Spectrahedral representations of plane hyperbolic curves

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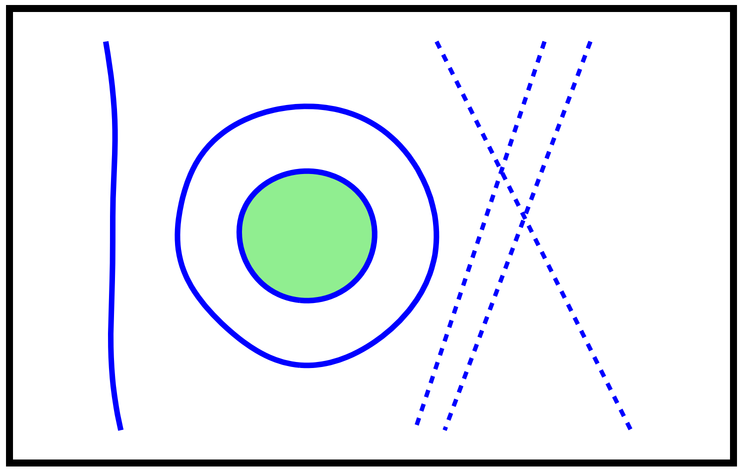

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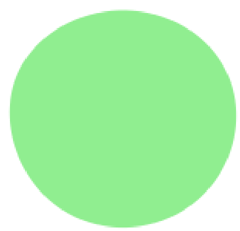
Convex semialgebraic set : $\Lambda_+(f,e) = \{a \in \mathbb{R}^n : f(a) \geq 0, \ldots\}$



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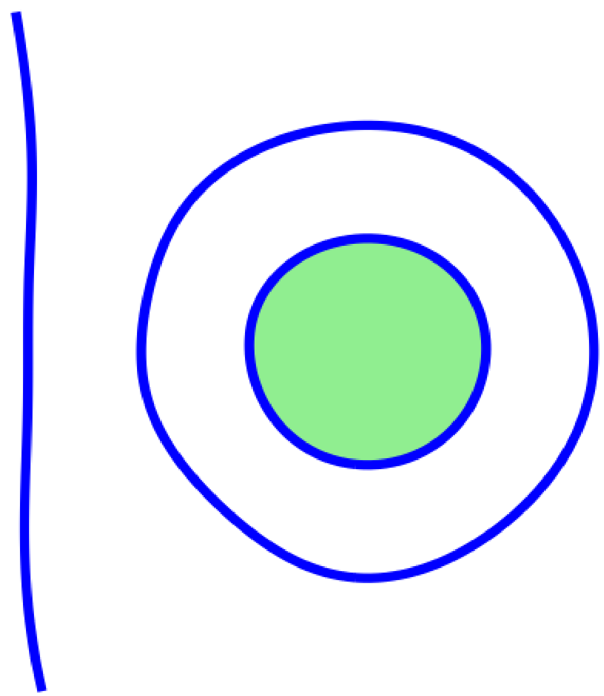

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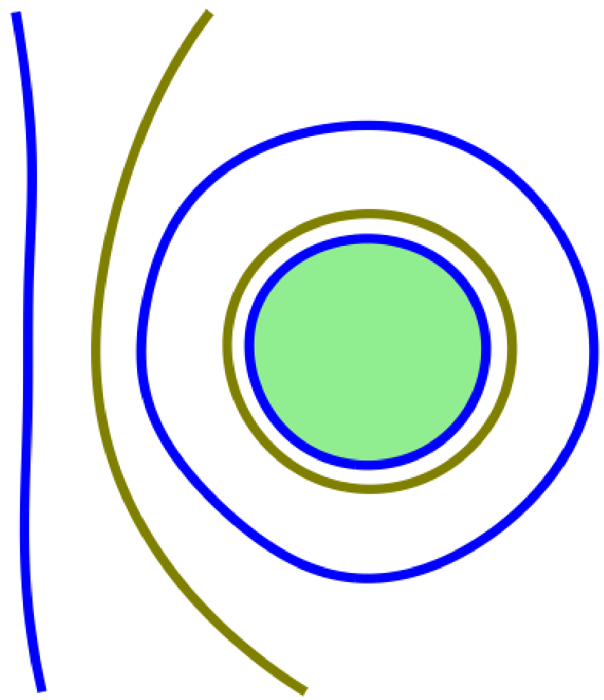

