UNIVERSITÉ DE LIMOGES

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MASTER THESIS OF MASTER ACSYON

HYPERBOLIC POLYNOMIALS AND LOCALLY POSITIVE MATRICES

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1 Abstract

In this thesis, i will present the basic theory of hyperbolic polynomials and an application to the study of locally positive matrices. I will give the definition of hyperbolic polynomials and hyperbolic optimization problem. In this internship i have focused on hyperbolic relaxations of the set of eigenvalues of locally positive matrices.

Keywords: Linear programming Optimization, Hyperbolic polynomials, Hyperbolic Programming, Relaxations SDP.

Contents

Introduction

The hyperbolic programming problem (HP) originates from the theory of partial differential equations studied by Lax Garding [1] that is a generalization of semidefinite programming (SDP). The problem (HP) is constrained by the hyperbolic cone of hyperbolic polynomial.
The hyperbolic programming problem as the following forms:

$$
\text{(HP)} \qquad \begin{aligned} c^* := \min \qquad & c^T x \\ \text{s.t.} \qquad & Ax = b. \\ & x \in \Lambda_+(f, e). \end{aligned}
$$

The main objectives of this internship is to study the hyperbolic cone of hyperbolic polynomials and the derivation of it. Also to know the special cases of hyperbolic cones that are usually used in optimization problems. We also examine relaxations of hyperbolic programs and locally positive semidefinite matrices.
In this thesis we have three chapters to discuss, in the first chapter we will introduce hyperbolic polynomials and some properties and examples with some graphics and we define the determinantal representation of a hyperbolic polynomial [2]. The second chapter is about hyperbolic optimization problems, we will concentrate on the study of the hyperbolicity cone of the problem and its connection with spectrahedra. The third chapter deals with the hyperbolic relaxation of locally positive matrices and some applications, we will focus on the paper [4] by G.BLEKHERMAN, SANTANU S. DEY, KEVIN SHU, and SHENGDING SUN.

2 Hyperbolic polynomials

Hyperbolic polynomials generalize characteristic polynomials of symmetric matrices to the multivariate case. So in this chapter we will study hyperbolic polynomials their characteristics and some examples.

2.1 Definitions and examples

Definition 1. (homogeneous polynomial) Suppose that f is a non constant polynomial on $\mathbb{R}^n$ and d is a positive integer. Then f is homogeneous of degree d, if $f(\alpha x) = \alpha^d f(x)$, for all $\alpha \in \mathbb{R}$ and every $x \in \mathbb{R}^n$.

Definition 2. Let $f \in \mathbb{R}[x]$ with $x = (x_1, \ldots, x_n)$ be a homogeneous polynomial of degree $d \in \mathbb{N}$, f is called hyperbolic with respect to direction $e \in \mathbb{R}^n$ if :
• $f(e) \neq 0$ and
• the univariate polynomial $Ch_a(t) = f(a - te) \in \mathbb{R}[t]$ has only real roots for every $a \in \mathbb{R}^n$.

$t \mapsto Ch_a(t)$ is called the characteristic polynomial of a.
We denote by $\mathcal{H}_{d,n}(e)$ the set of hyperbolic polynomials of degree d, with n variables and with respect to direction e.

Remark 1. If $p \in \mathbb{R}[x]$ is hyperbolic, then the number of variables n is superior or equal to 2. Because the polynomial p is homogeneous.

Proposition 1. *we can define in an identical way a hyperbolic polynomial by replacing* $\forall a \in \mathbb{R}^n t \to p(a - te)$ **has only real roots** *with* $\forall a \in \mathbb{R}^n t \to p(e - ta)$ **has only real roots** *in particular, it can also be replaced by* $\forall a \in \mathbb{R}^n t \to p(e + ta)$ **has only real roots** *or* $\forall a \in \mathbb{R}^n t \to p(a + te)$ **has only real roots.**

Proof. The proof is straight forward (see for instance [8]). □

Remark 2. The roots of $Ch_a(t)$ are called the eigenvalues of a where

$$\lambda_1(a) \leq ... \leq \lambda_d(a).$$

Next, we present some basic examples of hyperbolic polynomials :

Example 1. The homogeneous polynomial $f(x,y,z) = -x^2 - y^2 + z^2 \in \mathbb{R}[x,y,z]$ is hyperbolic with respect to direction $e = (0,0,1)$, Since we have $f(e) = 1 \neq 0$ and for all $a \in \mathbb{R}^3$:

$$\begin{aligned} \forall t \in \mathbb{R} : Ch_a(t) &= f(a - te) \\ &= f(a_1, a_2a_3 - t) \\ &= -a_1^2 - a_2^2 + (a_3 - t)^2. \end{aligned}$$

We compute the discriminant, and we obtain : $\Delta = 4a_1^2 + 4a_2^2 > 0$ if $(a_1, a_2) \neq (0,0)$ Then f has only two real roots.

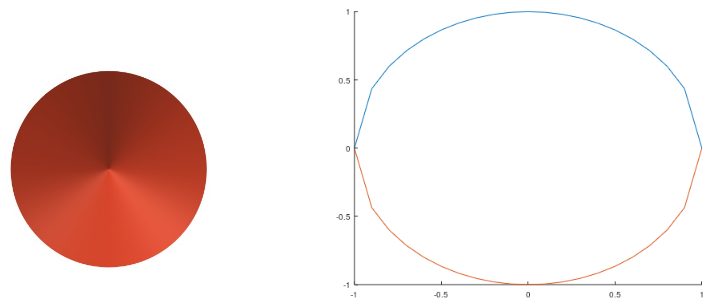

Figure 2.1: the set of zeros of the polynomial $f = -x^2 - y^2 + z^2$ (a) in $\mathbb{R}^3$, (b) in $\mathbb{R}^2$.

Example 2. $f = \det(X)$ and X is symmetric matrix and $X = (x_{i,j})_{1 \leq i \leq j \leq d}$, f is hyperbolic with respect to direction $e = I$, since we have $f(e) = 1 \neq 0$ and we have $f(A - te) = \det(A - tI)$, the roots of $\det(A - tI)$ are the eigenvalues of A.

