Bézier Curves: Topological Convergence of the Control Polygon

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Abstract. In terms of distance (e.g., Hausdorff distance), the control polygon of a Bézier curve converges to the curve via the de Casteljau subdivision. This convergence is widely studied in the literature. In this paper, we adopt a different point of view for the convergence problem: We study whether the control polygon preserves the topology of the associated curve. In this purpose, we deal with two main topological features of the Bézier curve, the existence of multiple points and the convexity.

§1. Introduction

The geometric properties of the Bézier curves need no longer to be introduced in CAGD. For many applications, the convergence of the control polygon to the associated curve via the de Casteljau subdivision is the most important of these properties. Namely, in many cases it is not necessary to deal with the curve itself, a "good enough" polygonal approximation suffices.

We are thus brought to study how good an approximation the control polygon can become by subdivision. If this question is asked in terms of distance (e.g., Hausdorff distance), the problem has been well studied in the literature and the answer is satisfying, especially for convex Bézier curves. But the question above can be also addressed from a topological point of view, that is, we can study whether the control polygon has the same topological characteristics as its associated curve. This problem is an important one among the emerging challenges in computational topology and, as far as we know, it has never been treated.

The two topological features that we shall deal with in this paper are the existence of multiple points and the convexity (we recall that a convex curve is a curve which lies on the boundary of its convex hull). Thus, there are three questions to be answered:

- can one associate a self-intersecting control polygon to any self-intersecting Bézier curve?

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– can one associate a non-self-intersecting control polygon to any non-self-intersecting Bézier curve?
– can one associate a convex control polygon to any convex Bézier curve?

It is obvious that the fourth question that can be formulated has no object, because a non-convex Bézier curve cannot have a convex control polygon.

The topology-preserving control polygon that we are looking for should be obtained by successive de Casteljau subdivisions of the initial control polygon. Moreover, any control polygon obtained from it by subdivision must also be topology-preserving. From this point on, we shall say that a topological property of a control polygon $\mathcal{P}$ is subdivision invariable if any control polygon obtained by subdivision from $\mathcal{P}$ satisfies this property. We notice that in this paper, we restrict our study to the case where the subdivision parameter is fixed and equal to $1/2$.

The layout of the paper is the following: In Section 2, we answer the first one of the three questions above. In Section 3, we briefly recall the definition and main properties of the hodograph of a Bézier curve. In Sections 4 and 5, we respectively address the two remaining questions. We remark that the answer to the second question depends on the regularity of the Bézier parametrisation of the curve. In Section 6, we give a few conclusions, as well as some remarks on the use of an adaptative subdivision parameter.

§2. Self-Intersecting Bézier Curve

In Figure 1, we can see an example of Bézier curve which has multiple points and such that its control polygon is not self-intersecting. As the control polygon converges to the curve in terms of distance, the proof of the following lemma is straightforward:

Lemma 1. *Let $\mathcal{B}$ be a self-intersecting Bézier curve. Then we can subdivide $\mathcal{B}$ until its control polygon becomes* subdivision invariably *self-intersecting.*

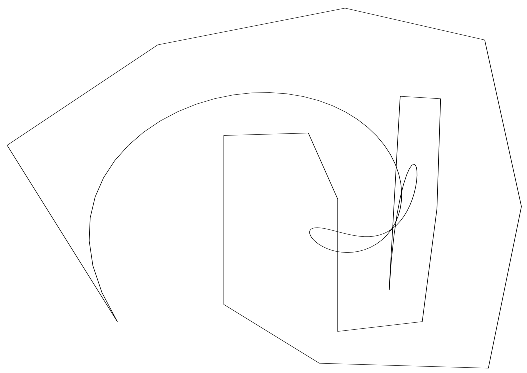

Fig. 1. Self-intersecting Bézier curve with non-self-intersecting control polygon.

For the other two questions of the introduction, our discussion is based on the connection between the studied Bézier curve and its hodograph. Therefore, let us briefly recall the definition and main properties of the last.

