

HOMEWORK / EXERCICES SESSION 4

Monomials and monomial order

Exercise 1. Let \prec be a monomial order, \mathbf{x}^α and \mathbf{x}^β be two monomials in \mathbb{M}_n . Show that $\mathbf{x}^\alpha | \mathbf{x}^\beta$ implies $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$.

Exercise 2. Let I be an ideal of $\mathbb{K}[x_1, \dots, x_n]$, show that $\text{in}(I) := \langle \text{LT}(I) \rangle = \{\text{LT}(p) | p \in I\}$ is an ideal (of course a monomial one).

Exercise 3. Find an instance of two monomial such that they do not form a Gröbner basis of the ideal they generate, i.e. find $I = \langle \mathbf{x}^\alpha, \mathbf{x}^\beta \rangle$ such that $\{\mathbf{x}^\alpha, \mathbf{x}^\beta\}$ is not a Gröbner of I .

Exercise 4. Let I_1, I_2 and I_3 be three monomial ideals. Prove that $(I_1 + I_2) \cap I_3 = (I_1 \cap I_3) + (I_2 \cap I_3)$.

Exercise 5. Let I_1 and I_2 be two monomial ideals, show that $I_1 \cap I_2$ is a monomial ideal (you can use precedings exercices).

Basics on Gröbner basis

Exercise 6. Let $f_1 = x^2y + z$ and $f_2 = xz + y \in \mathbb{K}[x, y, z]$.

1. Compute a Gröbner basis of $\langle f_1, f_2 \rangle$ for lex order with $x > y > z$.
2. Dividing $f = x^2z^3 - xy^2 - zy^2 + z^2$ by the computed Gröbner basis, show that $f \in \langle f_1, f_2 \rangle$. Would you be able to conclude without computing the Gröbner basis ?
3. Determines q_1 and $q_2 \in \mathbb{K}[x, y, z]$ such that $f = q_1 f_1 + q_2 f_2$.

Exercise 7. Let I be an ideal of $\mathbb{K}[x_1, \dots, x_n]$ and G be a finite subset of I . Show that G is a Gröbner basis of I with respect to a given monomial order \prec if and only if for each $f \in I$ the remainder of the division of f by G is 0.