Equivalence of Arrangements of Curves

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1 Introduction

For arrangements of hyperplanes in $I\!R^d$, the notion of equivalence is well understood and presents no ambiguity. Two definitions of it can be found in the literature, one given by Edelsbrunner and the other one given by Grünbaum. Edelsbrunner's definition is based on the position vectors of the faces of an arrangement and it implies the more general definition of Grünbaum, which relies solely on the incidence graphs of the two arrangements.

The two definitions can be naturally extended to arrangements of (simple unbounded or closed) Jordan curves in the plane (we remark that if we want to consider arrangements of Jordan arcs, the only pertinent definition is Grünbaum's one). Nevertheless, these definitions appear to be rather improper in this case, and in Section 2 we give examples showing their limitations.

We intuitively say that two arrangements of curves are equivalent if we can continuously and reversibly deform the partition of the plane defined by one of the sets of curves to obtain the partition of the plane defined by the other one, or if they are obtained one from another by reflection. In Section 3, we give the formal expression of this notion and discuss the relations between it and the two previous definitions.

2 Equivalence of arrangements

Let $\mathcal{H}^1 = \{\mathcal{H}_i^1\}_{1\leq i\leq n}$ and $\mathcal{H}^2 = \{\mathcal{H}_i^2\}_{1\leq i\leq n}$ be two sets of hyperplanes of $I\!R^d$. The definition of the equivalence of the arrangements $\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ given by Edelsbrunner [2] relies on the notion of *position vector* of the cells of an arrangement (that is, an element of $\{+,0,-\}^n$ such that the i^{th} component gives the position of the cell with respect to the i^{th} hyperplane for fixed +- and −-sides of this one):

Definition 2.1 *$\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ are equivalent if there exists a bijection $\phi : \mathcal{A}(\mathcal{H}^1) \longrightarrow \mathcal{A}(\mathcal{H}^2)$ such that the position vector $V_{\mathcal{A}(\mathcal{H}^1)}(f)$ of any cell f of the arrangement $\mathcal{A}(\mathcal{H}^1)$ is equal to the position vector $V_{\mathcal{A}(\mathcal{H}^2)}(\phi(f))$ of the cell $\phi(f)$ of the arrangement $\mathcal{A}(\mathcal{H}^2)$: $V_{\mathcal{A}(\mathcal{H}^1)}(f) = V_{\mathcal{A}(\mathcal{H}^2)}(\phi(f))$.*

We shall say that two arrangements which are equivalent in the sense of this definition are *E-equivalent.* The other definition, introduced by Grünbaum [5], is based uniquely upon the incidence graph of the two arrangements:

Definition 2.2 *$\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ are equivalent if there exists a bijection $\psi : \mathcal{A}(\mathcal{H}^1) \longrightarrow \mathcal{A}(\mathcal{H}^2)$ such that g is a subcell of f in $\mathcal{A}(\mathcal{H}^1)$ if and only if $\psi(g)$ is a subcell of $\psi(f)$ in $\mathcal{A}(\mathcal{H}^2)$.*

Two arrangements which are equivalent in the sense of this definition will be called *G-equivalent.* It is easy to prove the following remarks:

1. The E-equivalence implies the G-equivalence.
2. If $\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ are (E- or G-)equivalent and $\mathcal{A}(\mathcal{H}^1)$ is simple, then $\mathcal{A}(\mathcal{H}^2)$ is simple.
3. The bijections ϕ and ψ preserve the cells dimensions.
4. If $\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ are G-equivalent, there exists a numbering of the hyperplanes of $\mathcal{H}^1$ and $\mathcal{H}^2$ and a choice of the +-sides of these hyperplanes such that $\mathcal{A}(\mathcal{H}^1)$ and $\mathcal{A}(\mathcal{H}^2)$ are E-equivalent.