Example 3. : The homogeneous polynomial $f = x^4 + y^4 - z^4 \in \mathbb{R}[x,y,z]$ is not hyperbolic with respect to any point in $\mathbb{R}^3$. Since for $e = (0,0,1)$ we have $f(e) = -1 \neq 0$, we can check that for all $a \in \mathbb{R}^3, t \in \mathbb{R}$

$$\begin{aligned} Ch_a(t) &= f(a - te) \\ &= f(a_1, a_2a_3 - t) \\ &= a_1^4 + a_2^4 - (a_3 - t)^4. \end{aligned}$$

The characteristic polynomial has four roots : two real roots and two real complex for all $a \in \mathbb{R}^3$.

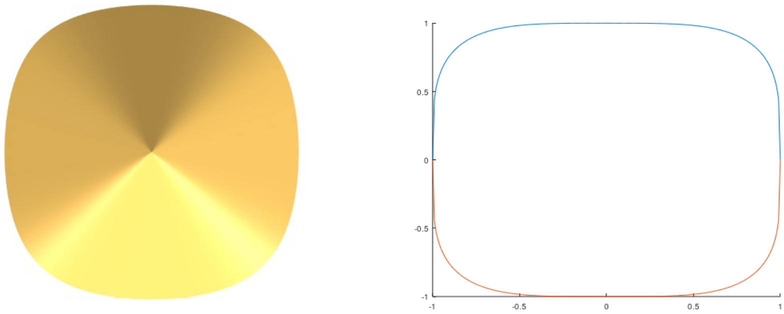

Figure 2.2: the set of zeros of the polynomial $f = x^4 + y^4 - z^4$ (a) in $\mathbb{R}^3$, (b) in $\mathbb{R}^2$.

Example 4. Let $f(x, y) = xy(x + y)$ is hyperbolic polynomial with respect to direction $e = (1, 1)$. Indeed we have that $f(e) = 2 \neq 0$ and for $a = (a_1, a_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$

$$\begin{aligned} Ch_a(t) &= f(te - a) \\ &= 2t^3 + t^2[-3(a_1 + a_2)] + t(4a_1a_2 + a_1^2 + a_2^2) - a_1a_2(a_1 + a_2). \end{aligned}$$

The characteristic polynomial has real roots.
For $a = (4, 3)$ the characteristic polynomial is $Ch_a = 2t^3 - 21t^2 + 73t - 84$. Its roots are real, see the figure 2.3 and figure 2.4

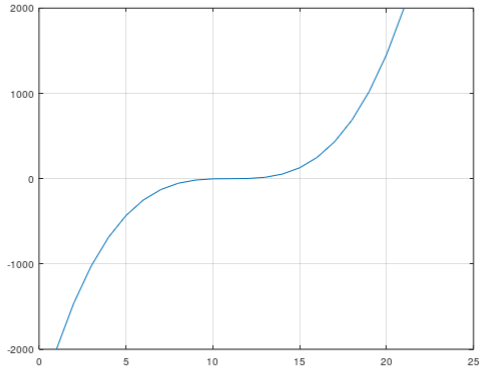

Figure 2.3: the set of zeros of the polynomial $f = xy(x + y)$ in $\mathbb{R}^2$.

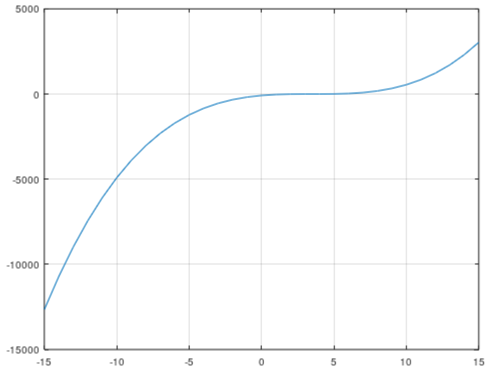

Figure 2.4: the set of zeros of the characteristic polynomial $Ch_a = 2t^3 - 21t^2 + 73t - 84$ in $\mathbb{R}^2$.

Example 5. The products of real linear forms $f = x_1...x_n$ is hyperbolic with respect to direction $e = \mathbb{1}_n$. Since we have $f(e) = 1 \neq 0$ and

$$\begin{aligned}\forall t \in \mathbb{R} \forall a \in \mathbb{R}^n : Ch_a(t) &= f(te - a)\\ &= (t - a_1) \cdots (t - a_n)\end{aligned}$$

$a_1, \cdots, a_n$ are the real roots of $Ch_a(t)$ then the result .

Property 1. If $p_1 \in \mathcal{H}_{d_1,n}(e)$ and $p_2 \in \mathcal{H}_{d_2,n}(e)$ Then : $p_1p_2 \in \mathcal{H}_{d_1+d_2,n}(e)$.

Proof. We show that p_1p_2 is hyperbolic polynomial with respect to direction

e, we have

$$\begin{array}{lll} p_1 \in \mathcal{H}_{d_1,n}(e) & \Longrightarrow & p_1(e) \neq 0 \\ p_2 \in \mathcal{H}_{d_2,n}(e) & \Longrightarrow & p_2(e) \neq 0. \end{array}$$

Then $p_1(e)p_2(e) \neq 0$
and $t \to Ch_{a,p_1}(t)$ $t \to Ch_{a,p_2}(t)$ have only real roots .
Then $\forall : t \to Ch_{a,p_1}(t)Ch_{a,p_2}(t)$ is also real-rooted. Where

$$\begin{aligned} ch_{a,p_1p_2}(t) &= (p_1p_2)(a-te) \\ &= p_1(a-te)p_2(a-te) \\ &= Ch_{a,p_1}(t)Ch_{a,p_2}(t). \end{aligned}$$

□

2.2 Definite Determinantal Representations

In this paragraph, we introduce some important results on hyperbolic polynomials that admit a determinantal representation.

