Computing the Combinatorial Structure of Arrangements of Curves Using Polygonal Approximations

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Résumé

Dans cet article, nous présentons une méthode de calcul du graphe d'incidence d'un arrangement de courbes. L'idée principale de notre approche est d'éviter la résolution déquations algébriques, car cette résolution ne peut pas être effectuée de manière exacte. Pour atteindre notre but, nous utilisons deux approximations polygonales pour chaque courbe de l'arrangement et nous établissons des conditions suffisantes pour léquivalence des trois arrangements obtenus. Ce processus fournit non seulement deux arrangements polygonaux ayant le même graphe d'incidence que l'arrangement de courbes, mais aussi une approximation de ce dernier arrangement en termes de distance. Cette méthode est concrétisée par un algorithme et nous présentons quelques résultats numériques et des exemples.

Mots-clef : géométrie algorithmique, arrangement, courbe de Bézier, équivalence combinatoire

Abstract

Computing with algebraic curves requires solving algebraic equations. There are involved symbolic and numerical methods that allow accurate performing of this solving. The geometry of the studied problem is reflected by the equations, but it plays no role in their solving.

In this paper, we treat the problem of the maximal topological information that can be obtained on an arrangement of curves without the use of algebraic equations. Namely, we prove that if the arrangement of curves is simple, then its incidence graph can be computed by dealing solely with two suitable polygonal approximations for each curve. We have conceived an algorithm based on this approach and we present numerical results and some examples.

Keywords: computational geometry, arrangement, Bézier curve, combinatorial equivalence

1. Introduction

Problems on arrangements represent one of the most important topics in computational geometry. Arrangements find numerous applications, ranging from the design of 2D drawing tools [11] to motion planning, point location, and visibility problems [6].

Arrangements of hyperplanes, especially arrangements of lines in the plane, have been widely studied. Satisfactory theoretical results (e.g. the zone theorem [8]) and algorithmical results (e.g. the study of the numerical stability [10]) have been found. The interest is now focused on arrangements in higher dimensions, or on planar arrangements of objects other than lines: segments [1], [13], [17] and (Jordan) curves [7], [11], [15], [19], [20], to give only a few examples. Dealing with arrangements of segments is more difficult than dealing with arrangements of lines because of the larger topological complexity of the cells. The case of curves raises also the fundamental problem of the computation with curves. Let us briefly recall some of the approaches of the problem of curve arrangements that can be found in the literature.

The construction of an arrangement can be done mainly in two ways: incrementally and using sweeping methods. For each of these two methods, elementary geometric operations must be performed on the curves defining the arrangement. Examples of these elementary operations are the computation of the intersection (generally, of cardinality greater than 1), the computation of the vertical tangents to a curve, etc. In [7], the authors consider these computations feasible accurately and in constant time, concentrating on the extension of the main theoretical results from arrangements of lines to arrangements of curves (zone theorem, combinatorial complexity of the faces). In [19], it is not the construction of an arrangement that is studied, but a closely related problem: how an arrangement of curves each pair of which intersect at most k times can be swept by an additional curve crossing each curve of the arrangement at most k times. As an application, some combinatorial results are presented. The assumption that the computations on curves are feasible accurately is done implicitly. In [20], the author dedicates a chapter to the study of the equivalence of arrangements of curves, establishing conditions of equivalence that allow to avoid the construction of the incidence graph.

In [15], the author has a much more application-oriented approach, by defining a strict framework for rounded arithmetics computations. So, he introduces the *approximate cylindrical decomposition* defined by a set of algebraic curves and gives an algorithm to compute it. In [11], the authors deal with arrangements of Bézier curves, a problem similar to the one we are studying in this paper. Yet, their approach is quite different: For each Bézier curve, they construct a *rounded interpolating polyline* and then compute the intersections of these polylines. The rounded interpolating polyline of a Bézier curve is an approximation in finite precision of what we shall call *carrier polygon*, polygonal line joining points of the Bézier curve. The authors use exact arithmetic for intersection computation. Thus, they can recover consistent topological information on the arrangement of polylines from the intersection computation. The topology of the polyline arrangement may not be the same as the topology of the arrangement of Bézier curves. This is what makes their approach different from the one presented in this paper, as our main concern is the topology of the curve arrangement.

In this paper, we present a new approach to the problems of arrangements of curves. We work in a strictly geometric framework, in which the only legal computations are those performed on linear objects. For this purpose, we shall use polygonal approximations of the curves defining the arrangement. Two important questions arise when such an approach is adopted. On the one hand, what kind of information can we obtain on the given arrangement of curves if we avoid algebraic equations? On the other hand, which are the restrictions we must impose on the arrangement of curves in order to assure that the required information can be provided by the arrangement of polygonal approximations?

On an arrangement, there are two different types of results. Firstly, there are topological (or combinatorial) characteristics expressed by the incidence graph. For example, this graph can be used to find the topological closure of a given face or all its neighbours. Secondly, it can be useful to have geometric information on the cells of the arrangement. That would allow to make a decision for questions such as the point location problem.

We are interested in the topological aspects of the problem. Namely, we want to compute the incidence graph of an arrangement of curves avoiding algebraic equations. In this framework, degenerate cases can not be treated. Indeed, if three curves have a common point, it generally cannot be found via polygonal approximations of the curves. Similarly, if two curves are tangent in a common point, algebraic equations must be generally used to detect the tangency. But on the other hand, our approach provides a robust algorithm for nondegenerate arrangements. Moreover, the method we propose detects the "almost" degenerate positions of the curves. If such a situation occurs, symbolic methods can be employed to obtain the exact local configuration of the arrangement.

The construction of the polygonal approximations is theoretically possible for Jordan curves as general as we want. Practically, the input of an algorithm computing the incidence graph should be more precise. We have thus chosen to deal with composite Bézier curves, a good geometric modeling tool that has been widely studied in the field of CAGD (see e.g. [9]). A subject similar to the one of our paper, but concerning only mutually nonintersecting composite Bézier curves, has been treated in [3]. The authors also extend their study to interval Bézier curves; the use of interval or lazy arithmetics might be of interest also in our problem.

Every curve will be approximated by two polygonal lines: the *control polygon* and the *carrier polygon*. Our approach is thus similar to the one presented in [4] and [5]. The authors introduce the carrier polygon and the *bounding polygon* and use them to solve classic problems in computational geometry, such as monotone decomposition, convex hull computation, intersection detection and computation, etc.

The outline of our paper is the following. In Section 2, we introduce the *polycurve*, a composite Bézier curve satisfying certain conditions. This is the object we shall deal with throughout the paper. We also define the control and carrier polygon of a polycurve.

In Section 3, we start by giving the basic definitions and notations concerning simple arrangements of polycurves and respectively of control and carrier polygons. The next topic that we deal with is the equivalence of arrangements, for which the definition and sufficient conditions are given. Finally, we prove the existence of an arrangement of control polygons equivalent to the arrangement of polycurves; this arrangement can be obtained via the de Casteljau subdivision.

From our point of view, the final result of Section 3 is purely theoretical. Indeed, to construct the arrangement of control polygons equivalent to the arrangement of polycurves, operations on Bézier curves are performed. In Section 4, we concentrate on establishing sufficient conditions of equivalence of the two arrangements performing operations solely on carrier and control polygons.

Section 5 briefly presents the algorithm constructing the arrangement of control polygons equivalent to the arrangement of polycurves. We remark that there is no real analysis of the complexity for the algorithm. This is due to the nature of the problem we process: The stop conditions cannot but be empirical.

Section 6 gives the numerical results obtained by running the implementation of our algorithm and a few examples and finally Section 7 concludes the paper.

2. The polycurve and its polygonal approximations

As we can see in the literature [7], [15], [19], [20], the problems on arrangements can be addressed for curves which are subject to very few constraints. But the use of general curves makes the theoretical results unsuitable for direct implementation. This is one of the reasons why we restrict our study to some particular piecewise convex parametric curves, not without remarking that these objects represent a large class of planar curved objects.

2.1. Point-controlled parametric curves

Let $\mathcal{C}$ be a parametric curve controlled by the set of points $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_m$. The polygonal line $\mathcal{P} = \mathrm{P}_0 \ldots \mathrm{P}_m$ is called *the control polygon* of the curve $\mathcal{C}$. We remark that the term "polygon" is improperly used here for an open polygonal line: The notion of "control polygon" is defined in this way in CAGD. The object that we introduce is a union of such parametric curves fulfilling certain geometric properties. Roughly, the required properties are the following:

- P_0 and P_m are the endpoints of the curve $\mathcal{C}$;
- the line $\mathrm{P}_0\mathrm{P}_1$ is tangent to $\mathcal{C}$ at P_0 and the line $\mathrm{P}_{m-1}\mathrm{P}_m$ is tangent to $\mathcal{C}$ at P_m;
- the curve is included in the convex hull of its control points;
- the curve and its control polygon satisfy the variation diminishing property (i.e., for any line Δ the cardinality of $\Delta \cap \mathcal{C}$ is less than or equal to the cardinality of $\Delta \cap \mathcal{P}$, see [9] for more details);
- there exists a method which allows to split the curve $\mathcal{C}$ in two curves, $\mathcal{C}'$ and $\mathcal{C}''$, such that:
  - $\mathcal{C}'$ and $\mathcal{C}''$ are respectively controlled by the points $\mathrm{P}'_0 = \mathrm{P}_0, \mathrm{P}'_1, \ldots, \mathrm{P}'_m$ and $\mathrm{P}''_0 = \mathrm{P}'_m, \mathrm{P}''_1, \ldots, \mathrm{P}''_m = \mathrm{P}_m$ and all these points are affine combinations of $\mathrm{P}_0, \ldots, \mathrm{P}_m$;
  - $\mathcal{C}'$ and its control polygon $\mathcal{P}'$ (respectively $\mathcal{C}''$ and its control polygon $\mathcal{P}''$) satisfy the same properties as $\mathcal{C}$ and if we iteratively apply this splitting method, the union of all the control polygons obtained converges to the curve $\mathcal{C}$.