Let now $\mathcal{G}^1 = \{\Gamma_i^1\}_{1\leq i\leq n}$ and $\mathcal{G}^2 = \{\Gamma_i^2\}_{1\leq i\leq n}$ be two sets of (simple unbounded or closed) Jordan curves in the plane such that $\text{card}(\Gamma_i^k \cap \Gamma_j^k) \leq s$ for a fixed $s \in I\!N$ ($i \neq j \in \{1,\ldots,n\}$, $k = 1,2$). We recall the definition of a simple arrangement of Jordan curves:

Definition 2.3 *Let $\mathcal{G} = \{\Gamma_i\}_{1\leq i\leq n}$ be a set of Jordan curves. We say that the arrangement $\mathcal{A}(\mathcal{G})$ is simple if*

1. *$\Gamma_i \cap \Gamma_j \cap \Gamma_k = \emptyset$ for all distinct $i,j,k \in \{1,\ldots,n\}$;*
2. *Γ_i and Γ_j cross each other transversely in every common point, for all $i \neq j \in \{1,\ldots,n\}$;*
3. *$\cup_{i=1}^n \Gamma_i$ is a connected set.*

Definitions 2.1 and 2.2 can be extended easily to introduce the equivalence of the arrangements $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$, but if s is greater than 1, then the first two of the previous remarks do not hold. Indeed, neither the E-equivalence nor the G-equivalence assures the preservation of the second condition of Definition 2.3 and

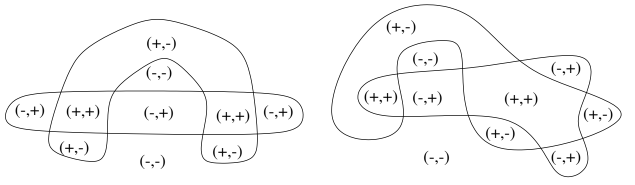

Figure 1: Two arrangements which are E-equivalent and not G-equivalent.

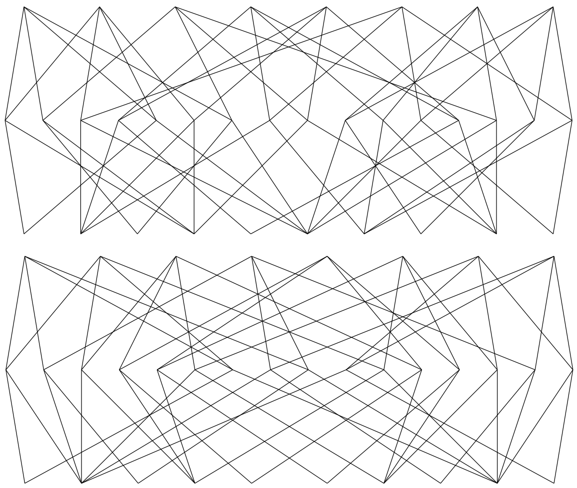

Figure 2: The incidence graphs of the two arrangements of Figure 1 are not isomorphic (e.g., there is a unique 6-arcs node in the first graph, whilst there are two such nodes in the second graph).

therefore the one of the simplicity (Remark 2). For Remark 1, we give in Figure 1 an example of two arrangements for which the bijection ϕ of Definition 2.1 can be constructed (the bounded region delimited by a closed curve represents its $+$-side) whereas the bijection ψ of Definition 2.2 cannot. Indeed, it is easy to see that the incidence graphs of the two arrangements, which are presented in Figure 2, are not isomorphic.

It is very easy to check that Remark 3 holds for arrangements of curves in the plane. For Remark 4, stronger hypotheses are required in this case:

Proposition 2.4 *If $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$ are simple and G-equivalent, then there exists a numbering of the curves of $\mathcal{G}^1$ and $\mathcal{G}^2$ and a choice of the $+$-sides of these curves such that the two arrangements are E-equivalent.*

Proof: Let $\psi : \mathcal{A}(\mathcal{G}^1) \longrightarrow \mathcal{A}(\mathcal{G}^2)$ be the bijection of Definition 2.2. We fix (arbitrarily) a numbering of the curves of $\mathcal{G}^1$, $\Gamma^1_1, \Gamma^1_2, \ldots, \Gamma^1_n$, and a $+$-side and a $-$-side for each Γ^1_i.

