Abstracts

On the projective geometry of conic feasibility problems

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(joint work with Rainer Sinn)

This extended abstract is based on the article [7] to which we refer for a more complete list of contributions on this topic.

Let $\mathbf{K} \subset U$ be a closed convex pointed cone with non-empty interior in a real vector space U, and let $L \subset U$ be an affine subspace. The *conic feasibility problem* is the algorithmic question of deciding whether $\mathbf{K} \cap L = \emptyset$ or $\mathbf{K} \cap L \neq \emptyset$.

Several classical questions can be cast as instances of conic feasibility problems, let us briefly mention a few. The existence of solutions to a system of affine inequalities

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i, \quad i = 1, \ldots, m$$

is the feasibility problem of *linear programming*, that is, the question whether an affine space $L \subset U = \mathbb{R}^n$ intersects the n-dimensional nonnegative orthant $\mathbf{K} = \mathbb{R}^n_+$. The solvability of a linear matrix inequality

$$A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0$$

for given real symmetric matrices $A_0, \ldots, A_n$, is the feasibility problem of *semidefinite programming*: in this case $\mathbf{K} = \mathbb{S}^d_+$, the cone of positive semidefinite real symmetric matrices, and L is an affine space in $U = \mathbb{S}^d$. Other important classes arise in combinatorial optimization (*e.g.* copositive, completely positive matrices), real algebraic geometry (moment cones, cone of positive polynomials, SOS cone) and in several other areas.

One classically makes the distinction between different shades, or *types*, of feasibility and infeasibility. A feasible conic program is called *strongly feasible* (resp. *weakly feasible*) if the affine space L contains (resp. does not contain) an interior point of $\mathbf{K}$. On the other side, an infeasible conic program is called *strongly infeasible* if the Euclidean distance $d(\mathbf{K}, L)$ is strictly positive, and it is called *weakly infeasible* if $d(\mathbf{K}, L) = 0$.

An example of a weakly feasible set in semidefinite programming is the Gram spectrahedron of a sum-of-squares polynomial which has no rational Gram matrices, for instance one of the polynomials given by Scheiderer in [9]:

$$x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z + 2x^2z^2 + xz^3 + yz^3 + z^4.$$

An example of weakly infeasible semidefinite program (see [8]) is given by the linear matrix inequality

$$\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} \succeq 0$$

since the previous inequality has no solutions, but the set $\left\{ \left[\begin{smallmatrix} 1/n & 1 \\ 1 & n \end{smallmatrix}\right] : n \in \mathbb{N} \right\}$ is included in the cone $\mathbf{K} = \mathbb{S}^2_+$ and has distance zero from the line $L = \{ \left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right] : x \in \mathbb{R} \}$.

Let us also mention that such degenerate programs might seem artificial, but actually can arise quite naturally in concrete situations, for instance in the case of Lasserre relaxations of the minimization of the Motzkin polynomial [10].

The existence of "weak types" makes the feasibility problem harder, since for these types, the feasibility is not robust under small perturbations of the input data: a weakly feasible program will become either strongly feasible or strongly infeasible after small perturbation, and the same for a weakly infeasible program. It is a typical behavior of numerical solvers that of giving a wrong answer in these degenerate situations.

The main contribution of our work [7] is the construction of a general framework for homogenizing a conic program. Indeed, different algorithms for solving conic (feasibility) problems are based on homogenization of the constraints, see for instance [11, 4] or [3, Ch. 4] and references therein. The idea at the core of our method is as follows: the vector space U is seen as an affine hyperplane in a vector space V, and $\mathbf{K}$ is *lifted* to a cone $\widehat{\mathbf{K}} \subset V$, such that $\widehat{\mathbf{K}} \cap U = \mathbf{K}$. The idea of this projective point of view is that the original cone $\mathbf{K}$ is now the part of the homogeneous cone $\widehat{\mathbf{K}}$ that *can be observed* if one sits in the affine chart U of $\mathbb{P}(V)$.

In this framework, it is possible to get information about the feasibility type of the original program $\mathbf{K} \cap L = \widehat{\mathbf{K}} \cap L$ and the *homogenized conic program* $\widehat{\mathbf{K}} \cap \operatorname{span}(L)$. One example is the following characterization of infeasibility:

$$\mathbf{K} \cap L = \emptyset \text{ and } (-\mathbf{K}) \cap L = \emptyset \quad \Leftrightarrow \quad \widehat{\mathbf{K}} \cap \operatorname{span}(L) \subset \operatorname{lin}(L) \tag{1}$$

where $\operatorname{lin}(L)$ is the direction of L: this tells us that L does not meet neither $\mathbf{K}$ nor $(-\mathbf{K})$ exactly when the homogenized feasible set lies at infinity.

Another interesting caracterization concerns a special subclass of strongly infeasible programs. A conic program $\mathbf{K} \cap L$ is called *stably infeasible* if there exists a neighbourhood $\mathcal{L}$ of L in the Grasmanniann of affine subspaces of U of the same dimension as L, such that if $L' \in \mathcal{L}$, then $\mathbf{K} \cap L' = \emptyset$. In other words, stably infeasible conic programs are strongly infeasible and keep this property after small perturbations. However it is easy to construct strongly infeasible conic programs that are not stable, already in the class of linear programming.

A second aspect that has been highlighted in [7] and in the talk at MFO, is to which extent the infeasibility can be certified over the field of definition of the conic program. Indeed, the expected output of a decision algorithm is a yes-no answer and in the case of conic infeasibility, one could also ask for a *rational infeasibility certificate*, that is, an element $\ell \in \mathbf{K}^*$ (in the dual cone $\mathbf{K}^* \subset V^\vee$), which is (nonnegative on $\mathbf{K}$ and) strictly negative on L, and that can be defined over the smallest field containing the input. The element ℓ is a separating hyperplane for $\mathbf{K}$ and L and the question is wheter such certificate can be made rational. Let us remarl that this is a natural question for infeasibility, since the same question has in general a "no" answer for the feasibility of non-linear conic programs, as already mentioned for the examples in [9].

In [7], we prove two main results concerning rationality of infeasibility certificates: first, every stably infeasible conic program admits a rational infeasibility certificate; second, we construct explicit strongly infeasible semidefinite programs that do not admit rational infeasibility certificates.

Finally, let us mention two last aspects of our contribution. The first concerns the property mentioned in (1), which is used in [7] to design a variant of the classical facial reduction algorithm [2], for the case of infeasible conic programs: if $\mathbf{K}$ is a nice cone, and if $\mathbf{K} \cap L = (-\mathbf{K}) \cap L = \emptyset$, then there exists a sequence of functionals $\ell_1, \ldots, \ell_k \in \mathbf{K}^*$, $k \leq 1 + \dim L$, that facially reduce the homogenized program by sending the feasible set $\widehat{\mathbf{K}} \cap \operatorname{span}(L)$ to infinity (see [7, Th. 3.4]).

The second concerns the complexity theory of semidefinite programming. The homogenization we have described allows us to give an alternative proof of the result by Ramana [6] that feasibility of semidefinite programming is in $\mathrm{coNP}_{\mathbb{R}}$ (Blum-Shub-Smale complexity model [1]), see [7, Sec. 4].

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