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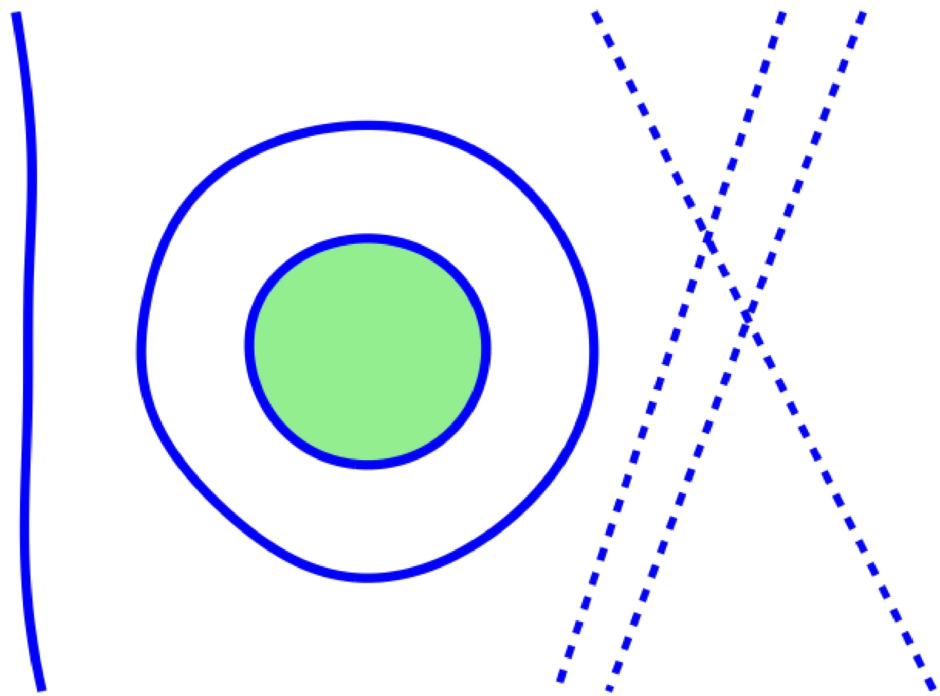
Interlacer : $g \in \mathbb{R}[x]_{\deg f - 1}$. Extra-factor : $\ell_1 \cdot \ell_2 \cdots \ell_s$



Hyperbolic polynomials (recap)

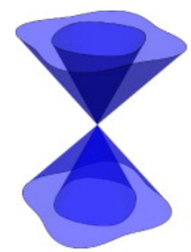
$f \in \mathbb{R}[x]_d$ is *hyperbolic w.r.t.* $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ if

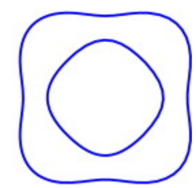
- $f(e) \neq 0$
- $\forall\, a \in \mathbb{R}^n \quad t \mapsto ch_a(t) := f(t\,e - a)$ has only real roots

$\Lambda_+(f, e) = \{a \in \mathbb{R}^n : ch_a(t) = 0 \Rightarrow t \geq 0\}$ *hyperbolicity cone*

$f = x_1 \cdots x_d$ $\qquad$ $f = \det X$, X symm.