3. Hodograph of a Bézier Curve

Let $\boldsymbol{c}$ be a $\mathcal{C}^1$ curve. Then its **hodograph** is the curve described by the endpoints of the tangent vectors of $\boldsymbol{c}$ translated to a common origin. Of course, the hodograph depends on the parameterisation of the curve. It is easy to see that the hodograph of a Bézier curve $\mathcal{B}$ of degree n is a Bézier curve $\mathcal{H}(\mathcal{B})$ of degree $n-1$. The control points of $\mathcal{H}(\mathcal{B})$ are the points

$$\frac{1}{n}(\mathrm{P}_{i+1} - \mathrm{P}_i) \qquad \text{for } i \in \{0, \ldots, n-1\},$$

where P_i are the control points of $\mathcal{B}$, $i \in \{0, \ldots, n\}$.

The properties of the hodograph that we are going to use do not depend on the scale at which this one is represented. Thus, we can consider that the control polygon of $\mathcal{H}(\mathcal{B})$ is $\mathcal{Q} = \mathrm{Q}_0 \ldots \mathrm{Q}_{n-1}$, where $\mathrm{Q}_i = \mathrm{P}_{i+1} - \mathrm{P}_i$ for all $i \in \{0, \ldots, n-1\}$. In Figure 2, we present an example of Bézier curve and its associated hodograph.

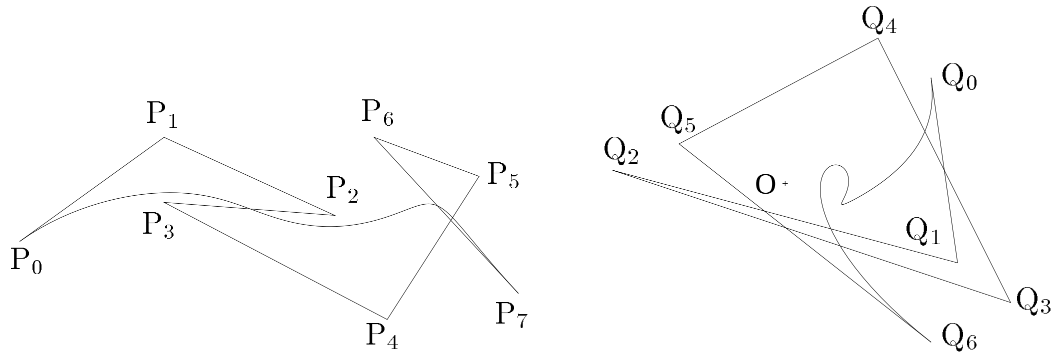

Fig. 2. A Bézier curve and its hodograph.

Let us now suppose that we subdivide the curve $\mathcal{B}$ using a parameter t_0. We denote by $\mathcal{B}'$ and $\mathcal{B}''$ the two resulting Bézier curves. Then we can see that the hodographs of $\mathcal{B}'$ and $\mathcal{B}''$ are respectively the two Bézier curves obtained from $\mathcal{H}(\mathcal{B})$ by subdivision of parameter t_0.

Now let us answer the two remaining questions on the topological properties of the control polygon.

§4. Non-Self-Intersecting Bézier Curve

As in the previous case, it is well known that there exist Bézier curves which have no multiple points and such that their control polygon is self-intersecting. Such a curve is presented in Figure 3.

As we have mentioned in the introduction, the existence of a control polygon with no multiple points depends on the regularity of the Bézier parameterisation: If the derivative does not vanish, then there exists a topology-preserving control polygon; otherwise, there are cases where any control polygon obtained by subdivision of parameter $1/2$ is self-intersecting.

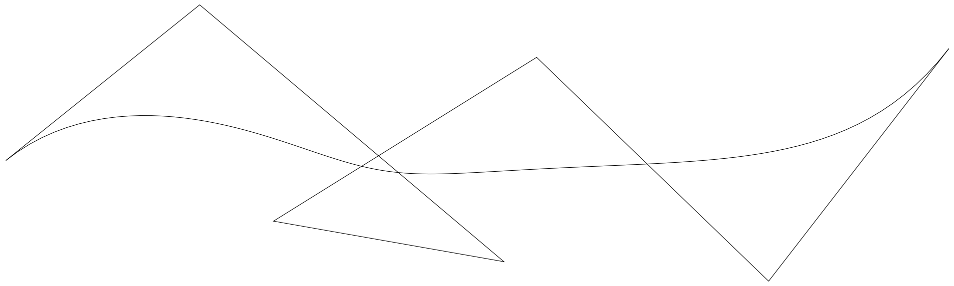

Fig. 3. Non-self-intersecting Bézier curve with self-intersecting control polygon.