These five conditions are satisfied by many types of parametric curves: Bézier and rational Bézier curves, B-splines and NURBS under certain conditions, etc. We have chosen, as a first step, Bézier curves for the implementation of the method. This is why, for clarity, we will restrict from now on the discussion to Bézier curves, even if it can be proven that the results are valid for any type of parametric curve fulfilling the conditions above.

The method of subdivision mentioned in the last condition will be the de Casteljau subdivision algorithm, and we have chosen a subdivision parameter equal to 1/2. The reasons of this choice are the following: Firstly, the parameter 1/2 is the only one assuring quadratic convergence of the control polygon to the Bézier curve, as shown in [2] (pp 34-35). Secondly, the computations are more accurate in this case; indeed, additions and divisions by 2 suffice for the computation of the new control polygons. On the other hand, if we do not want to use a fixed subdivision parameter, we cannot adapt this parameter properly without addressing algebraic equations, and as we have said in the introduction we do not consider algebraic equations.

For the same reason, all the Bézier curves we deal with are *completely convex*:

Definition 2.1. – *We say that a Bézier curve is completely convex if its control polygon is convex.*

Let us recall that a curve (or a polygonal chain) is called convex if it is simple and included in the boundary of its convex hull.

A completely convex Bézier curve is obviously convex, thanks to the variation diminishing property. So, we do not need to compute inflection points to obtain a decomposition of our object in piecewise convex curves. This decomposition is necessary, whereas the inflection points computation requires solving algebraic equations.

2.2. The polycurve

Let $\mathbb{B}$ be the set of all the simple planar curves that can be written as a (finite) union of completely convex Bézier curves. A curve $\mathcal{B} \in \mathbb{B}$ will be $\mathcal{B} = \cup_1^n \mathcal{B}_i$, where for all $i \in \{1, \ldots, n\}$ $\mathcal{B}_i$ is a completely convex Bézier curve with control polygon $\mathcal{P}(\mathcal{B}_i)$. On $\mathbb{B}$, we introduce the relation "obtained by de Casteljau subdivision/fusion of the composing Bézier curves", denoted "$\sim$", and it is easy to see that this is an equivalence relation.

Definition 2.2. – *We shall call polycurve either an equivalence class of the relation "obtained by de Casteljau subdivision/fusion of the composing Bézier curves" over $\mathbb{B}$ (an element of $\mathbb{B}/\sim$) or an element of $\mathbb{B}$.*

All the members of a polycurve (equivalence class) have the same geometric support and differ one from another by parametrization. This is why an element of $\mathbb{B}$ will also be called polycurve. Of course, we consider all the polycurves (elements of $\mathbb{B}$) which are equivalent through $\sim$ to be equal. By default, the name polycurve will apply from this point on to the objects of the set $\mathbb{B}$.

2.3. The control polygon and the carrier polygon of a polycurve

The two polygonal approximations of a polycurve that we introduce here are somehow similar to the *bounding* and *carrier* polygons defined and used in [4], [5].

Definition 2.3. – *The control polygon of a polycurve is the union of the control polygons of the Bézier curves composing the polycurve.*

The control polygon of a polycurve (equivalence class) is obviously not unique, two different members of a class of $\mathbb{B}/\sim$ have different control polygons. The same observation holds for the *carrier polygon* of the polycurve:

Definition 2.4. – *The endpoints of all the Bézier curves composing a polycurve, considered* *in order along the polycurve, define its carrier polygon.*

We remark that to any control polygon of a polycurve (equivalence class) is associated a unique carrier polygon. In Figure 1, we can see two different members of the same element of $\mathbb{B}/\!\sim$, for each one the control polygon being represented in solid line and the carrier polygon in dotted line.

2.4. The initial and the current polycurve

Let us recall that a polycurve $\mathcal{B} \in \mathbb{B}$ is written $\mathcal{B} = \cup_1^n \mathcal{B}_i$, where for all $i \in \{1, \ldots, n\}$ $\mathcal{B}_i$ is a completely convex Bézier curve with control polygon $\mathcal{P}(\mathcal{B}_i)$. Then the control polygon of the polycurve will be denoted by $\mathcal{P}(\mathcal{B})$ and its carrier polygon will be denoted by $\mathcal{C}(\mathcal{B})$. The number n of Bézier curves composing the polycurve will be called the *size* of the polycurve and will be denoted by $|\mathcal{B}|$.

We remark that a (connected and simple) union of Bézier curves is a polycurve. Thus, the notations introduced here will be used for any such object, even if it is not explicitely defined as a polycurve. For example, if $\mathcal{B}_i$ is a Bézier curve and its endpoints are Z_0 and Z_z, then its control polygon is $\mathcal{P}(\mathcal{B}_i)$ and its carrier polygon is $\mathcal{C}(\mathcal{B}_i) = [\mathrm{Z}_0 \mathrm{Z}_z]$.

Let $\mathcal{B}$ be a polycurve and n_0 be the minimal size of the members of the equivalence class of $\mathcal{B}$:

$$n_0 = \min\{|\mathcal{B}'| : \mathcal{B}' \sim \mathcal{B}\}$$

There exists a unique polycurve $\mathcal{B}^{ini}$ such that $\mathcal{B}^{ini} \sim \mathcal{B}$ and $|\mathcal{B}^{ini}| = n_0$.This polycurve will be called the *initial* polycurve of the given equivalence class. Any member of the class that we deal with will be the *current* polycurve at a given time (the initial polycurve is then the current polycurve at the very beginning, when no de Casteljau subdivision has been applied). All the polycurves will be denoted by $\mathcal{B} = \cup_1^n \mathcal{B}_i$, keeping in mind that n varies and that the Bézier curve $\mathcal{B}_i$ is generally not the same in two different expressions of a polycurve.

Let $\cup_1^n \mathcal{B}_i$ be the current expression of the polycurve $\mathcal{B}$. For $i \in \{1, \ldots, z\}$, we denote by $\widetilde{\mathcal{B}}_i$ the Bézier curve of the initial expression of $\mathcal{B}$ from which $\mathcal{B}_i$ has been obtained by subdivision. We consider the set of all the Bézier curves of the current composition of $\mathcal{B}$ that are obtained by subdivision from $\widetilde{\mathcal{B}}_i$; these curves have successive subscripts in the expression of $\mathcal{B}$ and their union differs from $\widetilde{\mathcal{B}}_i$ uniquely by parametrisation. This is why we shall consider that $\widetilde{\mathcal{B}}_i$ is a union of Bézier curves of the current polycurve. Thus, $\mathcal{P}(\widetilde{\mathcal{B}}_i)$ does not denote the control polygon of the initial Bézier curve, but a union of control polygons which changes when (some of) the Bézier curves composing $\widetilde{\mathcal{B}}_i$ are subdivided. The same remark holds for $\mathcal{C}(\widetilde{\mathcal{B}}_i)$.

The following figure illustrates all the notions that we have introduced in this section.

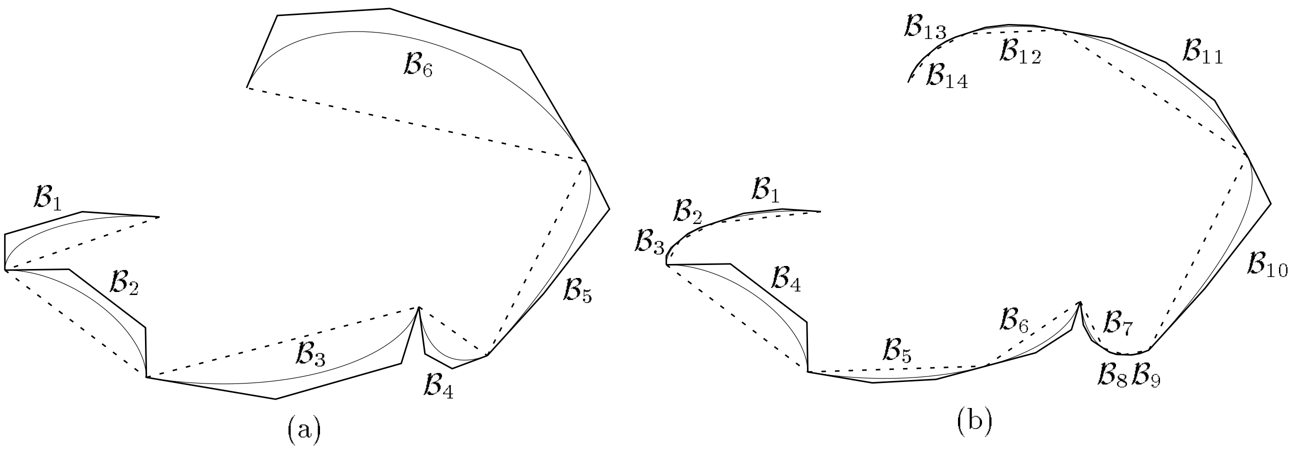


Figure 1. *The control polygon is in solid line and the carrier polygon is dotted.* (a) *The initial polycurve;* $n_0 = 6$. (b) *The current polycurve;* $n = 14$. *For example,* $\mathcal{B}_6^{ini} = \widetilde{\mathcal{B}}_{11} = \widetilde{\mathcal{B}}_{12} = \widetilde{\mathcal{B}}_{13} = \widetilde{\mathcal{B}}_{14} = \mathcal{B}_{11} \cup \mathcal{B}_{12} \cup \mathcal{B}_{13} \cup \mathcal{B}_{14}$, *but* $\mathcal{P}(\mathcal{B}_6^{ini}) \neq \mathcal{P}(\widetilde{\mathcal{B}}_{11}) = \mathcal{P}(\mathcal{B}_{11}) \cup \mathcal{P}(\mathcal{B}_{12}) \cup \mathcal{P}(\mathcal{B}_{13}) \cup \mathcal{P}(\mathcal{B}_{14})$.

