Practical session

In what follows the polynomial denoted $f(x)$ is assume to be squarefree of degree d.

The aim of this practical session is to design and implement a complete algorithm to approximate all the roots of an univariate suqarefree polynomials with only simple complex roots. The method we propose to study there is the homoty or path following method.

Sketch of the method: We want to compute approximations of all the roots of a given squarefree polynomial $f(x)$ of degree d. Denote $\mathcal{Z}(f) = \{\zeta_1, \ldots, \zeta_d\} \subset \mathbb{C}$ the set of its roots. Choose d distinct points $z_1, \ldots, z_d$ in $\mathbb{C}$, we define $g_z(x) = \prod_{i=1}^{d} (x - z_i)$ and $F(t, x) = t * f(x) + (1 - t) * g_z(x)$. Remark that $\mathcal{Z}(g_z) = \{z_1, \ldots, z_d\}$ and that $\mathcal{Z}(F(0, x)) = \mathcal{Z}(g_z)$ and $\mathcal{Z}(F(1, x)) = \mathcal{Z}(f)$. The goal is to move t from 0 to 1 and to follow the roots of $F(t, x)$ using the Newton method when t moves. To do it, we consider N a sufficiently large integer, and we denote $\delta_i = \frac{i}{N}$, for $i = 0 \ldots N$. So we deform g_z to f following the roots of the intermediate polynomials using the Newton method.

You must send us both your worksheet and you function file at the end of the session : `olivier.ruatta@unilim.fr`, `thomas.lickteig@unilim.fr`.

1. **Basics for real solving in Maple**

   Here we describe a function allowing to compute the real roots of a polynomial in a given compact interval. Let $-\infty < a < b < +\infty$ be two real number and $f(x)$ be a polynomial. We consider the following Maple function:

   ```
   MySolve := proc(f,x,a,b)
     RETURN({solve({f,x>=a,x<=b},x)});
   end;
   ```

   Test this function with:

   i. `f := x^2-1; a := 0; b := 2;`

   ii. `f :=x^3-.25*x^2-.875*x+.375; a := 0; b := 1;`

   iii. `f :=x^3-.25*x^2-.875*x+.375; a := -2; b := 2.`

   Using the help on the `solve` function (`?solve`), find what this function. Using the help on `nops` function (`?nops`), write a function `MyNbRealRoots(f,x,a,b)` computing the number of real roots in a given compact interval.

2. **Basics on plotting with Maple**

   Load the `plots` library (`with(plots);`). Consider the following Maple function:

   ```
   MyFuncPlot := proc(f,x,a,b,grid)
     RETURN(CURVES([seq([a+i*(b-a)/grid, evalf(eval(f,x=a+i*(b-
   a)/grid))],i=0..grid)]));
   end;
   ```

   Test the following instructions:

   i. `P := MyFuncPlot(x^2,x,-2,2,100): display(P);`

   ii. `P := MyFuncPlot(sin(x),x,-20,20,1000): display(P);`

   iii. `P := MyFuncPlot(x^3-x^2+x-3,x,-10,10,1000): Q := MyFuncPlot(x^2-x+1,x,-10,10,1000): display(P,Q);`

   Using the help on the `plots` library, find what this function do. You can also take a look to `plot,structure`.

3. **Basics for the Newton method**

We recall that the Newton iteration apply to f at z^{old} is given by $z^{\text{new}} \leftarrow z^{\text{old}} - \frac{f(z^{\text{old}})}{f'(z^{\text{old}})}$. Write a prodedure `MyNewtonMethod(f,x,z,prec)` taking as input polynomial `f` of the variable `x`, a (assumed good) starting point `z` for the Newton method and a precision `prec > 0` and returning an approximation of an actual zero of `f` with precision at least `prec`.

4. **Deciding if a complex coefficients polynomial as real root on an interval**

   Remark that if $f(x) = \sum_{i=0}^{d} f_i * x^i \in \mathbb{C}[x]$, then $f(x) = \hat{f}(x) + i * \tilde{f}(x) = \sum_{i=0}^{d} \mathrm{Re}(f_i) * x^i + i * \sum_{i=0}^{d} \mathrm{Im}(f_i) * x^i$ where both $\hat{f}(x)$ and $\tilde{f}(x) \in \mathbb{R}[x]$. Let $-\infty < a < b < +\infty$, show that $f(x)$ as a real root in $[a,b]$ if and only if $\tilde{f}(x)$ and $\hat{f}(x)$ have a common root in $[a,b]$, i.e., if and only if $\hat{f}(x) \wedge \tilde{f}(x)$ (it is a real coefficient polynomial) has a root in $[a,b]$. Using the `gcd` function of Maple and the `MyNbRealRoots` function write a function `HasRealRoot(f,x,a,b)` deciding when a complex coefficients polynomial $f(x)$ as a real root on a given compact interval $[a,b]$.

5. **Polynomial from its roots and parametrized polynomial**

   Taking d distinct values $z_1, \ldots, z_d$, make a function `PolFromRoots(Z,x)` returning the polynomial $g_z(x) = \prod_{i=1}^{d} (x - z_i)$ (think to the function `expand`). Build a function `Homotop(Z,f,t,x)` taking $Z = [z_1, \ldots, z_d]$ a list of d distinct complex points, a polynomial `f` of degree d of the variable `x` and returning $F(t,x) = t * f(x) + (1-t) * g_z(x)$.

6. **Characterizing good path**

   Read the help of the Maple function `Resultant(f,g,x)`. Compute $C(t) = \mathrm{Res}_x(F(t,x), \frac{d}{dx}F(t,x))$. Show that if $C(t)$ as no real root in $[0,1]$, then for all $t \in [0,1]$, the polynomial $F(t,x)$ is squarefree. It is to say that if $[z_1(t), \ldots, z_d(t)]$ are the roots of $F(t,x)$ then, for all $i = 1, \ldots, d$, $z_i(t)$ is a continuous path such that $z_i(0) = z_i$ and $z_i(1) = \zeta_j$ for $j \in \{1, \ldots, d\}$. Write a function `IsGoodPath(F,t,x)` deciding if for all $t \in [0,1]$, $F(t,x)$ is squarefree.

7. **Follow the path**

   Now, if every previous function is all right, you are ready to use the function `PathFollowing(Z,f,x,nbstep,prec)` that you can find in the file `TP1.fun`. Read carefully the code of this function. In this function, `Z` is a list of starting points allowing to define the initial polynomial during the homotopy, `f` is the polynomial we want to solve, `x` is the unknown, `nbstep` is tha number of step to be made during the homotopy and `prec` is the precision for the Newton iteration.

   i. Test this function with starting points `[3,4]` to solve $x^2 - 1$ using 20 steps and keeping precision of 10^{-5}.

   ii. Do the same test with satrting points `[I,-I]`. What happen in your opinion ?

   iii. Solve $x^2 - 8*x + 12$ starting from `[3+5*I,4+2*I]` with precision at lest 10^{-6}.

8. **Draw the paths**

   In the same file as in the previous ppoint, you will find the function `GraphPathFollowing(Z,f,x,nbstep,prec)` using the same argument that `PathFollowing` but it return a list of curves (so use `:` instead of `;` to store the result in a variable). To display the curves, use the `display` function.

   i. Draw the paths of the last exemple of the previous point.

   ii. Draw the path to solve $f(x) = 30 * I - (18 + 23 * I) * x + (9 + 3 * I) * x^2 - x^3$ starting from `[5+4*I,3*I,2-3*I]` using 30, 20, 10 and 5 steps. Comment you observations.

   iii. Find examples that work and fail with degree 4 polynomials.