4.2. Regular Bézier parameterisation

In order to prove the existence of a non-self-intersecting control polygon, let us first take a look at the angular properties of a self-intersecting polygonal chain.

Let $\mathcal{P} = \mathrm{P}_0\mathrm{P}_1 \ldots \mathrm{P}_n$ be a polygonal chain. For $i \in \{1, \ldots, n-1\}$, the angle $\widehat{\mathrm{P}}_i$ is the oriented angle between the oriented lines $\overrightarrow{\mathrm{P}_{i-1}\mathrm{P}_i}$ and $\overrightarrow{\mathrm{P}_i\mathrm{P}_{i+1}}$. If the segments $[\mathrm{P}_{i-1}\mathrm{P}_i]$ and $[\mathrm{P}_i\mathrm{P}_{i+1}]$ overlap, we set $\widehat{\mathrm{P}}_i = \pi$; thus, $\widehat{\mathrm{P}}_i \in (-\pi, \pi]$. In Figure 3, we show the signs of the angles associated to the vertices of a polygonal chain.

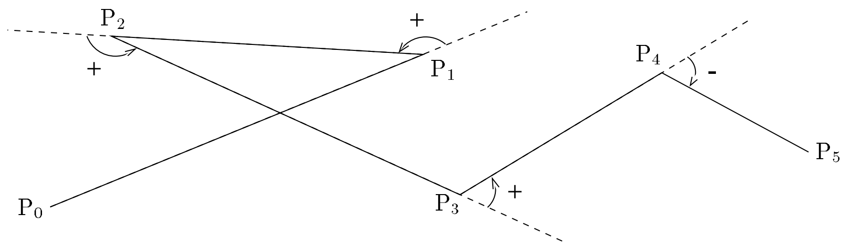

Fig. 4. Angles along a polygonal chain.

Using these notations, it is easy to prove the lemma below.

Lemma 2. *If $\mathcal{P}$ is self-intersecting, then at least one of the two following properties is fulfilled:*

1) *there exists $k \in \{1, \ldots, n-1\}$ such that $\widehat{\mathrm{P}}_k = \pi$;*
2) *there exist $k_1, k_2 \in \{1, \ldots, n-1\}$, $k_1 < k_2$, such that $\sum_{i=k_1}^{k_2} \widehat{\mathrm{P}}_i \notin [-\pi, \pi]$.*

Now we can state the main result of this subsection.

Proposition 3. *Let $\mathcal{B}$ be a Bézier curve with no multiple points and regular Bézier parameterisation. Then we can subdivide it until its control polygon becomes subdivision invariably non-self-intersecting.*

Proof: In this proof, all the subdivisions that we perform are concurrently applied to the curve $\mathcal{B}$ and to its hodograph.

It is obvious that we can subdivide $\mathcal{B}$ until any two distinct Bézier curves that we obtain have subdivision invariably disjoint control polygons. Once this property achieved, we subdivide until the control polygon of each sub-Bézier curve of the hodograph can be separated from the origin by a line. As the first derivative of $\mathcal{B}$ does not vanish, the hodograph does not pass through the origin, so this is possible. It is also clear that this property is subdivision invariable.

It is easy to see that if the control polygon of the hodograph is separated from the origin by a line, then the control polygon of the curve cannot be self-intersecting. Indeed, the segments that join the origin to the control points of the hodograph are parallel to the edges of the control polygon of the curve. Thus, two such edges cannot overlap (the segment joining the corresponding control points of the hodograph should pass by the origin, impossible) and the sum of any successive angles belongs to the interval $(-\pi, \pi)$. Using the previous lemma, we deduce that the control polygon of the curve has no multiple points.