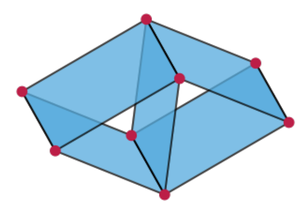
$ch_a(t) = \prod_i (te_i - a_i)$ $\qquad$ $ch_A(t) = \det(tI_d - A)$



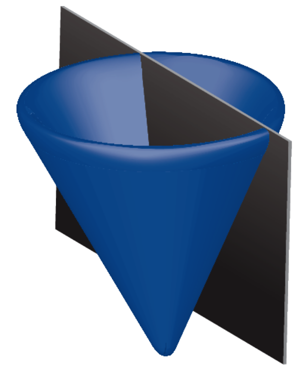


Special hp : $f = \det(x_1 A_1 + \cdots + x_n A_n)$, $A_1, \dots, A_n$ symm. with $e_1 A_1 + \cdots + e_n A_n \succ 0$, is hyp. with respect to $e = (e_1, \dots, e_n)$.

Brändén (2010): hyperbolic polynomials f without determinantal representations of type $f^k = \det(A(x))$ of any size



(Generalized) Lax Conjecture



Geometric version

Every hyperbolicity cone is a spectrahedron

$$\Lambda_+(f,e) = \{a \in \mathbb{R}^n : A_1 a_1 + \ldots + A_n a_n \succeq 0\}$$

Helton-Vinnikov – true for curves (even stronger : $f = \det A$)

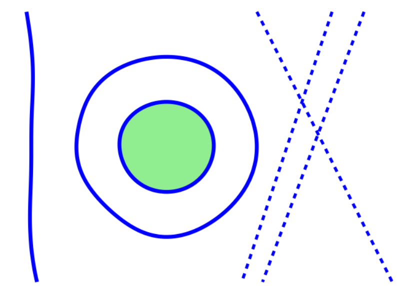
Brändén – GLC cannot be proved by means of det. represent. of f

Algebraic version

Let f be hyperbolic with respect to e. Then there exist a poly. q and matrices A_i such that

(1) $q \cdot f = \det(A(x))$

(2) $\Lambda_+(q,e) \supset \Lambda_+(f,e)$



$f =$ blue curve

$q =$ dashed curve

Kummer – (1) is true for real-smooth hyperbolic polynomials

Spectrahedral vs Determinantal representations

Given

A hyperbolic polynomial $f \in \mathbb{R}[x]_d$

A direction of hyperbolicity $e \in \mathbb{R}^n$

one would like to compute a spectrahedral representation

$$\Lambda_+(f, e) = \{a \in \mathbb{R}^n : A_1 a_1 + \ldots + A_n a_n \succeq 0\}$$

Can be achieved by computing determinantal representations

$f = \det A$	not necessary (Brändén counterexamples)
$f^k = \det A$	not necessary (Brändén counterexamples)
$q \cdot f = \det A$	open (equivalent to GLC)

Contact curves and Interlacers ($n = 3$)

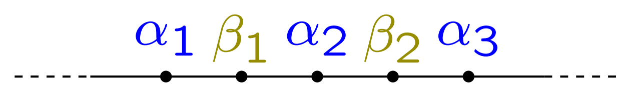
Let $f, g \in \mathbb{R}[x, y, z]$ be coprime. We say that

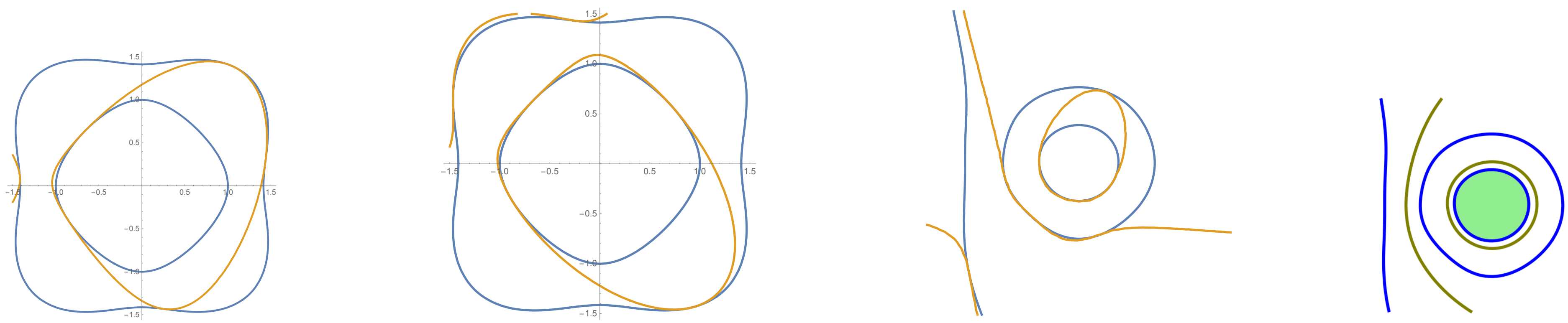
$p \in V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(g)$ is a *contact point* if $\mathrm{mult}_p(f, g)$ even

