Master of Mathematics ACSYON/SIC
Polynomial Systems

Université de Limoges
Faculté des Sciences et Techniques

HOMEWORK / PRACTICAL SESSION 1

In what follows the polynomial denoted $f(x)$ is assume to be squarefree.

Exercise 1. Let $f(x)=x^d+\sum_{i=0}^{d-1} f_i\, x^i$ and $g(x)$ be two polynomials. We denote $\mathbb{A}=\mathbb{K}[x]/(f)$.

1. Show that $\mathcal{M}_g : \left\{ \begin{array}{l} \mathbb{A} \longrightarrow \mathbb{A} \\ h \longmapsto g*h \end{array} \right.$ is a $\mathbb{K}$-linear map.
2. Find (the entries of) the matrix associated to $\mathcal{M}_g$ in the monomial basis.
3. Write the matrix of $\mathcal{M}_g$ in the Lagrange basis.
4. Give an polynomial condition depending only on the coefficients of the polynomials f and g vanishing if and only if the two polynomial share a root.

Exercise 2. Using the Lagrange basis, show that if a polynômial $f(x)$ divides a polynomial $g(x)$ then all the roots of $f(x)$ are also roots of $g(x)$.

Exercise 3. Let $f(x)=f_d*\prod_{i=1}^{d}(x-\zeta_i)\in\mathbb{C}[x]$ and let $\mathbb{A}=\mathbb{C}[x]/(f)$. We consider $g(x)\in\mathbb{A}$ and the following map:

$$\mathcal{P}: \left\{ \begin{array}{l} \mathbb{A} \longrightarrow \mathbb{A} \\ h \longmapsto \frac{h*h}{\|h*h\|_2} \end{array} \right.$$

where $\|h\|_2=\sqrt{\sum_{i=0}^{d-1}|h_i^2|}$ if $h(x)=\sum_{i=0}^{d-1} h_i\, x^i$. We consider the sequence defined by $H_i=\mathcal{P}(H_{i-1})$ and $H_0=h$. We assume that $|h(\zeta_1)|>|h(\zeta_i)|$ for all $i\in\{2,\ldots,d\}$.

1. Show that there exists a sequence of complex numbers (a_i) such that $|a_i|=1$ and such that the sequence (a_i*H_i) convergences.
2. Show that the $H=\lim_{i\longrightarrow\infty} a_i*H_i$ is an eigenvector of the map $\mathcal{M}_h: g\in\mathbb{A}\longmapsto h*g\in\mathbb{A}$.
3. What is the eigenvalue associated to this eigenvector.
4. We now assume that the roots of $f(x)$ are such that $|\zeta_1|>|\zeta_i|$ for all $i\in\{2,\ldots,d\}$. Give an iterative procedure to approximate ζ_1.

This method is called “power iteration” in numerical linear algebra.

Exercise 4. Let $f(x)=f_d*\prod_{i=1}^{d}(x-\zeta_i)\in\mathbb{C}[x]$, $\mathbb{A}=\mathbb{C}[x]/(f)$ and $g\in\mathbb{A}$.

1. Show that if $\mathcal{M}_g:\mathbb{A}\longrightarrow\mathbb{A}$ has maximal rank d then $f(x)$ and $g(x)$ have no common root.
2. Assume $\mathcal{M}_g$ invertible. Express the inverse element $g^{-1}\in\mathbb{A}$ in the monomial basis in terms of its expression in the Lagrangian basis and the Vandermonde matrix.
3. Assume now that $f_0\neq 0$ and that $|\zeta_1|<|\zeta_i|$ for all $i\in\{2,\ldots,d\}$. Using the previous exercice, give an algorithm to compute ζ_1.

In linear numerical algebra, this is the inverse power iteration.