Definition 3. A homogeneous polynomial $p \in \mathbb{R}[x]$ is said to be **determinantal** if $p(x)$ is the determinant of a matrix with a symmetric linear entries; i.e.,there exist matrices $M_1, \ldots, M_n \in \mathbb{R}^{d\times d}$ such that

$$p(\mathbf{x}) = \det\left(x_1M_1 + x_2M_2 + \cdots + x_nM_n\right).$$

Where $M_1, \ldots, M_n$ the coefficient matrices. The matrix $x_1M_1 + \cdots + x_nM_n$ is said to give a **symmetric determinantal representation** of p of size d. The determinantal representation is called definite if $M_1, \ldots, M_n$ is positive definite.

Corollary 1. *Let $M_1, \ldots, M_n$ are symmetric matrices of size d, such that $e_1M_1 + \cdots + e_nM_n \succ 0$ for a certain $e = (e_1, \ldots, e_n)$ then the polynomial $p = \det(x_1M_1 + \cdots + x_nM_n)$ is hyperbolic with respect to direction e. Where $M_1, \ldots, M_n$ are symmetric matrices.*

Theorem 1. *(Helton-Vinnikov)*
Every hyperbolic polynomial in three variables possesses a definite symmetric determinantal representation, i.e.

$$f = \det(XA + YB + ZC) \; and e_1A + e_2B + e_3C \succ 0$$

.

Example 6. Let $f(x,y,z) = z^2 - x^2 - y^2$. f is hyperbolic polynomial with respect to direction $e = (0,0,1)$, and is determinantal:

$$\begin{aligned} f(x,y,z) &= det \begin{pmatrix} z-x & y \\ y & z+x \end{pmatrix} \\ &= det \left(x \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$

and

$$e_1 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + e_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \succ 0$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are symmetric matrices.

3 Hyperbolic Optimization

To a given hyperbolic polynomial we can associate a convex cone called the hyperbolicity cone: polyhedra and spectrahedra are special examples of such cones. In this chapter, we define the hyperbolicity cone, the hyperbolic programming problem and its elements, and we will see the special examples of hyperbolic optimization problems and their relaxation.

3.1 Hyperbolicity cones

Definition 4. The open hyperbolicity cone of $f \in \mathbb{R}[x]_d$ with respect to direction e is

$$\begin{aligned} \Lambda_{++}(f,e) &= \{a \in \mathbb{R}^n / Ch_a(t) = 0 \Rightarrow t > 0\} \\ &= \{x \in \mathbb{R}^n : \lambda_{min}(x) > 0\}. \end{aligned}$$

The closure of Λ_{++} in the Euclidean topology is called the hyperbolicity cone. It is denoted by Λ_+ and equals

$$\Lambda_+ = \overline{\Lambda_{++}} = \{x \in \mathbb{R}^n : \lambda_{min}(x) \geq 0\}.$$

Theorem 2. *(Gårding 1959 [1])*
If p is hyperbolic with respect to direction e then $\Lambda_+(p,e)$ is convex.

Proof. Let $e, e' \in \Lambda_+(p,e)$ and let $x = \lambda e + (1-\lambda)e'$ and we prove that $x \in \Lambda_+(p,e)$ it is sufficient to show that the solutions of the characteristic polynomials $p(x - te)$ are positive since

$$\begin{aligned} p(x-te) &= p(\lambda e + (1-\lambda)e' - te) \\ &= (1-\lambda)^d p(e' - \frac{t-\lambda}{1-\lambda}e). \end{aligned}$$

It is sufficient that $\frac{t-\lambda}{1-\lambda} > 0$ then $x \in \Lambda_+(p,e)$ i.e. $\Lambda_+(p,e)$ is convex.

$\square$

Theorem 3. *(Gårding)*
Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e and $a \in \Lambda_+(p,e) \setminus \mathrm{V}(p)$ with $a \neq 0$,

where $\mathrm{V}(p) = \{a : p(a) = 0\}$. *Then a is also a hyperbolicity direction of p and* $\Lambda_+(p, e) = \Lambda_+(p, a)$.

Proof. Proof given in [3]. $\square$

One of the main motivations for studying hyperbolic polynomial and their hyperbolicity cones is that they provide a natural generalization for linear and semidefinite programming.

Remark 3. The hyperbolicity cone is not unique, it cones in pairs:

$$\Lambda_+(f, -e) = -\Lambda_+(f, e).$$

Example 7. We consider the polynomials of example 5 $p(x_1, x_2, \ldots, x_n) = x_1 x_2 ... x_n$ is hyperbolic with respect to direction $e = \mathbf{1}_n := (1, 1, \ldots, 1)$, and their hyperbolicity cone is the non-negative orthant $\mathbb{R}^n$. Since for $a \in \mathbb{R}^n$

$$p\left(a + t\mathbf{1}_n\right) = t^n + e_1(a)t^{n-1} + \cdots + e_{n-1}(a)t + e_n(a)$$

where $e_k(a) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k}$ is the elementary symmetric polynomial of degree k in the variables $a_1, a_2, \ldots, a_n$. And

$$\Lambda_+(p, e) = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : e_1(a) \geq 0, e_2(a) \geq 0, \ldots, e_n(a) \geq 0\}$$

is a polyhedron.

Example 8. The polynomial

$$p\left(x_1, x_2, \ldots, x_n\right) = x_n^2 - \sum_{k=1}^{n-1} x_k^2$$

is hyperbolic with respect to the direction $e = (0, 0, \ldots, 1) \in \mathbb{R}^n$. Its hyperbolicity cone is the Lorentz cone

$$\Lambda_+(p, e) = \left\{ (x_1, x_2, \ldots, x_n) : \sqrt{x_1^2 + \ldots + x_{n-1}^2} \leq x_n \right\}.$$

For $n = 3$ we find the hyperbolic polynomials $f(x_1 x_2 x_3) = x_3^2 - x_1^2 - x_2^2$ with respect to direction $e = (0, 0, 1)$ is the example 1 of chapter one and the hyperbolicity cone as

$$\Lambda_+(p, e) = \left\{ (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} \leq x_3 \right\}.$$

Example 9. Let $f = \det X$ is hyperbolic polynomial with respect to direction $e = I_d$, we have the hyperbolicity cone as :

$$\begin{aligned}\Lambda_+(detX, I_d) &= \{A \in \mathbf{S}^d / det(A - tI) \text{has only non negative roots } \} \\ &= \{A \in \mathbf{S}^d / \text{ The eigenvalue of A are positive } \} \\ &= \{A \in \mathbf{S}^d / A \succeq 0\} \\ &= Sym_+^d.\end{aligned}$$

Then the hyperbolicity cone of f is the set of symmetric positive semi-definite matrices.