Let i_0 be an index in $\{1, \ldots, n\}$ and let e_1 be an edge of $\mathcal{A}(\mathcal{G}^1)$ which lies on $\Gamma^1_{i_0}$. We denote by $\Gamma^2_{i_0}$ the curve of $\mathcal{G}^2$ on which lies $\psi(e_1)$. We want to prove that for any cell f of $\mathcal{A}(\mathcal{G}^1)$ which lies on $\Gamma^1_{i_0}$, $\psi(f)$ lies on $\Gamma^2_{i_0}$.

Let f be a (0- or 1-) cell of $\mathcal{A}(\mathcal{G}^1)$ which lies on $\Gamma^1_{i_0}$. Then there exist $f_1 = e_1, f_2, \ldots, f_m = f$ cells of $\mathcal{A}(\mathcal{G}^1)$ which lie on $\Gamma^1_{i_0}$ such that $\cup_{j=1}^m f_j$ is a simple curve (thus a connected set). The cell f_j is an edge if j is odd and a vertex otherwise. As ψ is a bijection and preserves the incidences, $\cup_{j=1}^m \psi(f_j)$ is also a simple curve.

We suppose that $m \geq 2$; if $m = 1$, then $f = e_1$ and $\psi(f)$ lies on $\Gamma^2_{i_0}$ by definition. For any $k \in I\!N^*$, $k \leq \frac{m}{2}$, if $\psi(f_{2k-1})$ lies on $\Gamma^2_{i_0}$, then $\psi(f_{2k})$ lies on $\Gamma^2_{i_0}$. Indeed, $\psi(f_{2k-1})$ is an edge and $\psi(f_{2k})$ is a vertex which is incident to it.

For any $k \in I\!N^*$, $k < \frac{m}{2}$, if $\psi(f_{2k-1})$ lies on $\Gamma^2_{i_0}$, then $\psi(f_{2k+1})$ lies on $\Gamma^2_{i_0}$. Indeed, $\psi(f_{2k-1})$ and $\psi(f_{2k+1})$ are both edges which are incident to the vertex $\psi(f_{2k})$. The arrangement $\mathcal{A}(\mathcal{G}^1)$ is simple, so there are exactly four faces in $\mathcal{A}(\mathcal{G}^1)$ such that f_{2k} is a point of their boundary. Moreover, two of these faces are incident to f_{2k-1} and the other two are incident to f_{2k+1}. $\mathcal{A}(\mathcal{G}^2)$ is also simple and ψ preserves the incidences, thus there are exactly four faces such that $\psi(f_{2k})$ is a point of their boundary; two of them are incident to $\psi(f_{2k-1})$ and the other two are incident to $\psi(f_{2k+1})$. It results that $\psi(f_{2k-1})$ and $\psi(f_{2k+1})$ lie on the same curve of $\mathcal{G}^2$, that is, $\Gamma^2_{i_0}$.

As $\psi(f_1)$ lies on $\Gamma^2_{i_0}$, the two remarks above imply that $\psi(f) = \psi(f_m)$ lies on $\Gamma^2_{i_0}$. Therefore the images by ψ of all the cells which lie on a same curve of $\mathcal{G}^1$ lie on the same curve of $\mathcal{G}^2$. We deduce that $\mathrm{card}(\mathcal{G}^1) = \mathrm{card}(\mathcal{G}^2)$ and we denote by Γ^2_i the curve on which lie the images of the cells that lie on Γ^1_i, for all $i \in \{1, \ldots, n\}$. Obviously, the converse is also true, that is,