g is a *contact curve* (resp. *real contact curve*) if every $p \in V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(g)$ (resp. $p \in V_{\mathbb{R}}(f) \cap V_{\mathbb{R}}(g)$) is contact

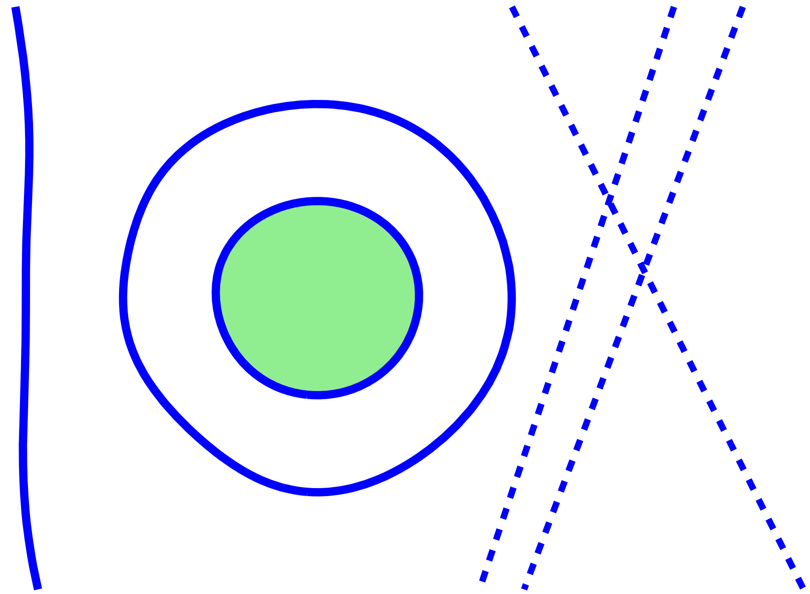
g is an *interlacer* if

$t \mapsto g(te + a)$ interlaces $t \mapsto f(te + a)$ for all a :





contact curve >> real contact curve << interlacers



Positive aspects of our contribution

The size of the det. repr. depends on $r =$ real contact points

There are (poss. large) spectrahedral representations over $\mathbb{Q}$

The multiplier is the simplest : $q = \ell_1 \cdot \ell_2 \cdots \ell_s$

For $r = d(d-1)/2$ (maximal) we get Helton-Vinnikov repr.

Extremal interlacers

We say that an interlacer is **extremal** if it is an extreme point of the set of interlacers (that is a cone). If f is smooth, any extremal interlacer has at least