We can see two special cases of hyperbolicity cones in the next section.

3.1.1 Polyhedra and Spectrahedra

Definition 5. Remind that a polyhedron $\mathcal{P}$ is the intersection of the convex cone of non-negative vectors $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_i \geq 0 \forall i\}$ with an affine subspace. By taking an affine basis for the subspace, we obtain

$$P = \{x : Ax \leq b\}.$$

A spectrahedron is the intersection of the set of positive semi-definite real matrices with an affine linear subspace of the symmetric matrices, i.e

$$S = \{x \in \mathbb{R}^n : M_0 + x_1 M_1 + \cdots + x_n M_n \succeq 0\}.$$

Theorem 4. *Every spectrahedron is a hyperbolicity cone.*

Remark 4. Every three-dimensional hyperbolicity cone is spectrahedral.

Generalised Lax Conjecture : Every hyperbolicity cone is a spectrahedron, that there exists $A_1, \ldots, A_n$ such that

$$\Lambda_+(f, e) = \{x \in \mathbb{R}^n : A_1 x_1 + \ldots + A_n x_n \succeq 0\}.$$

Example 10. Let the hyperbolic polynomial of example 9, $f = det(X)$ where $X = X_1 A_1 + ... + X_n A_n$. The hyperbolicity cone of f with respect to direction e is :

$$\Lambda_+(f,e) = \{ \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \in \mathbb{R}^n / X_1 A_1 + ... + X_n A_n \succeq 0\} = \mathcal{S}_A.$$

the hyperbolicity cone of f is the set of positive semidefinite matrices A is a spectrahedron.

Property 2. Let $p \in \mathcal{H}_{d_1,n}(e)$ and $q \in \mathcal{H}_{d_2,n}(e)$. Then

$$\Lambda_+(pq,e) = \Lambda_+(p,e) \cap \Lambda_+(pq,e).$$

Proof. We know that the product of polynomials p and q which are hyperbolic with respect to direction e is a hyperbolic polynomial w.r.t direction e and we know that the characteristic polynomial of the product is a product of characteristic polynomials where

$$\mathrm{ch}_{a;pq}(t) = \mathrm{ch}_{a;p}(t)\,\mathrm{ch}_{a,q}(t)$$

Then let

$$\begin{aligned} a \in \Lambda_+(pq,e) &\iff \text{ roots of}\mathrm{ch}_{a,pq}(t)\text{are non-negative} \\ &\iff \text{roots of } \mathrm{ch}_{a,p}(t)\text{and those of } \mathrm{ch}_{a,q}(t) \text{ are non-negative} \\ &\iff a \in \Lambda_+(p,e) \cap \Lambda_+(q,e). \end{aligned}$$

□

3.2 Hyperbolic Programming

The hyperbolic programming is a convex optimization problem defined as follows, we are given a homogeneous polynomial $p \in \mathbb{R}[x]_d$ hyperbolic with respect to direction $e \in \mathbb{R}^n$ with n variable $x = (x_1, \ldots, x_n)$

Then the hyperbolic program associated is

$$\begin{array}{llll} & c^* := \min & & c^T x \\ \text{(HP)} & & \text{s.t.} & Ax = b. \\ & & & x \in \Lambda_+(f,e). \end{array}$$

where $x \mapsto c^T x$ is a linear functional, A is a matrix of size $m \times n$ and $Ax = b$ is a system of linear equations.

Remark 5. Every semidefinite programming problem is a hyperbolic problem.

Example 11. We consider the semidefinite programming problem :

$$\max 3x + y$$
$$\text{s.t.} \quad \begin{pmatrix} 1+y & 2-2x-3y \\ 2-2x-3y & -1-x-y \end{pmatrix} \succeq 0.$$

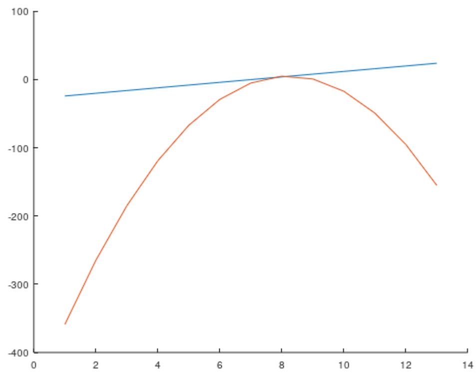

Figure 3.1: Feasible set and the line $3x + y = 0$

3.3 Derivative relaxations/Renegar derivatives

Definition 6. :
If p is hyperbolic with respect to direction e then the directional derivative $D_e p(x) = \frac{d}{dt} p(x+te)\Big|_{t=0}$ is hyperbolic in direction e.

Denote the hyperbolicity cone of $D_e p$ by $\Lambda_+(D_e p, e)$ (or $\Lambda'_+(p, e)$) (respectively $\Lambda_{++}(D_e p, e)$ or $\Lambda'_{++}(p, e)$).

we see that
$$D_e p(te - x) = \frac{d}{dt} p(te - x).$$

Remark 6. The roots of $Ch_{e,D_e p}(t)$ are the eigenvalues of $a$where

$$\lambda_1^{'}(a) \geq \dots \lambda_d^{'}(a)$$

The eigenvalues $\lambda_i'(x)$ with respect to p' of e thus interlace the eigenvalues with respect to p :

$$\lambda_1(x) \leq \lambda_1'(x) \leq \lambda_2(x) \leq \dots \leq \lambda_{n-1}'(x) \leq \lambda_n(x).$$

Where $\quad \left(\lambda_j(x) = \lambda_j'(x) \text{ or } \lambda_j'(x) = \lambda_{j+1}(x)\right) \Leftrightarrow \quad \lambda_j(x) = \lambda_j'(x) = \lambda_{j+1}(x).$