3. Arrangements of polycurves

Let $\Gamma = \{\gamma^{(k)}\}_{1 \leq k \leq n}$ be a set of Jordan arcs. The arrangement $\mathcal{A}(\Gamma)$ of Γ is the planar subdivision induced by the curves $\gamma^{(k)}$, that is, $\mathcal{A}(\Gamma)$ is a planar map whose vertices are the pairwise intersection points of the arcs of Γ, whose edges are maximal connected portions of the $\gamma^{(k)}$'s that do not contain a vertex, and whose faces are the connected components of $\mathbb{R}^2 \setminus \cup_1^n \gamma^{(k)}$. The vertices, edges, and faces of the arrangements represent respectively its 0-, 1-, and 2-cells.

Let f be a i-cell and g be a j-cell of $\mathcal{A}(\Gamma)$. If $i = j + 1$ and g lies on the boundary of f, then we say that g is a subcell of f. We also say that f and g define an incidence, or are incident to one another. Using this notion, we can define a useful representation of the arrangement $\mathcal{A}(\Gamma)$: The *incidence graph* of $\mathcal{A}(\Gamma)$ is a graph $g = (V, E)$ where there is a node in V for every cell of $\mathcal{A}(\Gamma)$ and an arc between two nodes if the corresponding cells are incident to one another.

In the following, we shall deal with a set of polycurves $\mathbf{B} = \left\{\mathcal{B}^{(k)}\right\}_{1 \leq k \leq n}$. For all $k \in \{1, \ldots, n\}$, the polycurve $\mathcal{B}^{(k)}$ of the set $\mathbf{B}$ will be written $\mathcal{B}^{(k)} = \cup_{i=1}^{n_k} \mathcal{B}_i^{(k)}$, where for all $k \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, n_k\}$ $\mathcal{B}_i^{(k)}$ is a completely convex Bézier curve with control polygon $\mathcal{P}(\mathcal{B}_i^{(k)})$. We shall also use the notations $\mathbf{P} = \{\mathcal{P}(\mathcal{B}^{(k)})\}_{1 \leq k \leq n}$ and $\mathbf{C} = \{\mathcal{C}(\mathcal{B}^{(k)})\}_{1 \leq k \leq n}$, where $\mathcal{P}(\mathcal{B}^{(k)}) = \cup_{i=1}^{n_k} \mathcal{P}(\mathcal{B}_i^{(k)})$ and $\mathcal{C}(\mathcal{B}^{(k)}) = \cup_{i=1}^{n_k} \mathcal{C}(\mathcal{B}_i^{(k)})$ for all $k \in \{1, \ldots, n\}$.

3.1. Simple arrangements

In the literature, degenerate positions of the objects defining the arrangement are excluded. On the one hand, the combinatorial results on degenerate arrangements are less representative than those obtained in the case of nondegenerate positions. On the other hand, the numerical computations can generate inconsistencies more often in the case of degenerate arrangements. A nondegenerate arrangement is usually called *simple*.

Definition 3.1. – *Let us suppose that all the polycurves of* $\mathbf{B}$ *are written in their initial composition. We will say that the arrangement* $\mathcal{A}(\mathbf{B})$ *is simple if and only if*

(3.1.i) *the endpoints of* $\mathcal{B}_i^{(k)}$ *do not lie on* $\mathcal{B}^{(\ell)}$ *for all* $\ell \neq k \in \{1,\dots,n\}$ *and* $i \in \{1,\dots,n_k\}$;

(3.1.ii) $\mathcal{B}^{(k)} \cap \mathcal{B}^{(\ell)} \cap \mathcal{B}^{(m)} = \emptyset$ *for all* $k, \ell, m \in \{1,\dots,n\}$ *distinct;*

(3.1.iii) *if* $\mathrm{P} \in \mathcal{B}^{(k)} \cap \mathcal{B}^{(\ell)}$ *is a vertex of* $\mathcal{A}(\mathbf{B})$, *then* $\mathcal{B}^{(k)}$ *and* $\mathcal{B}^{(\ell)}$ *cross transversely in* P;

(3.1.iv) $\cup_1^n \mathcal{B}^{(k)}$ *is connected.*

It is easy to see that this definition can be expressed with respect to any composition of the polycurves of $\mathbf{B}$ if we replace $\mathcal{B}_i^{(k)}$ with $\widetilde{\mathcal{B}}_i^{(k)}$ in the first condition.

3.2. Arrangements of control and carrier polygons

For all the possible expressions of the polycurves of the set $\mathbf{B}$, the arrangement $\mathcal{A}(\mathbf{B})$ is the same. This is not the case of the arrangements $\mathcal{A}(\mathbf{P})$ (of control polygons) and $\mathcal{A}(\mathbf{C})$ (of carrier polygons). We shall prove that these two arrangements converge, by subdivision, to $\mathcal{A}(\mathbf{B})$ from a combinatorial point of view. Let us first reformulate the notion of a simple arrangement in the case of arrangements of polygonal chains.

Definition 3.2. – *Let* $\mathbf{L} = \{\mathcal{L}^{(k)}\}_{1\leq k\leq n}$ *be a set of polygonal chains. We will say that the arrangement* $\mathcal{A}(\mathbf{L})$ *is simple iff*

(3.2.i) $\mathcal{L}^{(k)} \cap \mathcal{L}^{(\ell)} \cap \mathcal{L}^{(m)} = \emptyset$ *for all* $k, \ell, m \in \{1,\dots,n\}$ *distinct;*

(3.2.ii) *any two segments of* $\cup_1^n \mathcal{L}^{(k)}$ *lying on different polygonal chains and having a common point cross and do not overlap (they have exactly one common point);*

(3.2.iii) $\cup_1^n \mathcal{L}^{(k)}$ *is connected.*

This definition is identical to Definition 3.1, except for condition (3.1.i), which is not necessary in the case of polygonal chains. Indeed, we consider that we can determine whether a point belongs to a segment or not, decision impossible to take in the case of Bézier curves without the use of algebraic equations.

Lemma 3.3. – *If the arrangement* $\mathcal{A}(\mathbf{B})$ *is simple, we can obtain, via the de Casteljau subdivision, a set of control polygons* $\mathbf{P} = \{\mathcal{P}(\mathcal{B}^{(k)})\}_{1\leq k\leq n}$ *such that:*

(3.3.i) $\mathcal{A}(\mathbf{P})$ *is simple and any further subdivision leaves it simple;*

(3.3.ii) *if* $\mathbf{C} = \{\mathcal{C}(\mathcal{B}^{(k)})\}_{1\leq k\leq n}$ *is the set of carrier polygons associated to* $\mathbf{P} = \{\mathcal{P}(\mathcal{B}^{(k)})\}_{1\leq k\leq n}$, *then* $\mathcal{A}(\mathbf{C})$ *is simple and any further subdivision leaves it simple.*

Proof – It is easy to see that we can obtain, by subdivision, an expression of the set of polycurves $\mathbf{B}$ such that any Bézier curve appearing in this expression intersects the set of all the polycurves on which it does not lie at most once:

$$\mathrm{card}\Big(\mathcal{B}_i^{(k)} \cap \bigcup_{\ell\neq k} \mathcal{B}^{(\ell)}\Big) \leq 1 \qquad \forall k \in \{1,\dots,n\}\ i \in \{1,\dots,n_k\} \tag{1}$$

We take

$$\varepsilon_1 = \frac{1}{2}\min\left\{ \min_{\substack{\mathcal{B}_i^{(k)}\cap\mathcal{B}_j^{(\ell)}=\emptyset \\ k\neq\ell}} \mathrm{d}\left(\mathcal{B}_i^{(k)}, \mathcal{B}_j^{(\ell)}\right),\ \min_{\substack{\mathrm{P}\ \text{endpoint of}\ \widetilde{\mathcal{B}}_i^{(k)} \\ \ell\neq k\in\{1,\dots,n\}}} \mathrm{d}(\mathrm{P}, \mathcal{B}^{(\ell)}) \right\}$$

and we subdivide all the Bézier curves $\mathcal{B}_i^{(k)}$, $k \in \{1,\dots,n\}$, $i \in \{1,\dots,n_k\}$, until

$$\delta^{\mathrm{H}}\big(\mathcal{P}(\mathcal{B}_i^{(k)}), \mathcal{C}(\mathcal{B}_i^{(k)})\big) < \varepsilon_1$$

where δ^{H} is the Hausdorff distance. Thanks to the properties of Bézier curves, this is possible. We shall then have

$$\delta^{\mathrm{H}}\big(\mathcal{P}(\mathcal{B}_i^{(k)}),\mathcal{B}_i^{(k)}\big)<\varepsilon_1 \quad\text{and}\quad \delta^{\mathrm{H}}\big(\mathcal{C}(\mathcal{B}_i^{(k)}),\mathcal{B}_i^{(k)}\big)<\varepsilon_1 \quad \forall k\in\{1,\ldots,n\}\ i\in\{1,\ldots,n_k\}$$

and we claim that in the current expression the arrangements $\mathcal{A}(\mathbf{P})$ and $\mathcal{A}(\mathbf{C})$ are simple.