So, each control polygon of the sub-curves is non-self-intersecting and any two such polygons are disjoint. We conclude that the control polygon of $\mathcal{B}$ has no multiple points, and this is a subdivision invariable property. $\square$

We remark that in order to be sure of the subdivision invariability of the non-self-intersecting property, we really have to subdivide the curve until its hodograph satisfies the criterion of the proof above. Indeed, there are cases where an initialy non-self-intersecting control polygon can become self-intersecting after subdivision. In Figure 5, we present such an example. The first picture represents a Bézier cubic. The second one shows the upper half of the previous after subdivision and scaling. The third is obtained in the same way from the second and finally the forth presents the third after subdivision.

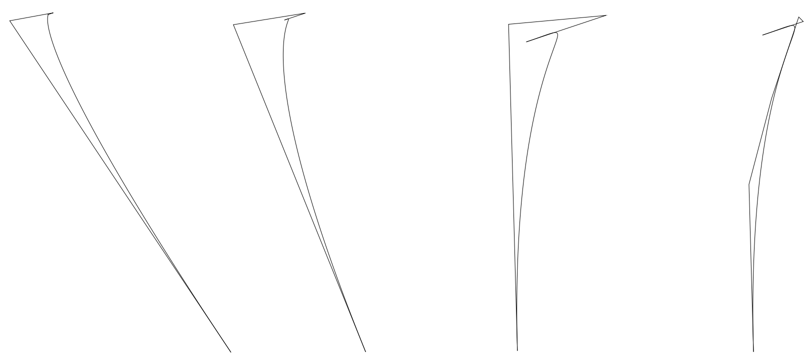

Fig. 5. A non-self-intersecting control polygon can become self-intersecting.

In Figure 6, we can see the hodograph of the (first) curve of Figure 5, with a zoom on the origin, which is very close of the upper endpoint of the hodograph and inside the region delimited by the hodograph and its control polygon up to the fourth subdivision.

4.3. Non-regular Bézier parameterisation

If the first derivative of the Bézier curve is allowed to vanish, there exist examples of non-self-intersecting curves with subdivision invariably self-intersecting

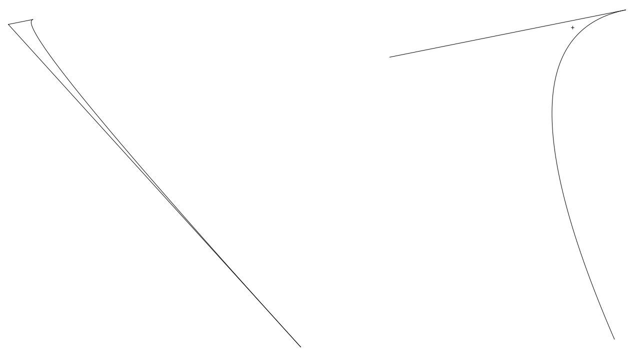

Fig. 6. The hodograph of the curve of Figure 5, with a zoom on the origin.

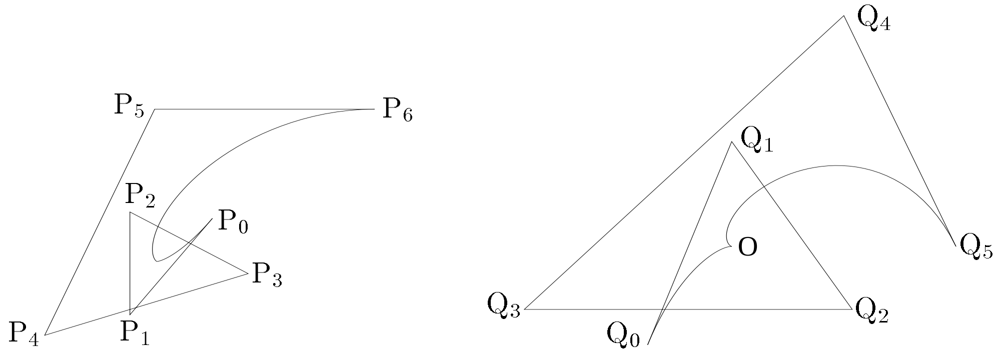

Fig. 7. Bézier curve with self-intersecting control polygon and its hodograph.

control polygon. Such an example is shown in Figure 7.