$$\Gamma^2_i = \bigcup_{f \subset \Gamma^1_i} \psi(f) \qquad (1)$$

Let us now fix again an index $i_0 \in \{1, \ldots, n\}$ and let f_1 be an edge of $\mathcal{A}(\mathcal{G}^1)$ which is on the $+$-side of $\Gamma^1_{i_0}$. Such an edge exists, thanks to Condition 3 of Definition 2.3. We fix the $+$-side of $\Gamma^2_{i_0}$ as the one that contains $\psi(f_1)$. We want to prove that $\psi(f)$ is on the $+$-side of $\Gamma^2_{i_0}$ for any cell $f \in \mathcal{A}(\mathcal{G}^1)$ which is on the $+$-side of $\Gamma^1_{i_0}$. It is obvious that the position of a face with respect to $\Gamma^2_{i_0}$ is given by the position of any of the edges which are incident to it and do not lie on $\Gamma^2_{i_0}$. Therefore we shall prove the statement above only for cells of dimension 0 and 1.

Let f be a 0- or a 1-cell of $\mathcal{A}(\mathcal{G}^1)$ which is on the $+$-side of $\Gamma^1_{i_0}$. Condition 3 of Definition 2.3 implies that there exist $f_2, f_3, \ldots, f_m = f$ cells of $\mathcal{A}(\mathcal{G}^1)$ of dimension 0 and 1 such that $\mathbf{c}^1 = \cup_{j=1}^m f_j$ is a simple curve. Obviously, $\mathbf{c}^2 = \cup_{j=1}^m \psi(f_j)$ is also a simple curve. Relation (1) implies that if $\mathbf{c}^1 \cap \Gamma^1_{i_0} = \emptyset$, then $\mathbf{c}^2 \cap \Gamma^2_{i_0} = \emptyset$. It immediately results that $\psi(f_j)$ is on the $+$-side of $\Gamma^2_{i_0}$ for all $j \in \{2, \ldots, m\}$.

If $\mathbf{c}^1 \cap \Gamma^1_{i_0} \neq \emptyset$, let us consider the indices j_1 and j_2 as follows:

$$j_1 = \min \left\{ j \mid f_j \text{ lies on } \Gamma^1_{i_0} \right\}$$
$$j_2 = \max \left\{ j \mid f_j \text{ lies on } \Gamma^1_{i_0} \right\}$$

$\mathcal{A}(\mathcal{G}^1)$ is simple and $\mathbf{c}^1$ is a simple curve, hence $j_1 < j_2$. It is also clear that f_{j_1} and f_{j_2} are vertices. Let $g_1 = f_{j_1}, g_2, \ldots, g_p = f_{j_2}$ be cells of $\mathcal{A}(\mathcal{G}^1)$ which lie on $\Gamma^1_{i_0}$ such that $\cup_{k=1}^{p} g_k$ is a simple curve. The choice of j_1 and j_2 implies that

$$\bigcup_{j=1}^{j_1-1} f_j \cap \Gamma^1_{i_0} = \emptyset \quad \text{and} \quad \bigcup_{j=j_2+1}^{m} f_j \cap \Gamma^1_{i_0} = \emptyset$$

So, f_j is on the +-side of $\Gamma^1_{i_0}$ for all j in $\{1, \ldots, j_1 - 1\} \cup \{j_2+1, \ldots, m\}$. We deduce that $\psi(f_j)$ is on the +-side of $\Gamma^2_{i_0}$ for all $j \in \{1, \ldots, j_1 - 1\}$ and $\psi(f_j)$ is on the same side of $\Gamma^2_{i_0}$ for all $j \in \{j_2+1, \ldots, m\}$, but it might be the $-$-side. To achieve our proof, it suffices to show that $\psi(f_{j_1-1})$ and $\psi(f_{j_2+1})$ are on the same side of $\Gamma^2_{i_0}$.

It is easy to see that p is odd and that the face g_k is a vertex if k is odd and an edge otherwise. We consider the edges $h_1 = f_{j_1-1}, h_3, \ldots, h_{p-2}, h_p = f_{j_2+1}$ that are on the +-side of $\Gamma^1_{i_0}$ and such that h_k is incident to g_k; for any $k \in \{1, 3, \ldots, p\}$ there exists a unique such edge.