Indeed, if we suppose for example that condition (3.2.i) is not met by the arrangement $\mathcal{A}(\mathbf{P})$, let $k,\ell,m\in\{1,\ldots,n\}$ all distinct and $h\in\{1,\ldots,n_k\}$, $i\in\{1,\ldots,n_\ell\}$, and $j\in\{1,\ldots,n_m\}$ be such that $\mathcal{P}(\mathcal{B}_h^{(k)})\cap\mathcal{P}(\mathcal{B}_i^{(\ell)})\cap\mathcal{P}(\mathcal{B}_j^{(m)})\neq\emptyset$ and let P be a point of this intersection. As $\mathcal{A}(\mathbf{B})$ fulfills (1), we cannot have $\mathcal{B}_h^{(k)}\cap\mathcal{B}_i^{(\ell)}\neq\emptyset$, $\mathcal{B}_i^{(\ell)}\cap\mathcal{B}_j^{(m)}\neq\emptyset$, and $\mathcal{B}_j^{(m)}\cap\mathcal{B}_h^{(k)}\neq\emptyset$ simultaneously; let us suppose $\mathcal{B}_h^{(k)}\cap\mathcal{B}_i^{(\ell)}=\emptyset$. Then we will have

$$\mathrm{d}\left(\mathcal{B}_h^{(k)},\mathcal{B}_i^{(\ell)}\right)\le\mathrm{d}\left(\mathrm{P},\mathcal{B}_h^{(k)}\right)+\mathrm{d}\left(\mathrm{P},\mathcal{B}_i^{(\ell)}\right)\le\delta^{\mathrm{H}}\left(\mathcal{P}(\mathcal{B}_h^{(k)}),\mathcal{B}_h^{(k)}\right)+\delta^{\mathrm{H}}\left(\mathcal{P}(\mathcal{B}_i^{(\ell)}),\mathcal{B}_i^{(\ell)}\right)<2\varepsilon_1$$

which contradicts the definition of ε_1.

The second part of the definition of ε_1 and the simplicity of $\mathcal{A}(\mathbf{B})$ similarly imply the conditions (3.2.ii) and (3.2.iii) for $\mathcal{A}(\mathbf{P})$ and the reasoning for $\mathcal{A}(\mathbf{C})$ is identical.

□

3.3. Equivalence of arrangements

As we mentioned at the beginning of the section, the notion of equivalence of arrangements is closely related to the one of incidence graph. This is expressed by the following definition:

Definition (Grünbaum [12]) 3.4. – *Let $\mathcal{M}$ and $\mathcal{N}$ be two sets of curves with $|\mathcal{M}|=|\mathcal{N}|$. We will say that $\mathcal{A}(\mathcal{M})$ and $\mathcal{A}(\mathcal{N})$ are equivalent if and only if there exists a bijection $\varphi:\mathcal{A}(\mathcal{M})\longrightarrow\mathcal{A}(\mathcal{N})$ preserving the incidences.*

Using the definition above, to decide whether $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ are equivalent we must construct the incidence graphs of the two and compare them. To simplify this decision, we give sufficient conditions of equivalence based only on the vertices of the two arrangements. But before, let us point out that we will use the notation $\mathcal{A}_0(\mathbf{B})$ for the set of vertices of $\mathcal{A}(\mathbf{B})$, $\mathcal{A}_1(\mathbf{B})$ for the edges and $\mathcal{A}_2(\mathbf{B})$ for the faces (the same notations are obviously valid for $\mathcal{A}(\mathbf{P})$ and $\mathcal{A}(\mathbf{C})$).

Hypotheses:

(H1) $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ are simple;

(H2) the endpoints of $\widetilde{\mathcal{B}}_i^{(k)}$ do not belong to $reg\left(\mathcal{B}_j^{(\ell)},\mathcal{P}(\mathcal{B}_j^{(\ell)})\right)$ for all $k\neq\ell\in\{1,\ldots,n\}$, $i\in\{1,\ldots,n_k\}$ and $j\in\{1,\ldots,n_\ell\}$, where $reg(\mathcal{B}_j^{(\ell)},\mathcal{P}(\mathcal{B}_j^{(\ell)}))$ is the open bounded region of the plane delimited by $\mathcal{B}_j^{(\ell)}\cup\mathcal{P}(\mathcal{B}_j^{(\ell)})$;

(H3) there exists a bijection $\varphi_0:\mathcal{A}_0(\mathbf{B})\longrightarrow\mathcal{A}_0(\mathbf{P})$ satisfying
- (H3.i) the vertex $v\in\mathcal{A}_0(\mathbf{B})$ belongs to $\mathcal{B}^{(k)}$ if and only if the vertex $\varphi_0(v)\in\mathcal{A}_0(\mathbf{P})$ belongs to $\mathcal{P}(\mathcal{B}^{(k)})$;
- (H3.ii) if $\{v_1,\ldots,v_m\}\subset\mathcal{A}_0(\mathbf{B})$ is the set of all vertices belonging to $\mathcal{B}^{(k)}$, then the order of these vertices on $\mathcal{B}^{(k)}$ and the order of the vertices $\{\varphi_0(v_1),\ldots,\varphi_0(v_m)\}\subset\mathcal{A}_0(\mathbf{P})$ on $\mathcal{P}(\mathcal{B}^{(k)})$ are identical.

Proposition 3.5. – *If the hypotheses (H1), (H2) and (H3) are fulfilled, $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ are equivalent.*

Proof – To prove the equivalence of the two arrangements, we construct the bijection φ of Definition 3.4.

As the function φ_0 of (H3) is a bijection, we deduce from (H3.i) that the number of vertices of $\mathcal{A}(\mathbf{B})$ belonging to $\mathcal{B}^{(k)}$ and the number of vertices of $\mathcal{A}(\mathbf{P})$ belonging to $\mathcal{P}(\mathcal{B}^{(k)})$ are equal for all $k \in \{1, \ldots, n\}$. Obviously, $\mathcal{B}^{(k)}$ is a closed curve if and only if $\mathcal{P}(\mathcal{B}^{(k)})$ is a closed polygonal chain, thus the number of edges of $\mathcal{A}(\mathbf{B})$ lying on $\mathcal{B}^{(k)}$ and the number of edges of $\mathcal{A}(\mathbf{P})$ lying on $\mathcal{P}(\mathcal{B}^{(k)})$ are also equal. Thanks to (H3.ii) we can establish a one-to-one correspondence φ_1 between the edges of $\mathcal{A}(\mathbf{B})$ and those of $\mathcal{A}(\mathbf{P})$ such that if the edge e and the vertex v define an incidence in $\mathcal{A}(\mathbf{B})$, then $\varphi_1(e)$ and $\varphi_0(v)$ define an incidence in $\mathcal{A}(\mathbf{P})$.

So, the restriction of φ to $\mathcal{A}_0(\mathbf{B}) \cup \mathcal{A}_1(\mathbf{B})$ is defined and satisfies the required conditions; we want now to define $\varphi_2 = \varphi\big|_{\mathcal{A}_2(\mathbf{B})}$.

Let c be a face of $\mathcal{A}(\mathbf{B})$ and $v_1 \in \mathcal{A}_0(\mathbf{B})$ be a vertex lying on the boundary of c. First, we suppose that the boundary of c is a simple closed curve δ_1, composed of vertices and edges of $\mathcal{A}(\mathbf{B})$. Let $\mathcal{L}_1 = \{v_1, e_1, \ldots, v_a, e_a\}$ be the counterclockwise ordered list of all these vertices and edges. The properties of φ_0 and φ_1 imply that the list $\mathcal{L}_2 = \{\varphi_0(v_1), \varphi_1(e_1), \ldots, \varphi_0(v_a), \varphi_1(e_a)\}$ defines a simple closed curve δ_2 and we prove that this curve is the boundary of a face of $\mathcal{A}(\mathbf{P})$. For this, we show that all the edges of $\mathcal{L}_2$ are incident to the same face.

The edges $\varphi_1(e_1)$ and $\varphi_1(e_2)$ are of course incident to a common face, let it be c'. The boundary of c being considered counterclockwisely, c is to the left of e_1 and e_2, as shown in Figure 2, and we claim that c' is to the left of $\varphi_1(e_1)$ and $\varphi_1(e_2)$.

