 \ell_2 \cdots \ell_c, \] the "extra factor", whose cone is a polyhedral set that strictly contains \( \Lambda_+(f, e) \)
Let \( f \in \mathbb{R}[x, y, z]_d \) be hyperbolic in direction \( e \). Then \( h \in \mathbb{R}[x, y, z]_{d-1} \) is a

- **contact curve** for \( f \) if every intersection point of \( V_\mathbb{C}(f) \) and \( V_\mathbb{C}(g) \) has even multiplicity

- **real contact curve** for \( f \) if every intersection point of \( V_\mathbb{R}(f) \) and \( V_\mathbb{R}(g) \) has even multiplicity

- **interlacer of \( f \) in direction \( e \)** if the roots \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d \) of \( f(te-a) \) and the roots \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{d-1} \) of \( h(te-a) \) interlace perfectly, namely

\[
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d
\]

contact curve >>>> real contact curve <<< interlacer
Dixon method (variant of Plaumann-Vinzant)

Essentially based on the property \( A \cdot A^{\text{adj}} = \det A \cdot I_d \)

Assume \( f = \det A \) and \( V_C(f) \) is smooth, then

\[
\text{co-rank}(A) = \text{rank}(A^{\text{adj}}) = 1 \mod \det A.
\]

INPUT

\( f \) (hyperbolic curve)

Sketch of the PROCEDURE:

\( m_{11} \leftarrow D_e f := e_1 \frac{\partial f}{\partial x} + e_2 \frac{\partial f}{\partial y} + e_3 \frac{\partial f}{\partial z} \)

\( D_e f \) special interlacer

split \( S \cup \overline{S} = V_C(f) \cap V_C(D_e f) \)

extend \( m_{11} \) to basis \( \langle m_{11} \ldots m_{1d} \rangle \) of polyn. vanishing on \( S \)

\( m_{jk} \leftarrow \text{solve } a_{11} a_{jk} - \overline{a_{1j}} a_{1k} = 0 \mod f \) for \( j \leq k \)

\( M \leftarrow (m_{jk}) \)

\( A \leftarrow M^{\text{adj}} / f^{d-2} \)

OUTPUT

\( A \) (satisfying \( f = \det A \) and \( A(e) \succ 0 \))
The set of interlacers

Precomputing an interlacer is “easier” than precomputing a contact curve.

This is because the set of interlacers is tractable:

\[(\text{Kummer-Plaumann-Vinzant 2013})\]

\[\text{Int}(f, e) = \{h \in \mathbb{R}[x, y, z]_{d-1} \mid h \text{ interlaces } f \text{ in direction } e\}\]

is a section of the cone of positive polynomials:

\[\text{Int}(f, e) = \{h \in \mathbb{R}[x, y, z]_{d-1} \mid W(f, h) := (D_e f) h - f(D_e h) \geq 0\}\]

that is, interlacers can be sampled through techniques based on sums of squares and SDP.
Main result

Let $f \in \mathbb{R}[x, y, z]_d$ be hyperbolic with respect to $e$, and let $h \in \text{Int}(f, e)$. Let $\ell_1, \ldots, \ell_c$ be the (real) lines joining the pairs of complex intersections of $f$ and $h$.

Then* there are $A_1, A_2, A_3 \in S^m$, with $m = (d^2 + d)/2 - r$, such that

- $\Lambda_+(f, e) = \{x \in \mathbb{R}^3 : xA_1 + yA_2 + zA_3 \succeq 0\}$
- $f \cdot \ell_1 \cdot \ell_2 \cdots \ell_c = \det(x_1A_1 + x_2A_2 + x_3A_3)$

* up to genericity assumptions on $g$:
1) no three intersection points of $f$ and $h$ aligned
2) no three of the lines pass through the same point
3) $f$ does not vanish over intersection points of two lines
Our variant

The **main point** is that the extra factor \( g = \ell_1 \cdots \ell_c \) corrects the failure of \( h \) to be a contact curve, by adding multiplicity to the complex intersections of \( f \) and \( h \).

**Positive aspects:**

1. The multiplier is the simplest we can imagine: product of linear forms
2. The size of the spectrahedral representation depends on \( r \), the number of real intersections.
3. For maximal \( r \), one gets the Helton-Vinnikov representation

Maximizing \( r \) means minimizing the size of the representation, and means that the interlacer is “special”.
Extremal interlacers

An extreme point of $Int(f, e)$ is called an *extremal interlacer*. It corresponds to interlacers with “many” real intersections with $f$.

If $f$ is smooth, any extremal interlacer has at least

$$\left\lceil \frac{(d + 1)d - 2}{4} \right\rceil$$

contact points (counted multiplicities).

<table>
<thead>
<tr>
<th>$d$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{(d+1)d-2}{4}$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>⋯</td>
</tr>
<tr>
<td>$\frac{d(d-1)}{2}$</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>⋯</td>
</tr>
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Quartic curves

Open question: can we always have six intersections?
One example

The cubic $f = x^3 + 2x^2y - xy^2 - 2y^3 - xz^2$ is hyperbolic with respect to $e = (1, 0, 0)$.

The two green interlacers have coefficients in $K$ with $|K : \mathbb{Q}| = 4$, and the dashed one is rational. The corresponding spectrahedral representation is

$$\frac{24}{125} (2x - y) \cdot f = \det \begin{pmatrix}
5x + 10y & -x - 2y & -4z & 2z \\
-x - 2y & x & 0 & 0 \\
-4z & 0 & 4x + 2y & -2x - 4y \\
2z & 0 & -2x - 4y & 4x + 2y
\end{pmatrix}.$$
We can prove that there are (possibly large) \textbf{rational} spectrahedral representations (even in the case when there are no rational determinantal representations).

Let \( f \in \mathbb{Q}[x, y, z] \) be a hyperbolic curve, with smooth real zero set. Then its hyperbolicity cone has a rational spectrahedral representation, of size at most \( \binom{d+1}{2} \).
Example

\[ f = y^2z - (x^3 - 6xz^2 - 3z^3) \] has no rational \( 3 \times 3 \) determinantal representation, but it has a rational generalized representation, which yields the spectrahedral representation

\[
\Lambda_+(f, e) = \left\{ \begin{pmatrix} 3z & y & -x - z & -3x + z \\ y & -x + 2z & 0 & -y \\ -x - z & 0 & z & x + 4z \\ -3x + z & -y & x + 4z & -x + 18z \end{pmatrix} \succeq 0 \right\}.
\]

The extra-factor is a line and the interlacer has two contact points:
References

This talk is based on

“Spectrahedral representations of plane hyperbolic curves”

Related papers:

“Computing Hermitian determinantal representations of hyperbolic curves”

“Testing hyperbolicity of real polynomials”