Geometrically: the derivative relaxation is larger than the original cone :

$$\Lambda_+(p,e) \subseteq \Lambda_+(D_e p,e)$$

i.e., the derivative cone $\Lambda_+(D_e p,e)$ is a relaxation of $\Lambda_+(p,e)$.
This gives a nesting sequence of convex hyperbolicity cones :

$$\Lambda_+(f,e) \subset \Lambda_+\left(D_e f,e\right) \subset \cdots \subset \wedge_+\left(D_e^{(d-1)} f,e\right)$$

(the last one is a half-space) giving a sequence of lower bounds for the linear function to optimize:

$$\min_{\Lambda_+(f,e)} c^T x \geq \min_{\wedge_+(D_e f,e)} c^T x \geq \cdots \geq \min_{\wedge_+\left(D_e^{(d-1)} f,e\right)} c^T x.$$

Example 12. Let $f(x_1,x_2,x_3) = x_1 x_2 x_3$. f is hyperbolic polynomial in direction $e = (1,1,1) \in \mathbb{R}^3$ and the hyperbolicity cone is $\mathbb{R}_+^3$. This is a polyhedron, we will prove that

$$\Lambda_+(p,e) \subset \Lambda_+\left(D_e f,e\right) \subset \wedge_+\left(D_e^{(2)} f,e\right).$$

where $\wedge_+\left(D_e^{(2)} f,e\right)$ is a closed half-space. We will take the directional derivatives of p as follows :

$$D_e f = \sum_{i=1}^{3} e_i \frac{\partial f}{\partial x_i} = x_2 x_3 + x_1 x_3 + x_1 x_2.$$

$$D_e^{(2)} f = 2x_1 + 2x_2 + 2x_3.$$

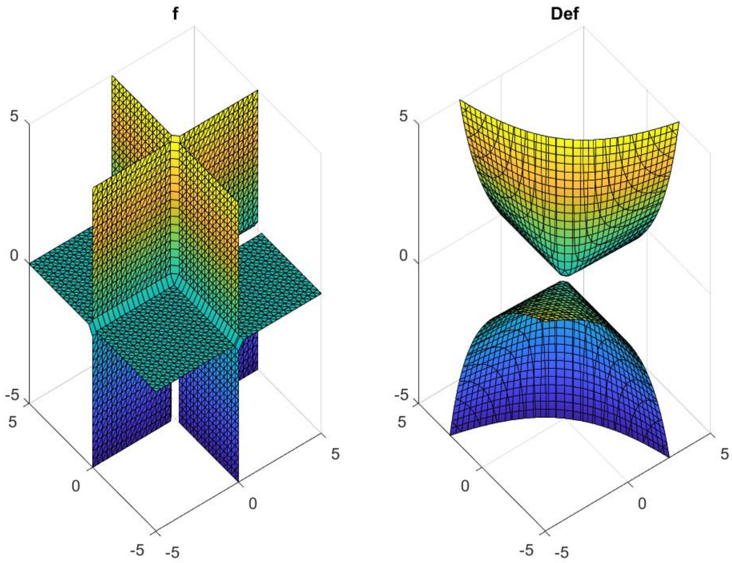

Figure 3.2: The derivative cones of the hyperbolic polynomial :
(a)$\Lambda_+(f,e)$ (b) $\Lambda_+(D_e fe)$.

and their hyperbolicity cone is a spectrahedron.

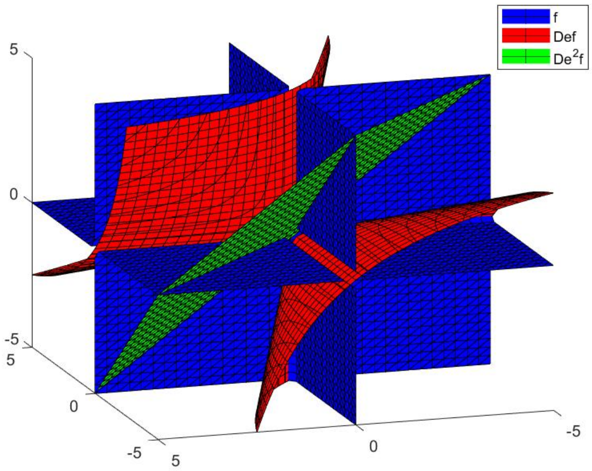

Figure 3.3: All the derivatives cones of f in $\mathbb{R}^3$.

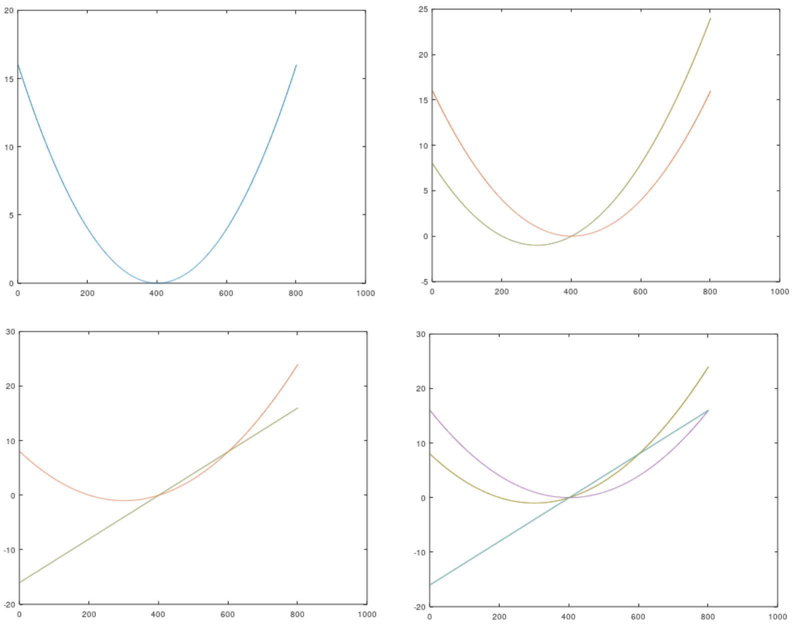

Figure 3.4: The hyperbolicity cones associated to the derivatives of the hyperbolicity polynomials intersected with a plane $L = \{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ in $\mathbb{R}^2$, (a) $\Lambda_+(f, e)$, (b) $\Lambda_+(f, e) \subset \Lambda_+(D_e f, e)$, (c) $\Lambda_+(D_e f, e) \subset \Lambda_+(D_e^{(2)} f, e)$, (d) all the derivative cones.