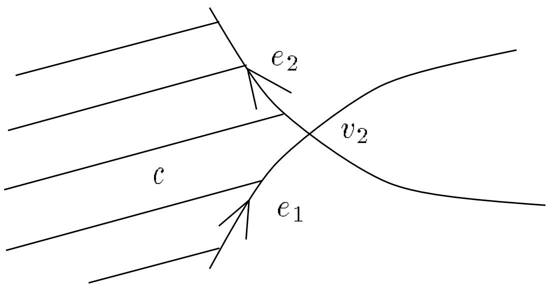


Figure 2. *The face c is to the left regarding the edges e_1 and e_2; the edge e_2 is "the first to the left" from v_2 as we arrive on e_1.*

When we arrive in v_2 on e_1, e_2 is "the first to the left". Let $k \neq \ell \in \{1, \ldots, n\}$ be such that $e_1 \subset \mathcal{B}^{(k)}$ and $e_2 \subset \mathcal{B}^{(\ell)}$. If the vertex v_3 precedes the vertex v_2 on $\mathcal{B}^{(\ell)}$, then condition (H3.ii) implies that $\varphi_0(v_3)$ precedes $\varphi_0(v_2)$ on $\mathcal{P}(\mathcal{B}^{(\ell)})$. On the other hand, condition (H2) implies that if the polycurve $\mathcal{B}^{(k)}$ crosses the polycurve $\mathcal{B}^{(\ell)}$ from left to right, then the control polygon $\mathcal{P}(\mathcal{B}^{(k)})$ crosses the control polygon $\mathcal{P}(\mathcal{B}^{(\ell)})$ from left to right. Thus, we can deduce that $\varphi_1(e_2)$ is "the first to the left" from $\varphi_0(v_2)$ as we arrive on $\varphi_1(e_1)$ and therefore the face c' is to the left regarding $\varphi_1(e_1)$ and $\varphi_1(e_2)$.

By the same reasoning, we obtain that the edge $\varphi_1(e_i)$ is the first to the left from $\varphi_0(v_i)$ as we arrive on $\varphi_1(e_{i-1})$ and consequently, by induction, we deduce that $\varphi_1(e_i)$ is incident to c', for all $i \in \{3, \ldots, a\}$.

So, for any face c of $\mathcal{A}(\mathbf{B})$ bounded by a simple closed curve there exists a face c' of $\mathcal{A}(\mathbf{P})$ bounded by the image through φ_0 and φ_1 of the boundary of c. Let us now suppose that the boundary of the cell c is not a simple close curve. Then there exists at least one edge e of $\mathcal{A}(\mathbf{B})$ which is incident to c and to no other cell of the arrangement. The edge e is either a *free* edge (edge that is incident to a unique vertex, containing an endpoint of a polycurve), or a *traversing* edge, as we can see in Figure 3. We remark that there can not exist more than one traversing edge incident to a given cell.

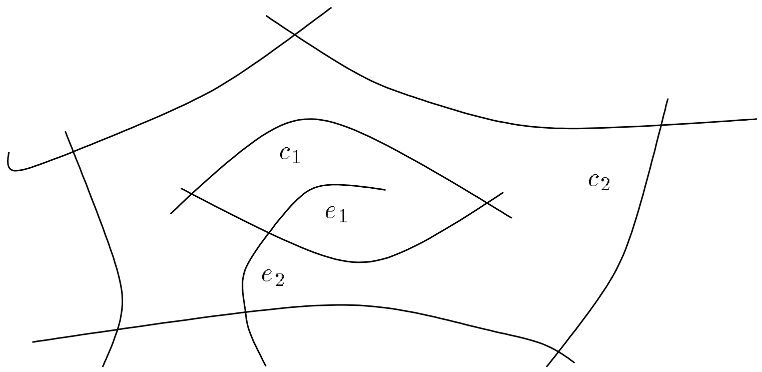


Figure 3. e_1 *is a free edge incident to* c_1 *and* e_2 *is a traversing edge incident to* c_2*.*

It is obvious that if e is a free egde, then $\varphi_1(e)$ is a free edge. It is also easy to see that if e is a traversing edge, then $\varphi_1(e)$ is a traversing edge. Indeed, in this case the set $\mathcal{A}_0(\mathbf{B}) \cup \mathcal{A}_1(\mathbf{B}) \setminus \{e\}$ is not connected and thus the set $\mathcal{A}_0(\mathbf{P}) \cup \mathcal{A}_1(\mathbf{P}) \setminus \{\varphi_1(e)\}$ is not connected. This implies that $\varphi_1(e)$ is a traversing edge.

Using these remarks, the case where the boundary of the face c is not a simple closed curve can be treated in the same way as the case that we have presented. We conclude that we can define a one-to-one map function $\varphi_2 : \mathcal{A}_2(\mathbf{B}) \longrightarrow \mathcal{A}_2(\mathbf{P})$ such that the function φ satisfying $\varphi\big|_{\mathcal{A}_i(\mathbf{B})} = \varphi_i$ for $i = 0, 1, 2$ preserves the incidences. But φ_0 and φ_1 are bijective, so the reasoning above can be made starting with the faces of $\mathcal{A}(\mathbf{P})$. Thus, we can state that φ is bijective.

$\square$

3.4. Existence of the solution

In this subsection, we present the result justifying, from a theoretical point of view, the method proposed in our paper. To prove this result, we shall use the followin lemma:

Lemma 3.6. – *Let* $\mathcal{Q}$ *and* $\mathcal{R}$ *be two completely convex Bézier curves satisfying* $\mathrm{card}(\mathcal{Q} \cap \mathcal{R}) = 1$ *and crossing transversely. Then we can obtain, by de Casteljau subdivision, control polygons of* $\mathcal{Q}$ *and* $\mathcal{R}$ *such that their intersection contains exactly one point and any further subdivision leaves this property unchanged.*

Proof – Let P be the intersection point of $\mathcal{Q}$ and $\mathcal{R}$. We denote the tangent to $\mathcal{Q}$ at the point P by $\tan_{\mathrm{P}}(\mathcal{Q})$; similarly, $\tan_{\mathrm{P}}(\mathcal{R})$ is the tangent to $\mathcal{R}$ at P. As the two curves cross transversely in P, there exists a unique common supporting line Δ of $\mathcal{Q}$ and $\mathcal{R}$ satisfying

$$\Delta \cap \mathcal{Q} \subset \mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{R}), \mathcal{R}) \quad \text{and} \quad \Delta \cap \mathcal{R} \subset \mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{Q}), \mathcal{Q})$$

where $\mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{Q}), \mathcal{Q})$ is the open half-plane delimited by $\tan_{\mathrm{P}}(\mathcal{Q})$ and containing no point of $\mathcal{Q}$ and similarly for $\mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{R}), \mathcal{R})$. Let O be the orthogonal projection of P on Δ. Λ is the parallel to $\tan_{\mathrm{P}}(\mathcal{Q})$ passing through O and Υ is the parallel to $\tan_{\mathrm{P}}(\mathcal{R})$ passing through O, as shown in Figure 4. We also introduce the notations $\{\mathrm{M}\} = \Upsilon \cap \mathcal{Q}$, $\{\mathrm{N}\} = \Lambda \cap \mathcal{R}$, $\{\mathrm{T}_{\mathcal{Q}}\} = \Delta \cap \mathcal{Q}$, and $\{\mathrm{T}_{\mathcal{R}}\} = \Delta \cap \mathcal{R}$.

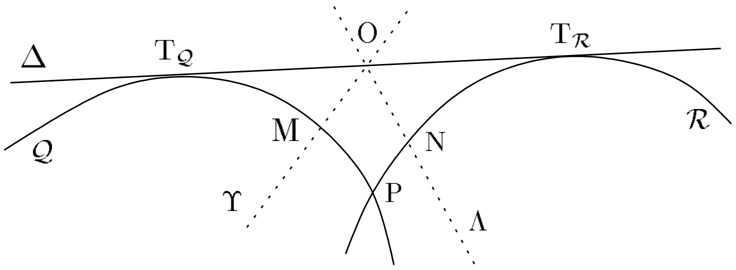


Figure 4. *Definition of the lines* Λ *and* Υ.

We subdivide $\mathcal{Q}$ until $\mathcal{P}(\mathcal{Q}) \subset \mathrm{HP}(\Lambda, \mathcal{Q})$ and $\mathcal{R}$ until $\mathcal{P}(\mathcal{R}) \subset \mathrm{HP}(\Upsilon, \mathcal{R})$, where $\mathrm{HP}(\Lambda, \mathcal{Q})$ is the open half-plane delimited by Λ and such that its closure contains $\mathcal{Q}$ and similarly for $\mathrm{HP}(\Upsilon, \mathcal{R})$. It is obvious that $\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R}) \neq \emptyset$. Moreover, $\mathcal{P}(\mathcal{Q})$ and $\mathcal{P}(\mathcal{R})$ intersect in the curved rhombus defined by Λ, Υ, $\mathcal{Q}$, and $\mathcal{R}$, denoted by $\diamondsuit$PMON. If we suppose $\mathrm{card}\big(\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R})\big) \geq 2$, then there exist two points A_1 and A_2 in the interior of $\diamondsuit$PMON which are common to both control polygons.