Let us suppose that the edge $\psi(h_{k_0})$ is on the +-side of $\Gamma^2_{i_0}$ for a $k_0 \in \{1, 3, \ldots, p-2\}$. The face that is incident to $\psi(h_{k_0})$ and $\psi(g_{k_0+1})$ is then on the +-side of $\Gamma^2_{i_0}$. Such a face exists, because $\psi(h_{k_0})$ and $\psi(g_{k_0+1})$ are both incident to the vertex $\psi(g_{k_0})$ and do not lie on the same curve of $\mathcal{G}^2$. Moreover, this face is the image by ψ of the face of $\mathcal{G}^1$ that is incident to h_{k_0} and g_{k_0+1}, let us denote it by c. By the choice of the edges h_k, h_{k_0+2} is also incident to c, thus $\psi(h_{k_0+2})$ is incident to $\psi(c)$. We deduce that $\psi(h_{k_0+2})$ is on the +-side of $\Gamma^2_{i_0}$.

As $\psi(h_1) = \psi(f_{j_1-1})$ is on the +-side of $\Gamma^2_{i_0}$, we can conclude that $\psi(f) = \psi(f_m)$ is on the +-side of $\Gamma^2_{i_0}$. Hence all the cells situated on the +-side of $\Gamma^1_{i_0}$ have images situated on the +-side of $\Gamma^2_{i_0}$. If for all the curves of $\mathcal{G}^2$ we fix the +-side in the same way we did for $\Gamma^2_{i_0}$, it is very easy to see that the bijection ψ preserves the position vectors.

□

The two arrangements of Figure 1 are obviously not equivalent in an intuitive sense of this notion, so Grünbaum's definition is more appropriate for arrangements of curves than Edelsbrunner's definition. Yet, there exist arrangements of the plane which have the same incidence graph and cannot be considered equivalent, as we can see in Figure 3.

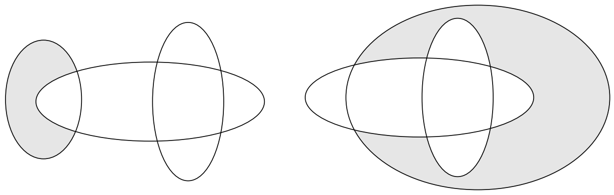

Figure 3: The two arrangements have the same incidence graph (the grey face of an arrangement corresponds to the unbounded face of the other), but they do not "seem" equivalent.

3 An algebraic topology approach

So, none of the two given definitions of the equivalence of arrangements is satisfactory for the case of planar Jordan curves. In algebraic topology, two figures are considered to be identical if they can be mapped one onto the other by a bijective bicontinuous function, that is, a *homeomorphism*. In the case of arrangements, this is expressed as follows:

Definition 3.1 *We say that the arrangements $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$ are equivalent if there exists a homeomorphism $h : I\!R^2 \longrightarrow I\!R^2$ such that $h(f) \in \mathcal{A}(\mathcal{G}^2)$ for all $f \in \mathcal{A}(\mathcal{G}^1)$.*

Two arrangements equivalent in the sense of this definition will be called *T-equivalent*. It is obvious that the homeomorphism h conserves the dimensions of the cells (because it is bijective and continuous). We can also prove that it conserves the incidence graph of the arrangement $\mathcal{A}(\mathcal{G}^1)$, so we have the following result:

Lemma 3.2 *If two arrangements are T-equivalent, then they are G-equivalent.*

Proof: We define $\psi : \mathcal{A}(\mathcal{G}^1) \longrightarrow \mathcal{A}(\mathcal{G}^2)$ by $\psi(f) = h(f)$ for all $f \in \mathcal{A}(\mathcal{G}^1)$. We want to prove that ψ satisfies the conditions of Definition 2.2.