$$\left\lceil \frac{(d+1)d-2}{4} \right\rceil$$

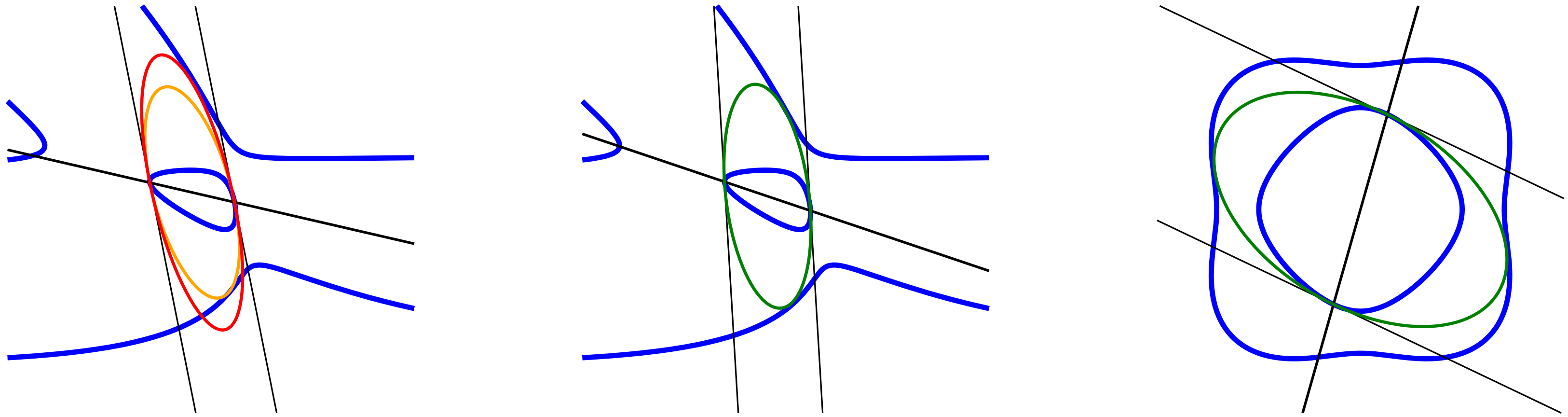
contact points (counted multiplicities).

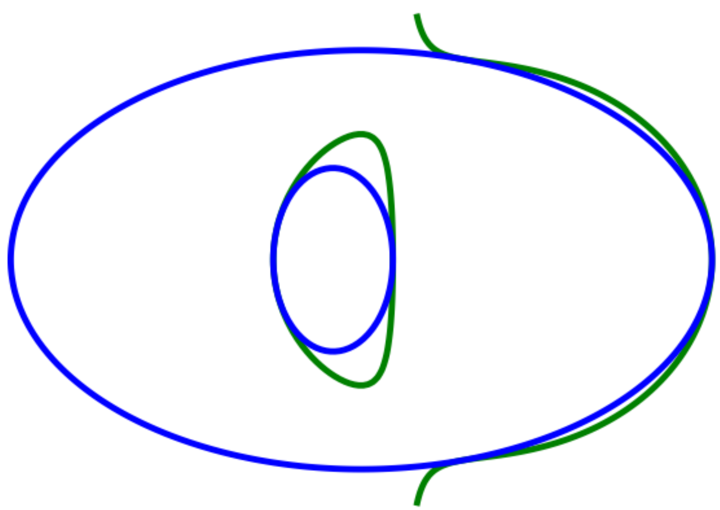
Expected number of real contact points compared with the number of points for a full contact curve:

d	2	3	4	5	6	$\cdots$
$\left\lceil \frac{(d+1)d-2}{4} \right\rceil$	1	3	5	7	10	$\cdots$
$\frac{d(d-1)}{2}$	1	3	6	10	15	$\cdots$

Not clear whether there always exist interlacers with $\frac{d(d-1)}{2}$ real contact points.

The case of quartics





Main result

Let f be hyperbolic with respect to e, and let g be an interlacer of f with r real contact points.

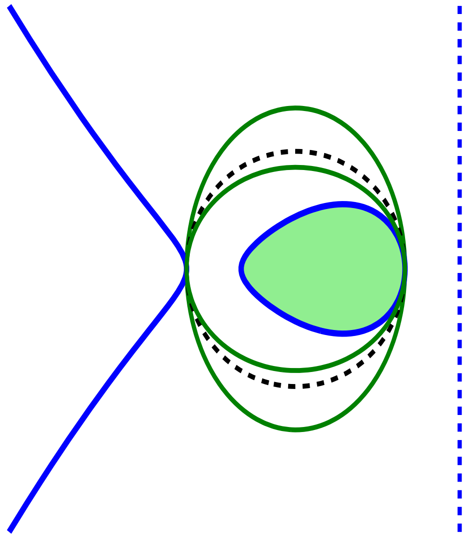
Idea. Fix the lack of g being a contact curve by increasing the multiplicity of the complex intersections of f with g. Let ℓ_i be the line through the conjugate points $p_i, \overline{p}_i \in V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(g)$.