Let $\mathrm{A}_1\mathrm{A}_2$ be the line passing through A_1 and A_2. As $\mathcal{Q}$ and $\mathcal{R}$ were, initially, completely convex Bézier curves, all the intersection points of $\mathrm{A}_1\mathrm{A}_2$ and $\mathcal{Q}$ (respectively $\mathcal{R}$) lie on the segment $[\mathrm{A}_1\mathrm{A}_2]$. We can deduce that $\mathrm{A}_1\mathrm{A}_2 \cap \mathcal{Q} \subset \mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ and $\mathrm{A}_1\mathrm{A}_2 \cap \mathcal{R} \subset \mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$. Moreover, $\mathrm{A}_1\mathrm{A}_2$ either is tangent to $\mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ or intersects $\mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ in two points. Indeed, the points A_1 and A_2 belong to $\diamondsuit$PMON and thus to the curve triangle $\mathrm{T}_{\mathcal{Q}}\mathrm{T}_{\mathcal{R}}\mathrm{P}$. It also results that $\mathrm{A}_1\mathrm{A}_2$ intersects the open segment $(\mathrm{T}_{\mathcal{Q}}\mathrm{T}_{\mathcal{R}})$ in exactly one point.

On the other hand, $\mathrm{A}_1\mathrm{A}_2$ either is tangent to $\mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$ or intersects $\mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$ in two points. To summarize, there exists a line which is tangent to $\mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ or intersects it in two points, is tangent to $\mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$ or intersects it in two points and intersects $(\mathrm{T}_{\mathcal{Q}}\mathrm{T}_{\mathcal{R}})$ in one point. This is obviously a contradiction, because the curves $\mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ and $\mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$ are convex and $\mathrm{T}_{\mathcal{Q}}\mathrm{T}_{\mathcal{R}}$ is tangent to them.

The supposition that $\mathrm{card}\,\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R}) > 1$ brought us to a contradiction and hence it is false. We deduce that the two control polygons intersect in exactly one point. Also, by construction, it is obvious that any further subdivision applied to the Bézier curves that currently compose $\mathcal{Q}$ and $\mathcal{R}$ leaves this property unchanged.

$\square$

We can now state and prove the theorem that we announced at the beginning of this section.

Theorem 3.7. – *If $\mathcal{A}(\mathbf{B})$ is simple, then we can obtain via the de Casteljau subdivision a set of control polygons $\mathbf{P} = \{\mathcal{P}(\mathcal{B}^{(k)})\}_{1\leq k\leq n}$ such that $\mathcal{A}(\mathbf{P})$ is simple and equivalent to $\mathcal{A}(\mathbf{B})$ and any further subdivision leaves these two properties unchanged.*

Proof – To prove this theorem, we show that a set of control polygons satisfying hypotheses (H1), (H2) and (H3) can be obtained and then we apply Proposition 3.5.

The existence of the simple arrangement $\mathcal{A}(\mathbf{P})$ is proven by Lemma 3.3, and we construct it as shown in the proof of that result. It is easy to see that, by construction, $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ fulfill (H1) and (H2). We also recall that the construction method given in the proof of Lemma 3.3 implies that, in its current expression, the set of polycurves $\mathbf{B}$ satisfies

$$\mathrm{card}\Big(\mathcal{B}_i^{(k)} \cap \bigcup_{\ell\neq k} \mathcal{B}^{(\ell)}\Big) \leq 1 \qquad \text{for all } k \in \{1,\ldots,n\},\ i \in \{1,\ldots,n_k\}$$

Let $\mathcal{B}_i^{(k)}$ and $\mathcal{B}_j^{(\ell)}$ be two Bézier curves of this expression of the arrangement. If they are disjoint, we subdivide them until their control polygons are also disjoint. If $\mathrm{card}(\mathcal{B}_i^{(k)} \cap \mathcal{B}_j^{(\ell)}) = 1$, they satisfy the conditions of Lemma 3.6; we subdivide them until their control polygons intersect in exactly one point. For the arrangement of control polygons $\mathcal{A}(\{\mathcal{P}(\mathcal{B}^{(k)})\}_{1\leq k\leq n})$ thus obtained, we define $\varphi_0 : \mathcal{A}(\mathbf{B}) \longrightarrow \mathcal{A}(\{\mathcal{P}(\mathcal{B}^{(k)})\}_{1\leq k\leq n})$ that associates to the intersection point of two Bézier curves of $\mathcal{A}(\mathbf{B})$ the intersection point of their control polygons. It is easy to see that φ_0 is well defined and a bijection; also, it fulfills hypothesis (H3.i). To prove that it fulfills hypothesis (H3.ii), it suffices to remark that the order of the Bézier curves $\mathcal{B}_1^{(k)}, \mathcal{B}_2^{(k)}, \ldots, \mathcal{B}_{n_k}^{(k)}$ in the composition of the polycurve $\mathcal{B}^{(k)}$ and the order of the control polygons $\mathcal{P}(\mathcal{B}_1^{(k)}), \mathcal{P}(\mathcal{B}_2^{(k)}), \ldots, \mathcal{P}(\mathcal{B}_{n_k}^{(k)})$ in the composition of $\mathcal{P}(\mathcal{B}^{(k)})$ are identical for all $k \in \{1, \ldots, n\}$.

$\square$

4. Sufficient conditions of equivalence

The result stated by Theorem 3.7 is very intuitive, thanks to the properties of the completely convex Bézier curves. But, as we have already mentioned, it is easy to see that it cannot be used to conceive an algorithm constructing the arrangement $\mathcal{A}(\mathbf{P})$ equivalent to $\mathcal{A}(\mathbf{B})$ without solving algebraic equations. So, we state a less intuitive property, namely that we can use only the polygonal approximations of the polycurves to verify if the arrangements are equivalent. The conditions of equivalence based on the sets of polygonal chains $\mathbf{P}$ and $\mathbf{C}$ are sufficient and not necessary. Yet, for any simple arrangement of polycurves we can obtain, by subdivision, arrangements of control polygons (respectively of carrier polygons) that fulfill the required properties. We have established two conditions. The first one assures that the number of vertices of $\mathcal{A}(\mathbf{B})$ that belong to a polycurve $\mathcal{B}^{(k)}$ is the same as the number of vertices of $\mathcal{A}(\mathbf{P})$ that belong to $\mathcal{P}(\mathcal{B}^{(k)})$. The second one assures that the orders of these vertices respectively on $\mathcal{B}^{(k)}$ and $\mathcal{P}(\mathcal{B}^{(k)})$ are identical. Proposition 4.5 states the sufficiency of the given conditions.

4.1. First condition: intersection of Bézier curves

Let $\mathcal{Q}$ and $\mathcal{R}$ be two completely convex Bézier curves with control polygons respectively $\mathcal{P}(\mathcal{Q}) = \mathrm{Q}_0\mathrm{Q}_1\ldots\mathrm{Q}_q$ and $\mathcal{P}(\mathcal{R}) = \mathrm{R}_0\mathrm{R}_1\ldots\mathrm{R}_r$. It is known that generally there is no relation between $\mathrm{card}(\mathcal{Q}\cap\mathcal{R})$ and $\mathrm{card}\big(\mathcal{P}(\mathcal{Q})\cap\mathcal{P}(\mathcal{R})\big)$, as we can see in Figure 5.

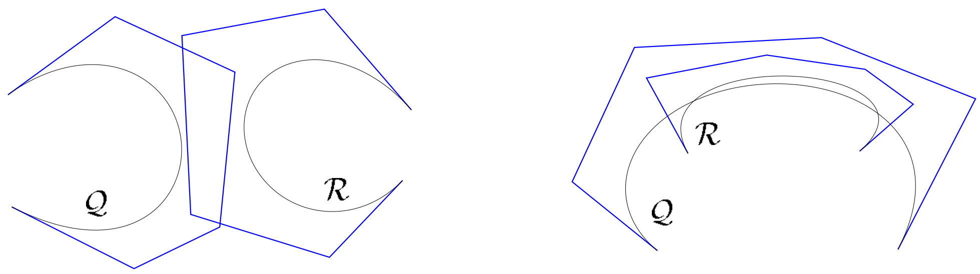


Figure 5. $\mathrm{card}(\mathcal{Q}\cap\mathcal{R})$ *and* $\mathrm{card}\big(\mathcal{P}(\mathcal{Q})\cap\mathcal{P}(\mathcal{R})\big)$ *are generally not related.*

Let us recall that $reg(c_1, \ldots, c_m)$ denotes the open bounded region delimited by the simple closed curve $\cup_1^m c_i$. If $\overline{reg\big(\mathcal{P}(\mathcal{Q}),\mathcal{C}(\mathcal{Q})\big)}$ and $\overline{reg\big(\mathcal{P}(\mathcal{R}),\mathcal{C}(\mathcal{R})\big)}$ are disjoint, it is obvious that $\mathcal{Q} \cap \mathcal{R} = \emptyset$. If the two Bézier curves have more than one common point, we could find no criterion to assure $\text{card}(\mathcal{Q} \cap \mathcal{R}) = \text{card}\big(\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R})\big)$. That is why we look for conditions which assure $\text{card}(\mathcal{Q} \cap \mathcal{R}) = 1$. The strongest condition that we can impose to the control and carrier polygons of the two Bézier curves regarding only their intersections is:

$$\left.\begin{array}{lr} \text{card}\big(\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R})\big) = \text{card}\big(\mathcal{C}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R})\big) = 1 & \text{(2a)}[\mathcal{Q},\mathcal{R}] \\ \text{card}\big(\mathcal{P}(\mathcal{Q}) \cap \mathcal{C}(\mathcal{R})\big) = \text{card}\big(\mathcal{C}(\mathcal{Q}) \cap \mathcal{C}(\mathcal{R})\big) = 1 & \text{(2b)}[\mathcal{Q},\mathcal{R}] \end{array}\right\} \quad (2)[\mathcal{Q},\mathcal{R}]$$

If we consider that the regions $reg\,(\mathcal{P}(\mathcal{Q}), \mathcal{C}(\mathcal{Q}))$ and $reg\,(\mathcal{P}(\mathcal{R}), \mathcal{C}(\mathcal{R}))$ represent "thick curves", then, roughly speaking, conditions $(2)[\mathcal{Q},\mathcal{R}]$ mean that these thick curves cross once. This is the property that should be fulfilled by the Bézier curves $\mathcal{Q}$ and $\mathcal{R}$. Conditions (2) are, taken together, symmetric in $\mathcal{Q}$ and $\mathcal{R}$. The following figure presents examples of curves fulfilling one or both of conditions (2).