It is clear that ψ is one-to-one, because h is one-to-one. If we suppose that ψ is not onto $\mathcal{A}(\mathcal{G}^2)$, then there exists $g \in \mathcal{A}(\mathcal{G}^2)$ such that $h^{-1}(g) \notin \mathcal{A}(\mathcal{G}^1)$. As h is onto $I\!R^2$, $h^{-1}(g)$ cannot be empty; let P be a point of $h^{-1}(g)$. $\mathcal{A}(\mathcal{G}^1)$ is a partition of the plane, so there exists a cell $f \in \mathcal{A}(\mathcal{G}^1)$ such that $\mathrm{P} \in f$. Then $h(\mathrm{P}) \in h(f) = \psi(f)$, that is, $h(\mathrm{P})$ belongs to a cell of $\mathcal{A}(\mathcal{G}^2)$ which is the image by ψ of a cell of $\mathcal{A}(\mathcal{G}^1)$. On the other hand, $h(\mathrm{P}) \in h(h^{-1}(g)) = g$, that is, $h(\mathrm{P})$ belongs to a cell of $\mathcal{A}(\mathcal{G}^2)$ which is not the image by ψ of a cell of $\mathcal{A}(\mathcal{G}^1)$. It results that $h(\mathrm{P})$ belongs to two distinct cells of $\mathcal{A}(\mathcal{G}^2)$, which contradicts the definition of an arrangement. We deduce that ψ is onto $\mathcal{A}(\mathcal{G}^2)$, thus bijective. We also remark that $\psi^{-1} : \mathcal{A}(\mathcal{G}^2) \longrightarrow \mathcal{A}(\mathcal{G}^1)$ is defined by $\psi^{-1}(g) = h^{-1}(g)$ for all $g \in \mathcal{A}(\mathcal{G}^2)$.

Let now $f_1, f_2 \in \mathcal{A}(\mathcal{G}^1)$ be such that f_1 is a subcell of f_2.

- If f_2 is an edge, f_1 belongs to its 1-dimensional boundary. As h is continuous, $h(f_1)$ belongs to the 1-dimensional boundary of $h(f_2)$, so $\psi(f_1)$ is a subface of $\psi(f_2)$.
- If f_2 is a cell, f_1 lies on its boundary (in $\mathbb{R}^2$). As h is continuous, $h(f_1)$ lies on the boundary of $h(f_2)$. Hence $\psi(f_1)$ is a subcell of $\psi(f_2)$, because it cannot be a vertex.

Similarly, if g_1 is a subcell of g_2 in $\mathcal{A}(\mathcal{G}^2)$, then $\psi^{-1}(g_1)$ is a subcell of $\psi^{-1}(g_2)$ in $\mathcal{A}(\mathcal{G}^1)$. Therefore we can conclude that ψ fulfills the conditions of Definition 2.2 and consequently $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$ are G-equivalent.

$\square$

In fact, Definition 3.1 completely captures the intuitive notion of equivalence, although it may not yet be clear why. Indeed, we think that two equivalent arrangements can be obtained one from another by a continuous deformation of the plane such that at each moment of the deformation the intermediate arrangement is "similar" to the initial arrangement. This can be formally expressed by saying that the homeomorphism h of Definition 3.1 is *isotopic* to the identity.

On the other hand, the two arrangements of Figure 4 are also "identical", that is, the homeomorphism h can be a reflection, or isotopic to a reflection. We can now use a result which is well-known in algebraic topology: Any homeomorphism of $\mathbb{R}^2$ is isotopic to either the identity or a reflection [6]. Thus Definition 3.1 is enough to express the intuitive notion of equivalence.