The derivative of the curve vanishes for $t = 1/3$. The point of the hodograph corresponding to this value of the parameter is the origin, where the hodograph presents a cusp. It is obvious that this point will never become the common endpoint of two successive curves if we use the subdivision parameter $t_0 = 1/2$. Then the sub-Bézier curve containing this point will have a self-interseting control polygon, no matter how many subdivisions are applied to the curve.

To prove this statement, let us study the evolution of the hodograph. All the angles of its control polygon are negative and this property is obviously subdivision invariable. Therefore, as a Bézier curve is a subset of the convex hull of its control polygon, the control polygon of the sub-curve containing the origin must "turn arround" this one, as shown in Figure 8. But if the control polygon of the hodograph has this configuration, then the sum of the angles of the control polygon of the curve (which are also all negative) is smaller than -2π. It results that the control polygon of the curve is either a spiral or self-intersecting. As the (sub-)curve is convex after the first subdivision, its control polygon cannot be a spiral, thus it is self-intersecting.

§4. Convex Bézier Curve

It is well known that there exist convex Bézier curves which have non-convex

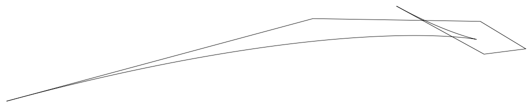

Fig. 8. Scaled representation of the sub-curve of the hodograph containing the origin after three subdivisions.

control polygon. It is less known (and less intuitive) that there exist convex Bézier curves which have *subdvision invariably* non-convex control polygon. Nevertheless, such curves exist, and an example is given in Figure 9.

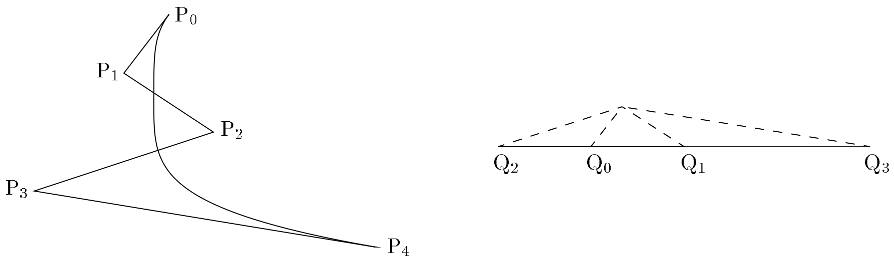

Fig. 9. Convex Bézier curve with non-convex control polygon and its hodograph.

The point corresponding to the parameter $t = 1/3$ is a flat point of the curve, the second and third derivatives vanish here. As we have also seen in the previous section, this point will always belong to the interior of a sub-Bézier curve if we use the subdivision of parameter $t_0 = 1/2$. Then the curve containing this point will have a non-convex control polygon, no matter how many subdivisions we apply.

Indeed, les us take a look at the corresponding point of the hodograph. It is not a multiple point (the parameterisation does not go back and forth on the segment that represents the curve), but the first derivative of the hodograph vanishes for $t = 1/3$. That implies that the points Q_0, Q_1, Q_2, and Q_3 cannot be well-ordered (if they were, the y-coordinate of the hodograph should be strictly increasing, thus the y-derivative greater than 0). This property will be fulfilled by all the sub-curves of the hodograph which contain the point initially corresponding to $t = 1/3$. But if the control points of the hodograph are not well-ordered, the slope of the edges of the control polygon of the curve cannot vary monotonously. Consequently, the last is not convex.

§5. Conclusion

The results that we have presented in this paper are somehow surprising. As the control polygon has very good convergence properties in terms of distance, we might be prone to expect equally good topological convergence. Perhaps the most unexpected is the possible behaviour of the control polygon of a convex Bézier curve. Of course, all the "bad" examples that we have given correspond to some kind of degeneracy of the Bézier parameterisation, which usually are not encountered in practice. Nevertheless, it is interesting to note that these cases exist.

These problems of preservation of topology cannot be solved by the use of an adaptative parameter subdivision. The reason is that in order to find the adapted parameter we must solve algebraic equations corresponding to the special points of the Bézier curve (where the derivatives vanish). As the only way of solving such equations is the use of numerical methods, we generally do not find *exactly* the good parameter. We can still prove that for convex Bézier curves the use of the suitable subdivision parameter would generate convex control polygons.

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