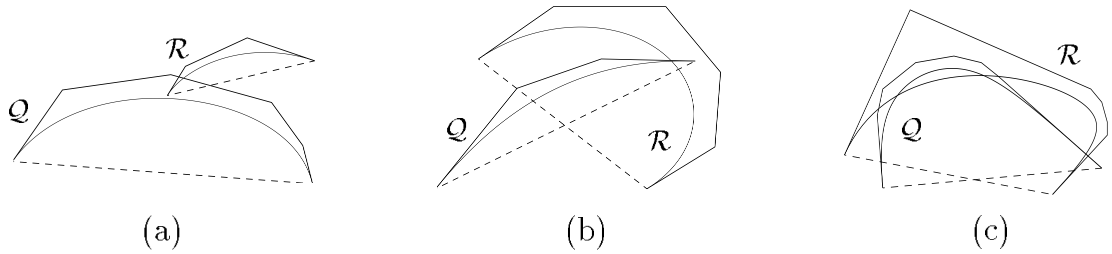


Figure 6. (a) $(2a)[\mathcal{R},\mathcal{Q}]$ *is satisfied, but* $(2a)[\mathcal{Q},\mathcal{R}]$ *is not;* (b) $(2b)[\mathcal{Q},\mathcal{R}]$ *is satisfied and* $(2b)[\mathcal{R},\mathcal{Q}]$ *is not;* (c) $(2)[\mathcal{Q},\mathcal{R}]$ *are both satisfied.*

As we can see in Figure 6(c), conditions $(2)[\mathcal{Q},\mathcal{R}]$ are not sufficient for the equality $\text{card}(\mathcal{Q} \cap \mathcal{R}) = 1$. Supposing that $\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R}) = [\mathrm{Q}_i\mathrm{Q}_{i+1}] \cap [\mathrm{R}_j\mathrm{R}_{j+1}]$, we will require the supplementary property illustrated by Figure 7: The two endpoints of $\mathcal{P}(\mathcal{Q})$ do not belong to the same of the two closed half-planes delimited by $\mathrm{R}_j\mathrm{R}_{j+1}$, and symmetrically in $\mathcal{Q}$ and $\mathcal{R}$.

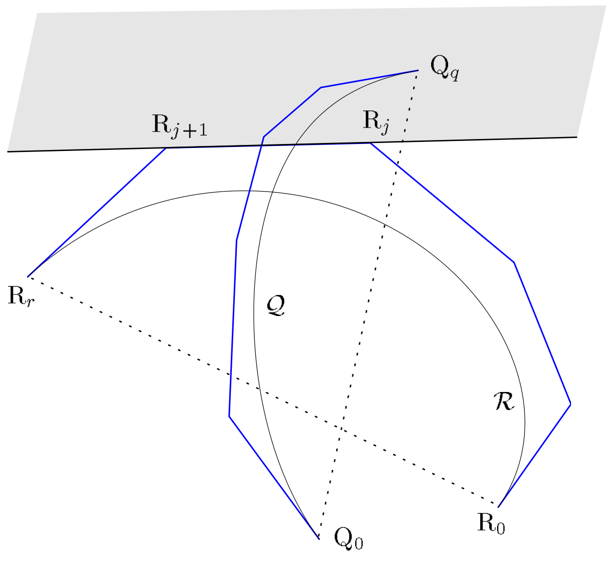


Figure 7. *Illustration of the condition* $(3)[\mathcal{Q},\mathcal{R}]$*; the condition* $(3)[\mathcal{R},\mathcal{Q}]$ *is obtained symmetrically.*

This condition is formally given by the relations (3)$[\mathcal{Q},\mathcal{R}]$ and (3)$[\mathcal{R},\mathcal{Q}]$:

$$\mathrm{Q}_0 \in \mathrm{R}_j\mathrm{R}_{j+1} \quad \text{or} \quad \mathrm{Q}_q \in \overline{\mathrm{HP}_{opp}(\mathrm{R}_j\mathrm{R}_{j+1},\mathrm{Q}_0)} \qquad (3)[\mathcal{Q},\mathcal{R}]$$

We recall that if a point P does not belong to a line Δ, $\mathrm{HP}(\Delta,\mathrm{P})$ denotes the open half-plane delimited by Δ which contains P and $\mathrm{HP}_{opp}(\Delta,\mathrm{P}) = \mathbb{R}^2 \setminus \overline{\mathrm{HP}(\Delta,\mathrm{P})}$.

As the conditions (2)$[\mathcal{Q},\mathcal{R}]$ are fulfilled, exactly one of the endpoints of $\mathcal{Q}$ either belongs to the segment $[\mathrm{R}_0\mathrm{R}_r]$ or does not belong to $\overline{\mathrm{HP}(\mathrm{R}_0\mathrm{R}_r,\mathcal{R})}$, the closed half-plane delimited by $\mathrm{R}_0\mathrm{R}_r$ and containing $\mathcal{R}$. Let us suppose that this endpoint is Q_0, as shown in Figure 7. Then it is easy to see that condition (3)$[\mathcal{Q},\mathcal{R}]$ implies $\mathrm{Q}_q \notin \mathrm{HP}(\mathrm{R}_j\mathrm{R}_{j+1},\mathcal{R})$.

Proposition 4.1. – *If* (2)$[\mathcal{Q},\mathcal{R}]$, (3)$[\mathcal{Q},\mathcal{R}]$ *and* (3)$[\mathcal{R},\mathcal{Q}]$ *are satisfied, then* $\mathrm{card}(\mathcal{Q}\cap\mathcal{R}) = 1$.

Proof – Conditions (2)$[\mathcal{Q},\mathcal{R}]$ have as an immediate consequence the fact that $\mathcal{Q}$ and $\mathcal{R}$ are not disjoint. Let us suppose that $\mathrm{card}(\mathcal{Q}\cap\mathcal{R}) \geq 2$ and let P_1 and P_2 be two of the common points of $\mathcal{Q}$ and $\mathcal{R}$. The region delimited by $\mathcal{Q}\big|_{(\mathrm{P}_1,\mathrm{P}_2)}$ and $\mathcal{R}\big|_{(\mathrm{P}_1,\mathrm{P}_2)}$ cannot be convex, because if it is convex then $[\mathrm{Q}_0\mathrm{Q}_q]\cap[\mathrm{R}_0\mathrm{R}_r] = \emptyset$, which contradicts (2b)$[\mathcal{Q},\mathcal{R}]$.

To fix the notations, we suppose that $\mathrm{Q}_0 \in \mathrm{HP}_{opp}(\mathrm{R}_0\mathrm{R}_r,\mathcal{R})$ and $\mathrm{R}_0 \in \mathrm{HP}_{opp}(\mathrm{Q}_0\mathrm{Q}_q,\mathcal{Q})$, as shown in Figure 8. Let Λ and Λ' be the two supporting lines from R_r to $\mathcal{Q}$ and we take $\mathrm{T}_\mathcal{Q},\mathrm{T}'_\mathcal{Q} \in \mathcal{Q}$ such that $\Lambda = \mathrm{R}_r\mathrm{T}_\mathcal{Q}$ and $\Lambda' = \mathrm{R}_r\mathrm{T}'_\mathcal{Q}$. One (and only one) of the points $\mathrm{T}_\mathcal{Q}$ and $\mathrm{T}'_\mathcal{Q}$ belongs to $\mathrm{HP}_{opp}(\mathrm{R}_0\mathrm{R}_r,\mathcal{R})$; let $\mathrm{T}'_\mathcal{Q}$ be this point. Symmetrically, we define the lines $\Upsilon = \mathrm{Q}_q\mathrm{T}_\mathcal{R}$ and $\Upsilon' = \mathrm{Q}_q\mathrm{T}'_\mathcal{R}$ with $\mathrm{T}'_\mathcal{R} \in \mathrm{HP}_{opp}(\mathrm{Q}_0\mathrm{Q}_q,\mathcal{Q})$.

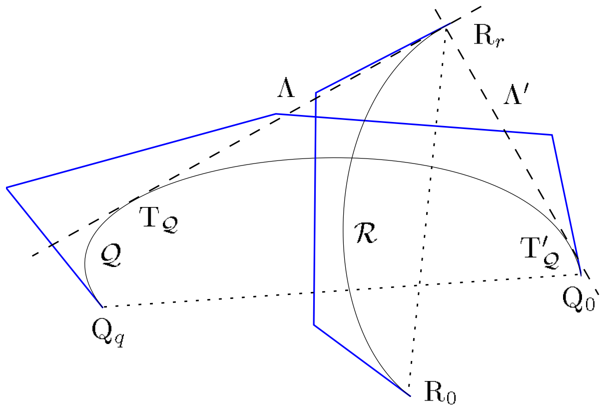


Figure 8. *Definition of the supporting lines* Λ *and* Λ'.