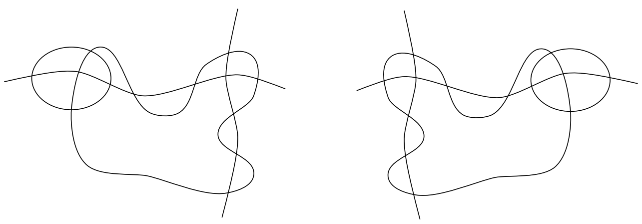

Figure 4: Two reflected arrangements are equivalent.

The converse of Lemma 3.2 is not true, an obvious counter-example being shown in Figure 3. This is due to the fact that the plane is not a compact manifold; so, if we consider curves lying in $\mathbb{R}^2 \cup \{\infty\}$ (or on the sphere, or in the projective plane), then Definitions 3.1 and 2.2 are equivalent.

Moreover, we can state the following:

Lemma 3.3 *If the arrangements $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$ are G-equivalent and if in the arrangement $\mathcal{A}(\mathcal{G}^1)$ there exists an unbounded edge, then $\mathcal{A}(\mathcal{G}^1)$ and $\mathcal{A}(\mathcal{G}^2)$ are T-equivalent.*

Proof: In the following, we shall denote by $\mathcal{A}_i(\mathcal{G})$ the set of i-cells of the arrangement $\mathcal{A}(\mathcal{G})$.

Let $\psi : \mathcal{A}(\mathcal{G}^1) \longrightarrow \mathcal{A}(\mathcal{G}^2)$ be the bijection of Definition 2.2. If $e \in \mathcal{A}_1(\mathcal{G}^1)$ is a bounded edge of $\mathcal{A}(\mathcal{G}^1)$, then $\psi(e)$ is also a bounded edge; indeed, an unbounded edge is incident to a unique vertex and $\psi(e)$ must be incident to two vertices. Similarly, if $e \in \mathcal{A}_1(\mathcal{G}^1)$ is unbounded, then $\psi(e)$ is unbounded.

As in the arrangement $\mathcal{A}(\mathcal{G}^1)$ there exists an unbounded edge, any unbounded face is incident to at least one (and at most two) unbounded edge(s). We deduce that $\psi(f)$ is an unbounded face if and only if f is an unbounded face.

We want to define the homeomorphism h of Definition 3.1.

For any vertex $v \in \mathcal{A}_0(\mathcal{G}^1)$, $h(v) = \psi(v)$. It is clear that $h\big|_{\mathcal{A}_0(\mathcal{G}^1)} : \mathcal{A}_0(\mathcal{G}^1) \longrightarrow \mathcal{A}_0(\mathcal{G}^2)$ is a bijection.

For any bounded edge $e \in \mathcal{A}_1(\mathcal{G}^1)$, we define $h\big|_e : e \longrightarrow \psi(e)$ as a bicontinuous bijection which preserves the orientation: If v_1 and v_2 are the vertices incident to e, then

$$\lim_{\substack{\mathrm{P}\in e \\ \mathrm{P}\longrightarrow v_1}} h(\mathrm{P}) = h(v_1) \text{ and } \lim_{\substack{\mathrm{P}\in e \\ \mathrm{P}\longrightarrow v_2}} h(\mathrm{P}) = h(v_2)$$

$h\big|_e$ can be defined using the arc-length parameterization of e and $\psi(e)$.

For any unbounded edge $e \in \mathcal{A}_1(\mathcal{G}^1)$ incident to a vertex v, we define $h\big|_e : e \longrightarrow \psi(e)$ such that the length of the arc $v\mathrm{P}$ is equal to the length of the arc $\psi(v)h(\mathrm{P})$ for any $\mathrm{P} \in e$. It is obvious that $h\big|_e$ is a bicontinuous bijection.