The homogeneous polynomial $D_e f$ is hyperbolic with respect to direction $e = (1,1,1)$ and it can be written as a determinantal representation with tree variable :

$$yz + xz + xy = \det \begin{bmatrix} x+z & z \\ z & y+z \end{bmatrix}$$
$$= \det \left(x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

and

$$e_1 A + e_2 B + e_3 C \succeq 0$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

To find the first derivative cone of polyhedron. SANYAL suggested the following theorem in his article [5]

Theorem 5. *(Sanyal)*
Let $P = \{x \in \mathbb{R}^n : \ell_i(x) \geq 0, \text{for } i \in 1, \dots, d\}$ be a fulldimensional polyhedral cone. Let $e \in \text{int } (P)$ and assume that $\ell_i(e) = 1$ for all i. Then the first derivative cone is given by all $x \in \mathbb{R}^n$ satisfying

$$\begin{bmatrix} \ell_1(x) + \ell_n(x) & \ell_n(x) & \cdots & \ell_n(x) \\ \ell_n(x) & \ell_2(x) + \ell_n(x) & \cdots & \ell_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ \ell_n(x) & \ell_n(x) & \cdots & \ell_{n-1}(x) + \ell_n(x) \end{bmatrix} \succeq 0.$$

Example 13. Let $f(x_1, x_2, x_3) = (x_1 - x_3)(x_1 + x_3)(x_2 - x_3)(x_2 + x_3) \in \mathbb{R}[x_1, x_2, x_3]$ the hyperbolic polynomial with respect to direction $e = (1,1,0)$, since $f(e) = 1 \neq 0$ and $Ch_a(f) = f(te + a)$. Then we have

$$\Lambda_+(f, e) = \{a \in \mathbb{R}^3 / -a_1 + a_3 \geq 0; -a_1 - a_3 \geq 0; -a_2 + a_3 \geq 0; -a_2 - a_3 \geq 0\}$$

is polyhedron.
Let $l_1(x) = x_1 - x_3, l_2(x) = x_1 + x_3, l_3(x) = x_2 - x_3, l_4(x) = x_2 + x_3$, then by theorem of Sanyal the hyperbolicity cone of first derivative cone is :

$$\Lambda_+(D_e f, e) = \{a \in \mathbb{R}^3 / M(a) \succeq 0\}$$

is spectrahedron, where

$$M(a) = \begin{pmatrix} l_1 + l_4 & l_4 & l_4 \\ l_4 & l_2 + l_4 & l_4 \\ l_4 & l_4 & l_3 + l_4 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & a_2 + a_3 & a_2 + a_3 \\ a_2 + a_3 & a_1 + a_2 + 2a_3 & a_2 + a_3 \\ a_2 + a_3 & a_2 + a_3 & 2a_2 \end{pmatrix} \succeq 0.$$

4 Hyperbolic relaxation of Locally Positive Matrices

Most of the information in this chapter is taken on the paper [4] by G.Blekherman,Santanu S. Dey, Kevin Shu, and Shengding Sun.

We consider a positive semidefinite relaxation of $n \times n$ symmetric matrices over a collection of sub-matrices on all $k \times k$ principal sub-matrices. We call a matrix in this class k-locally positive semidefinite. In this chapter we study the k-locally positive semidefinite matrices and their hyperbolic relaxation, with some applications.

4.1 k-locally positive semidefinite matrices

Definition 7. We recall that a symmetric matrix $X \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if

$$\boldsymbol{u}^\top X \boldsymbol{u} \geq 0, \quad \forall \boldsymbol{u} \in \mathbb{R}^n.$$

Definition 8. A principal submatrix of a square matrix A is the matrix obtained by deleting any k rows and the corresponding k columns.

Theorem 6. *We define the set of k-locally positive semidefinite matrices :*

$$\mathcal{S}^{n,k} := \{X \in \mathrm{Sym}_n \mid \textit{ every } k \times k \textit{ principal submatrix of } X \textit{ is PSD } \}. \qquad (1)$$

Remark 7. • $\mathcal{S}^{n,k}$ is larger than the cone of positive semidefinite matrices, so $\mathcal{S}^n = \mathcal{S}^{n,n} \subset \mathcal{S}^{n,k}$.
• $\mathcal{S}^{n,k}$ is the cone of quadratic forms non-negative on all k dimensional coordinate sub-spaces.

Example 14. The symmetric matrix

$$X = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \notin \mathcal{S}^{2,2}$$

because $det(X) < 0$ but $X \in \mathcal{S}^{2,1}$.
if $v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \implies v^T X v = 2v_1^2 \geq 0$
if $v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \implies v^T X v = v_2^2 \geq 0$

Property 3. $\mathcal{S}^{n,k}$ is a closed convex cone.

Proof. we consider

$$\mathcal{S}^{n,k} = \{A \in \mathrm{Sym}_n \mid A_I \succeq 0, \forall I \subseteq \{1, ..., n\}, |I| = k\}$$

where A_I the principal sub-matrix corresponding to the rows and columns indexed by I .
Let $A, B \in \mathcal{S}^{n,k}$ and $\lambda, \mu \in \mathbb{R}_+$
we prove that $(\lambda A + \mu B)_I \succeq 0, \forall I \subseteq \{1, \ldots, n\}$
we have $A \in \mathcal{S}^{n,k} \implies \lambda A \in \mathcal{S}^{n,k}$ and $B \in \mathcal{S}^{n,k} \implies \mu B \in \mathcal{S}^{n,k}$.
So the combination

$$(\lambda A + \mu B)_I = \lambda A_I + \mu B_I \succeq 0.$$

because $\mathbb{S}_k^+$ is convex cone containing $\lambda A_I + \mu B_I \succeq 0$. □

Definition 9. The factor width of a real symmetric matrix A is the smallest integer k such that there exists a real matrix U, where $A = UU^T$, and each column of U contains at most k non zeros.