Theorem. If f defines a smooth curve, and up to genericity assumptions on g there are real matrices A, B, C of size $m = (d^2+d-2r)/2$ such that $\Lambda_+(f,e) = \{xA+yB+zC \succeq 0\}$. Moreover

$$f \cdot \ell_1 \ell_2 \cdots \ell_s = \det(xA + yB + zC).$$

One example

The cubic $f = x^3 + 2x^2y - xy^2 - 2y^3 - xz^2$ is hyperbolic with respect to $e = (1, 0, 0)$.



By exact computation one gets :

Two interlacers living in $K[x, y, z]$ with $|K : \mathbb{Q}| = 4$ (green curves)

One rational interlacer (dashed green)

Corresponding spectrahedral representation:

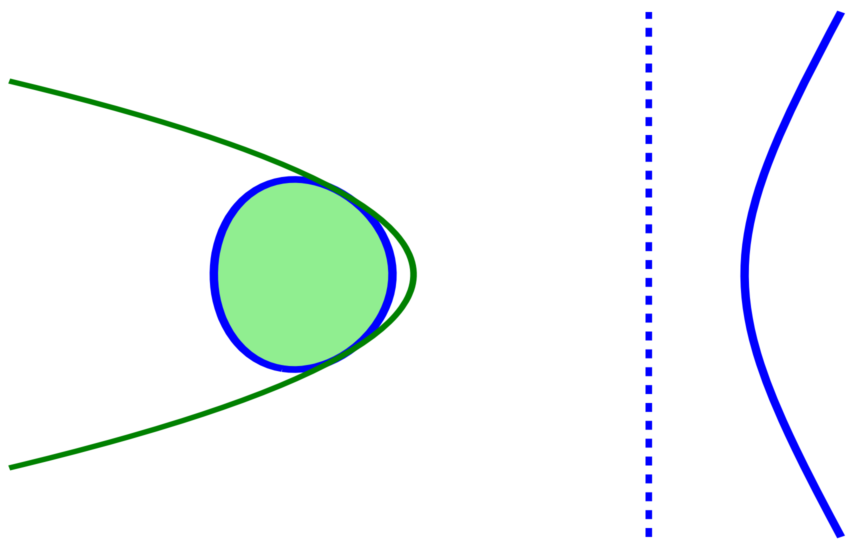
$$\frac{24}{125} f \cdot (2x - y) = \det \begin{pmatrix} 5x + 10y & -x - 2y & -4z & 2z \\ -x - 2y & x & 0 & 0 \\ -4z & 0 & 4x + 2y & -2x - 4y \\ 2z & 0 & -2x - 4y & 4x + 2y \end{pmatrix}.$$

Rational spectrahedral representation

$f = y^2z - (x^3 - 6x - 3)$ has no rational 3×3 determinantal representation (because $x^3 - 6x - 3$ is irreducible). But :

$$\Lambda_+(f,e) = \left\{ \begin{pmatrix} 3z & y & -x-z & -3x+z \\ y & -x+2z & 0 & -y \\ -x-z & 0 & z & x+4z \\ -3x+z & -y & x+4z & -x+18z \end{pmatrix} \succeq 0 \right\}.$$

Extra-factor is a line and the interlacer has two contact points:



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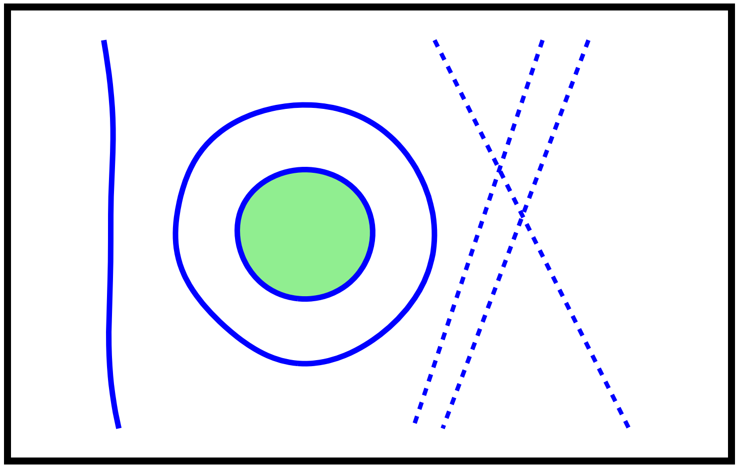