Any bounded face $f \in \mathcal{A}_2(\mathcal{G}^1)$ is homeomorphic to the open unit disk. Let us denote by $f_1, f_2, \ldots, f_{2m}$ the cells that lie, in this order, on the boundary of f, considering that f_{2k-1} is a vertex and f_{2k} is an edge. Then the homeomorphism $h_1 : f \longrightarrow S^1$ can be taken such that the continuous extension of h_1 to $\overline{f}$ associates to f_{2k} an ($\mathbb{R}$-open) arc of the unit circle of length $2\pi/m$. Similarly, $\psi(f)$ is homeomorphic to the open unit disk and the homeomorphism $h_2 : \psi(f) \longrightarrow S^1$ can be taken such that the continuous extension of h_2 to $\overline{\psi(f)}$ associates to $\psi(f_{2k})$ an arc of the unit circle of length $2\pi/m$. Then f and $\psi(f)$ are homeomorphic by $h\big|_f = h_1 \circ h_2^{-1}$.

Any unbounded face $f \in \mathcal{A}_2(\mathcal{G}^1)$ is homeomorphic to an open half-plane (e.g., the positive ordinate half-plane, denoted $H_{y>0}$). Let us denote by $f_1, f_2, \ldots, f_{2m+1}$ the cells that lie, in this order, on the boundary of f. It is clear that f_{2k+1} is an edge and f_{2k} is a vertex. Then the homeomorphism $h_1 : f \longrightarrow H_{y>0}$ can be taken such that the continuous extension of h_1 to $\overline{f}$ associates to f_1 the half-line $\{(x,y) \mid x < 1, y = 0\}$, to f_{2m+1} the half-line $\{(x,y) \mid x > m, y = 0\}$, and to f_{2k+1} the open segment of the X-axis $(k, k+1)$ for all $k \in \{1, \ldots, m-1\}$. The homeomorphism

$h_2 : \psi(f) \longrightarrow H_{y>0}$ can be defined similarly and it immediately results that f and $\psi(f)$ are homeomorphic by $h\Big|_f = h_1 \circ h_2^{-1}$.

As ψ conserves the incidences, it is now very easy to see that the application h that we have piecewise defined is a homeomorphism fulfilling the condition of Definition 3.1.

□

4 Conclusion

The definition that we propose for the equivalence of two arrangements of planar Jordan curves captures the intuitive idea of the notion better than the existing definitions, although being closely related to the definition given by Grünbaum. The price to pay for the precision of Definition 3.1 is that it has a *continuous* expression (vs. the *discrete* expression of Definitions 2.1 and 2.2); so, our result is a theoretical one rather than an algorithmic one. Nevertheless, Lemma 3.3 identifies the cases where Grünbaum's definition can fail, thus allowing us to use it safely.

References

[1] Boissonnat, Jean-Daniel, and Yvinec, Mariette, `Algorithmic Geometry`, Cambridge University Press, UK, 1998.

[2] Edelsbrunner, Herbert, `Algorithms in Combinatorial Geometry`, 1987, Springer-Verlag, Heidelberg.

[3] Edelsbrunner, Herbert, Guibas, Leonidas, Pach, Janos, Pollack, Richard, Seidel, Raimund, and Sharir, Micha, *Arrangements of Curves in the Plane - Topology, Combinatorics, and Algorithms*, Theoretical Computer Science 92, 1992, 319-336.

[4] Guibas, Leonidas, and Stolfi, Jorge, *Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams*, ACM Transactions on Graphics, Vol. 4, No. 2, 1985, 74-123.

[5] Grünbaum, Branco, `Arrangements and spreads`, Conference board of the mathematical sciences, AMS, Providence, RI, no 10, 1972.

[6] Kirby, Robion, *Stable Homeomorphisms and the Annulus Conjecture*, Annals of Mathematics, Vol. 29, No. 3, 1969, 575-582.

[7] Moise, Edwin, `Geometric Topology in Dimensions 2 and 3`, Springer-Verlag, New York, 1977.

[8] Stillwell, John, `Classical Topology and Combinatorial Group Theory`, Springer-Verlag, New York, 1980.

[9] Vo Phi, Khanh, `Contributions à l'étude des arrangements : équivalences combinatoires et perturbations`, PhD thesis, Université Joseph Fourier, Grenoble, 1994.