Corollary 2. $(\mathcal{S}^{n,k})^*$ *the dual cone of* $\mathcal{S}^{n,k}$ *is the set of symmetric matrices with factor width* k.

The main objective is to understand the properties of the eigenvalues of k-locally positive semidefinite matrices.

4.2 The hyperbolic relaxation

We will construct a natural relaxation of the set of eigenvalues of matrices in $\mathcal{S}^{n,k}$. Let X be a matrix of $n \times n$, we recall the definition of the characteristic polynomial of X :

$$p_X(t) = \det(X - tI) = \sum_{k=0}^{n} (-1)^{n-k} c_k^n(X) t^{n-k}.$$

Where

$$c_k^n(X) = \sum_{\substack{S \subseteq [n] \\ |S| = k}} \det\left(X|_S\right).$$

Where $X|_S$ is the principal submatrix of X obtained by restricting X to the rows and columns contained in S. And $c_k^n(X)$ is a coefficient in the characteristic polynomial of X. Notice that if $X \in S^{n,k}$, then for all $S \subseteq [n]$ with $|S| \leq k$, $X|_S$ is PSD and in particular, then $\det(X|_S) \geq 0$. This implies that $c_k^n(X) \geq 0$.

Let introduce the set

$$\Lambda_+(e_k^n, \mathbb{1}) = \{\lambda \in \mathbb{R}^n : e_k^n(\lambda - t\mathbb{1}) = 0 \implies t \geq 0\}.$$

Where

$$e_k^n(\lambda) = \sum_{\substack{s \subseteq [n] \\ |s|=k}} \prod_{i \in S} x_i.$$

and $\lambda(X)$ be the vector of eigenvalues of X.
The set $\Lambda_+(e_k^n, \mathbb{1})$ is the hyperbolicity cone of the polynomial e_k^n with respect to the all ones vector.
We can rewrite the first observation for the following :

$$(2) \quad X \in \mathcal{S}^{n,k} \implies \lambda(X) \in \Lambda_+(e_k^n, \mathbb{1})$$

where the set $\Lambda_+(e_k^n, \mathbb{1})$ to be the hyperbolicity cone of the hyperbolic polynomial e_k^n with respect to the all ones vector $\mathbb{1}$. we will refer to $\Lambda_+(e_k^n, \mathbb{1})$ as the hyperbolic relaxation for the eigenvalues of $\mathcal{S}^{n,k}$.
The cone $\Lambda_+(e_k^n, \mathbb{1})$ also called the $(n-k)^{th}$ Renegar derivative of the positive semidefinite see [3] since we have :

$$e_k^n(x) = \sum_{\{i_1 \ldots i_k\} \in \{1, \ldots n\}} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

So

$$\begin{aligned}
e_n^n(x) &= x_1 x_2 \ldots x_n. \\
D_{\mathbb{1}} e_n^n(x) &= x_2 \ldots x_n + x_1 x_3 \ldots x_n + \cdots + x_1 x_2 \ldots x_{n-1} \\
&= e_{n-1}^n(x). \\
D_{\mathbb{1}}^2 e_n^n(x) &= x_3 \ldots x_n + x_2 \ldots x_{n-1} + x_3 \ldots x_n + x_1 x_3 \ldots x_{n-1} + \cdots + x_1 x_2 + \cdots + x_{n-2} \\
&= e_{n-2}^n(x). \\
D_{\mathbb{1}}^3 e_n^n(x) &= e_{n-3}^n(x).
\end{aligned}$$

Therefore, we conclude that

$$D_{\mathbb{1}}^{n-k} e_n^n(x) = e_k^n(x).$$

and

$$\Lambda_+(e_k^n, \mathbb{1}) = D_{\mathbb{1}}^{n-k}(\Lambda_+(e_n^n, \mathbb{1})).$$

Thus $\Lambda_+(e_k^n, \mathbb{1})$ is the $(n-k)^{th}$ Renegar derivative of the PSD.

We notice that when $k = n$ then $\Lambda_+(e_k^n, \mathbb{1}) = \mathbb{R}_+^n$, hence for $k = n$ we have that $\Lambda_+(e_k^n, \mathbb{1})$ is precisely the set of eigenvalue of matrices in$\mathcal{S}^{n,k}$, i.e.

$$\Lambda_+(e_k^n, \mathbb{1}) = \{\lambda(X) : X \in \mathcal{S}^{n,k}\}.$$

and when $k = n-1$ we have seen the equivalent of the expression (2) since we have the following theorem :

Theorem 7 (See the paper [4])**.** *If k is one of* $1, n-1$ *or* n*, and* $x \in H\left(e_k^n\right)$*, then x is the vector of eigenvalues of some matrix in* $\mathcal{S}^{n,k}$*.*

Proof. the proof given in [6] . $\square$

We have an important example of k-locally positive semi-definite matrices $\mathcal{S}^{n,k}$ and not in $\mathcal{S}^n$.

Example 15. Let

$$G(n,k) = \frac{k}{k-1}I - \frac{1}{k-1}\overrightarrow{1}\,\overrightarrow{1}^{\top}.$$

Where $\overrightarrow{1}$ indicates the all ones vector of dimension n.
For $n = 3, k = 2$ we have

$$G(3,2) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

We see that $G(3,2) \notin \mathcal{S}^{3,3}$because the determinant of $G(3,2)$ is non positive Then $G(3,2) \in \mathcal{S}^{3,2}$.

Corollary 3. *Let* $n-1 > k > 2$ *or* $(n,k) = (4,2)$*. Suppose that* $M \in \mathcal{S}^{n,k}$*, M is (n,k)-locally singular, and M is invertible. Then M must be diagonally congruent to* $G(n,k)$*.*

We can measure the distance between a matrix in $\mathcal{S}^{n,k}$ and the cone of PSD matrices, is by considering the smallest eigenvalue of such a matrix. We will see more details in the following theorem and corollary.