on the arXiv : `https://arxiv.org/abs/1807.10901`

Pac. J. Math. – in press

Genericity assumptions on g

No three of the intersection points of f with g lie on a line

No three of the ℓ_i pass through the same point

f does not vanish on any point where two of the ℓ_i intersect

Dixon process (Hermitian version with interlacer)

Essentially based on the property $\boxed{A \cdot A^{adj} = \det A \cdot \mathrm{Id}_d}$

Remark : if $f = \det A$ and $V_{\mathbb{C}}(f)$ is smooth, then

$$\text{co-rank}(A) = \text{rank}(A^{adj}) = 1 \mod \det A.$$

INPUT

f hyperbolic with resp. to e

Sketch of the PROCEDURE :

$m_{11} \leftarrow D_e f := e_1 \frac{\partial f}{\partial x} + e_2 \frac{\partial f}{\partial y} + e_3 \frac{\partial f}{\partial z}$ # interlacer

split $S \cup \overline{S} = V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(D_e f)$

extend m_{11} to basis $\langle m_{11} \dots m_{1d} \rangle = V(\langle S \rangle)$

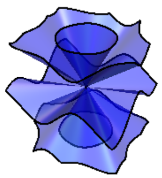
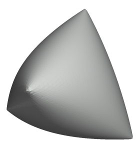
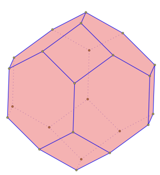
$m_{jk} \leftarrow$ solve $a_{11} a_{jk} - a_{1j} a_{1k} \in \langle f \rangle$ for $j \leq k$ # rank $= 1$

$M \leftarrow (m_{jk})$

$A \leftarrow M^{adj} / f^{d-2}$

OUTPUT A (satisfying $f = \det A$ and $A(e) \succ 0$)

Optimization viewpoint

Feasible set	*name*	**Optimization**	**Polynomial**
	Hyperbolicity Cone	HP	Hyperbolic polynomial
	Spectrahedron	SDP	$f = \det A(x)$
	Polyhedron	LP	$f = \prod \ell_i(x)$

Well-posedness of the Cauchy Problem

The *Cauchy problem*: Given $f \in \mathbb{R}[x]_{\leq d}$ and $\Omega \subset \mathbb{R}^n$ open :

Given $p \in C^\infty(\Omega)$ compute $u \in C^\infty(\Omega)$ such that

$$f(\partial_1, \ldots, \partial_n)u = p$$

Theorem (Lax, Mizohata). Decompose $f = \sum_{i \leq d} f_i$ with $f_i \in \mathbb{R}[x]_i$. If the Cauchy problem is well-posed (existence/uniqueness of solutions) then f_d is hyperbolic.

Example. The *Wave operator* in $(\partial_t^2 - \sum_i \partial_i^2)u = p$ corresponds to the polynomial $f = x_n^2 - \sum_{i=1}^{n-1} x_i^2$ (Minkowski). Its hyp. cone is the **second-order (or Lorentz) cone**

$$\Lambda_+(f, e) = \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq \sqrt{x_1^2 + \cdots + x_n^2}\}$$

Lax conjecture

Differential Equations, Difference Equations and Matrix Theory*†

P. D. LAX

Peter Lax conjectured in 1957 that this is true for polynomials in three (homogeneous) variables:

$$f = \det(x_1 A_1 + x_2 A_2 + x_3 A_3)$$

Helton-Vinnikov Theorem: The Lax conjecture is true, and the representation is of optimal size (size of A_i = degree of f).

Counting dimensions: Same statement is false for ≥ 4 variables.