Homework / Exercices session 2

Exercise 1. This is a "theoretical" exercice. The aim is t use resultants to understand the geometry of cubic polynomials with multiple roots. Here, $f(x) = x^3 + a * x^2 + b * x + c$.

i. Show that $f(x)$ has at least a multiple root if and only if $27 * c^2 - 18 * c * a * b - a^2 * b^2 + 4 * a^3 c + 4b^3 = 0$.

ii. Using resultant, show that $f(x)$ has a triple root if and only if $9 * a * b - 27 * c - 2 * a^2 = 0$.

iii. Using only derivation, show that $f(x)$ has a triple root if and only if $x = -\frac{a}{3}$ is the triple root of $f(x)$.

iv. Deduce from the previous question that $f(x)$ has a triple root if and only if $b = \frac{a^2}{3}$ and $c = \frac{a^3}{27}$.

Exercise 2. Consider the following polynomials: $P(X, Y) = \frac{X^2}{2} + Y^2 - 1$ and $Q(X, Y) = X - Y$. The polynomial $P(X, Y)$ define an ellipse in the plane and $Q(X, Y)$ is the first diagonal. The aim of this exercise is first to determine if the curves have common points and to compute then if it is the case. Here, we denote $\mathbb{K} = \mathrm{Frac}(\mathbb{Q}[X])$.

i. Compute $R(X) = \mathrm{Res}_Y(P, Q)$ (here both P and Q are considered as polynomials in $\mathbb{K}[Y]$).

ii. Compute ζ_1 and ζ_2, the two roots of $R(X)$. What are you able to say about $P(\zeta_i, Y)$ and $Q(\zeta_i, Y)$, for $i \in \{1, 2\}$. Deduce two intersection points of the curves.

iii. Show that the curves have no other intersection point.