Theorem 8. *Let $\mathcal{F}$ be the class of function $F : \mathrm{Sym}_n \to \mathbb{R}$ so that F is a unitarily invariant matrix norm and Let $k \in \{2, \ldots, n\}$. $\tilde{G}(n,k) = \frac{G(n,k)}{F(G(n,k))}$. For any $M \in \mathcal{S}^{n,k}$ with $F(M) = 1$, the minimum eigenvalue of M is at least as large as the minimum eigenvalue of $G(n,k)$, that is,*

$$\lambda_1(M) \geq \lambda_1(\tilde{G}(n,k)) \quad \textit{for all} \quad M \in \mathcal{S}^{n,k}$$

such that $\quad F(M) = 1.$

Corollary 4. *Let* $\overline{\mathrm{dist}}\left(\mathcal{S}^{n,k}, \mathcal{S}^n\right) = \max_{A \in \mathcal{S}^{n,k} : \|A\|_F = 1} \left(\min_{G \in \mathcal{S}^n} \|A - G\|_F\right).$ *then we have that* $\overline{\mathrm{dist}}\left(\mathcal{S}^{n,k}, \mathcal{S}^n\right) \leq \frac{(n-k)^{3/2}}{\sqrt{(n-k)^2 + (n-1)k^2}}.$

Property 4. We have the important result :

$$X \in \mathcal{S}^{n,k} \text{ if and only if } \lambda(X) \in \Lambda_+(e_k^n, \mathbb{1}).$$

The result is true for $k = n$ and for $k = n - 1$
On the other hand $\{\lambda(X) : X \in \mathbb{S}^{n,k}\}$ is strictly contained in $\Lambda_+(e_k^n, \mathbb{1})$ for $2 < k < n - 1$ and for $k = 2, n = 4$.

Proof. Proof given in [4]. □

We end the chapter with a real example of the k-locally positive semidefinite in an optimization problem.

Example 16. Rank-constrained SDP relaxation problem of MAX-CUT. See the paper [6] [7].
We consider the standard semidefinite programming problem with additional rank constraints, as follows :

$$\begin{array}{ll} \min_X & < C, X > \\ \text{s.t.} & < A_i X >= b_i i = 1, \ldots, m \\ & X \succeq 0 \\ & rank(X) \leq k. \end{array}$$

Where the matrices A_i and C may be assumed to be symmetric without loss of generality, X symmetric matrices of size n, $b \in \mathbb{R}^m$ and k integer.
This problem is called rank-constrained positive semidefinite. One example is the relaxation of max-cut of a rank-constrained positive semidefinite relaxation.
The max-cut problem is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph $G = (V, E)$ with the set of vertices $V = \{x_1 \ldots x_n\}$ and edges set E, the problem consists in finding a partition of V into two parts so as to maximize the number of crossing edges that are cut by the partition.
Let G be a graph and $V = \{1, \ldots, n\}$ the set of vertices associated. Let the vector $x \in \{-1, 1\}$ represent any cut in the graph via the interpretation that the sets $\{i : x_i = +1\}$ and $\{i : x_i = -1\}$ specify a partition of the vertices set of the graph. Then the max-cut can be written as follows:

$$\begin{array}{ll} \max_X & x^T Q x \\ \text{s.t.} & {x_i}^2 = 1 \text{ for } i \in [n] \end{array}$$

To obtain an SDP relaxation,we now formulate max-cut in $\mathcal{S}^n$. We consider the change of variable $X = xx^T, x \in \{-1, 1\}^n$. Then $X \in \mathcal{S}^n$ and

$$\begin{aligned} x^T Q x &= trace(x^T Q x) \\ &= trace(Q x x^T) \\ &= trace(QX) \end{aligned}$$

Then the max-cut is equivalent to:

$$(\text{MC}) \quad \begin{array}{ll} \max_X & Q \bullet X \\ \text{s.t.} & \text{diag}(X) = e \\ & rank(X) = 1 \\ & X \succeq 0 X \in \mathcal{S}^n. \end{array}$$

We replace that the constraint $rank(X) = 1$ and $X \succeq 0 X \in \mathcal{S}^n$ in this problem (MC) by the constraint $X \in \mathcal{S}^{n,k}$ is the set of k-locally semidefinite symmetric matrices. And the problem is therefore as follows:

$$(\text{MC}) \quad \begin{array}{ll} \max_X & Q \bullet X \\ \text{s.t.} & \text{diag}(X) = e \\ & X \in \mathcal{S}^{n,k}. \end{array}$$

it is an efficient relaxation to the set of $\mathcal{S}^{n,k}$. which makes it easier to obtain the solutions of original problem.

5 Conclusion

The first section of this project focused on the study of hyperbolic polynomials in the context of optimization problems, and we saw The determinantal representation of hyperbolic polynomials. In the second section we studied hyperbolic optimization problems and their relaxation, and we saw some applications to the derivative relaxation of hyperbolic polynomials. In the last section, we used the hyperbolic relaxations of the set of locally positive semidefinite matrices and we proposed ”rank-constrained SDP relaxation problem of MAX-CUT ” an application of k-locally positive semidefinite in an optimization problem.We ask some open questions : What are the limitations of relaxation and how can we deal with them while solving problems ?

References

[1] L. Garding. An inequality for hyperbolic polynomials. ˚ J. Math. Mech., 8(6):957–965, 1959.

[2] S. Naldi and D. Plaumann. Symbolic computation in hyperbolic programming. J. Alg. Appl., 17(10):1850192, 2018.

[3] James Renegar. Hyperbolic programs, and their derivative relaxations. Foundations of Computational Mathematics, 6:59–79, 2006.

[4] G.BLEKHERMAN, SANTANU S. DEY, KEVIN SHU, and SHENGDING SUN. Hyperbolic relaxation of k-locally positive semidefinite matrices. arXiv:2012.04031v1 [math.OC] 7 Des 2020.

[5] R. Sanyal. On the derivative cones of polyhedral cones. Advances in Geometry, 13(2):315–321, 2013.

[6] Miguel F. Anjos and Jean B.Lasserre, Introduction to Semidefinite,Conic and Polynomial Optimization (2016), pp.1-12.

[7] S.Naldi. Solving rank-constrained semidefinite programs in exact arithmetic — J. Symb. Comput. 85C:206–223 (2018).

[8] A. POLIATYKIN, Hyperbolic Polynomials and Hyperbolic Programming,2020.