Obviously, no intersection point of $\mathcal{Q}$ and $\mathcal{R}$ can belong to $\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}'_\mathcal{Q})}$. Moreover, only one intersection point of the two curves can lie on $\mathcal{Q}\big|_{(\mathrm{T}'_\mathcal{Q},\mathrm{T}_\mathcal{Q})}$, because otherwise there would exist two points of $\mathcal{Q}\cap\mathcal{R}$, let us call them P_1 and P_2, such that the region delimited by $\mathcal{Q}\big|_{(\mathrm{P}_1,\mathrm{P}_2)}$ and $\mathcal{R}\big|_{(\mathrm{P}_1,\mathrm{P}_2)}$ is convex, contradiction. We deduce that on $\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}_\mathcal{Q})}$ lies at most one point of intersection of $\mathcal{Q}$ and $\mathcal{R}$; similarly, on $\mathcal{R}\big|_{(\mathrm{R}_0,\mathrm{T}_\mathcal{R})}$ lies at most one intersection point of $\mathcal{Q}$ and $\mathcal{R}$.

$\mathcal{Q}$ is completely convex and $\mathrm{card}\big(\mathcal{C}(\mathcal{R})\cap\mathcal{C}(\mathcal{Q})\big) = \mathrm{card}\big(\mathcal{P}(\mathcal{R})\cap\mathcal{P}(\mathcal{Q})\big) = 1$. It is then easy to see that $\mathrm{card}\big(\mathcal{C}(\mathcal{R})\cap\mathcal{Q}\big) = 1$. In the same way, $\mathrm{card}\big(\mathcal{C}(\mathcal{Q})\cap\mathcal{R}\big) = 1$. As $\mathcal{Q}\cup\mathcal{C}(\mathcal{Q})$ and $\mathcal{R}\cup\mathcal{C}(\mathcal{R})$ are simple closed curves, the cardinality of their intersection must be even (they cannot be tangent one to the other, because the arrangement they form is simple).

Then the relation $\operatorname{card}\big(\mathcal{C}(\mathcal{Q})\cap\mathcal{C}(\mathcal{R})\big)=1$ implies, together with the two equalities above, that $\operatorname{card}(\mathcal{Q}\cap\mathcal{R})$ is odd. We supposed that this cardinality is greater than 1, therefore it must be at least 3. Let P_1, P_2 and P_3 be three distinct intersection points of the two curves and let us study the possible positions of these points on $\mathcal{Q}$ and $\mathcal{R}$.

First case: $\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}_\mathcal{Q})}\cap\mathcal{R}\big|_{(\mathrm{R}_0,\mathrm{T}_\mathcal{R})}\neq\emptyset$; let us suppose that the intersection point of these two pieces of curve is P_1. Then we must have $\mathrm{P}_2,\mathrm{P}_3\in\mathcal{Q}\big|_{(\mathrm{T}_\mathcal{Q},\mathrm{Q}_q)}\cap\mathcal{R}\big|_{(\mathrm{T}_\mathcal{R},\mathrm{R}_r)}$, situation presented in Figure 9. But this position obviously leads to a contradiction, because $\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}_\mathcal{Q})}$ and $\mathcal{R}\big|_{(\mathrm{R}_0,\mathrm{T}_\mathcal{R})}$ must also intersect and conditions (3)$[\mathcal{Q},\mathcal{R}]$ and (3)$[\mathcal{R},\mathcal{Q}]$ cannot be both fulfilled.

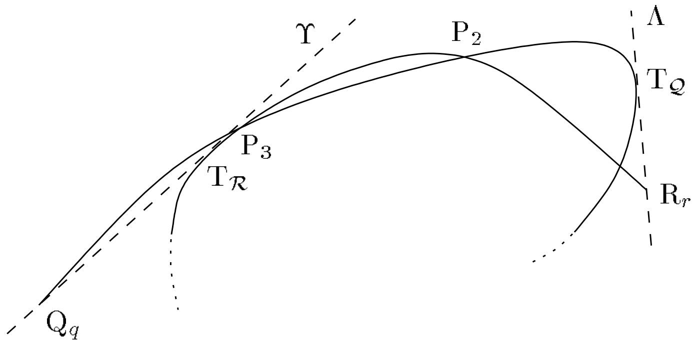


Figure 9. *Relative position of the two curves in the first case.*

Second case: $\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}_\mathcal{Q})}\cap\mathcal{R}\big|_{(\mathrm{R}_0,\mathrm{T}_\mathcal{R})}=\emptyset$; then we will have $\mathrm{P}_1\in\mathcal{Q}\big|_{(\mathrm{Q}_0,\mathrm{T}_\mathcal{Q})}\cap\mathcal{R}\big|_{(\mathrm{T}_\mathcal{R},\mathrm{R}_r)}$, $\mathrm{P}_2\in\mathcal{Q}\big|_{(\mathrm{T}_\mathcal{Q},\mathrm{Q}_q)}\cap\mathcal{R}\big|_{(\mathrm{T}_\mathcal{R},\mathrm{R}_r)}$ and $\mathrm{P}_3\in\mathcal{R}\big|_{(\mathrm{T}_\mathcal{Q},\mathrm{Q}_q)}\cap\mathcal{R}\big|_{(\mathrm{R}_0,\mathrm{T}_\mathcal{R})}$, situation presented in Figure 10 and that immediately leads to the same contradiction as the first case.

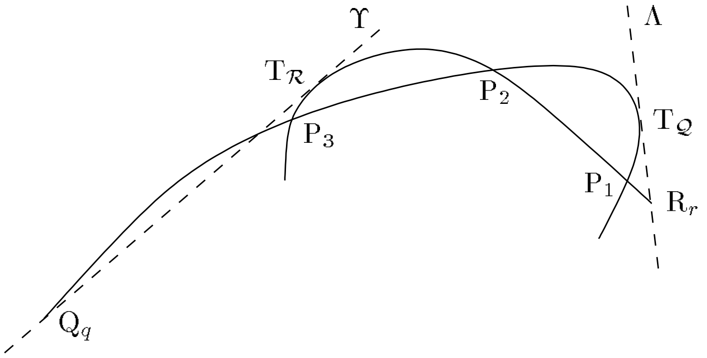


Figure 10. *Relative position of the two curves in the second case.*

To conclude, $\operatorname{card}(\mathcal{Q}\cap\mathcal{R})$ must be less than 3 and as it has to be odd, we obtain that it is equal to 1.

□

4.2. Second condition: simple triangles

The conditions we have given in the previous subsection are not sufficient for the equivalence of $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$. This affirmation is illustrated by the following figure, where conditions (2) and (3) are met for any pair of the set $\{\mathcal{Q},\mathcal{R},\mathcal{S}\}$ and hypothesis (H3.ii) is not fulfilled. Indeed, $\mathcal{Q}$ crosses $\mathcal{R}$ first and then $\mathcal{S}$, while $\mathcal{P}(\mathcal{Q})$ crosses $\mathcal{P}(\mathcal{S})$ first and only afterwards $\mathcal{P}(\mathcal{R})$ ($\mathcal{Q}$ and $\mathcal{P}(\mathcal{Q})$ are went through top-down). Of course, hypotheses (H1), (H2) and (H3) are only sufficient for the equivalence of the two arrangements and not necessary – besides, we can see that for the case of Figure 11 $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ are equivalent –, but as they are the criterion of equivalence we use, we are looking for conditions assuring that they are fulfilled.

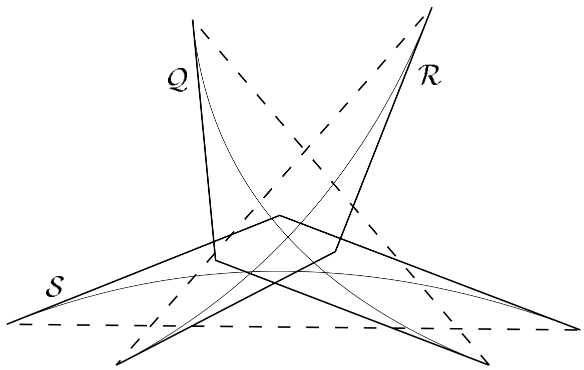


Figure 11. *The conditions of the previous subsection are fulfilled, but (H3.ii) is not.*

This kind of situation can arise whenever a *simple triangle* exists in the arrangement:

Definition 4.2. – *We will call simple triangle of $\mathcal{A}(\mathbf{B})$ any triple of Bézier curves $(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ satisfying*

$$\widetilde{\mathcal{Q}} \cap \widetilde{\mathcal{R}} \neq \emptyset \qquad \widetilde{\mathcal{R}} \cap \widetilde{\mathcal{S}} \neq \emptyset \qquad \widetilde{\mathcal{S}} \cap \widetilde{\mathcal{Q}} \neq \emptyset$$

where no two of $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ lie on the same polycurve. The simple triangle will be denoted by $\mathcal{T}_{\mathbf{B}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$.

Similarly, we define a simple triangle of $\mathcal{A}(\mathbf{P})$ and a simple triangle of $\mathcal{A}(\mathbf{C})$. If conditions (2)[$\mathcal{Q}, \mathcal{R}$] and (3)[$\mathcal{Q}, \mathcal{R}$] are satisfied for any two Bézier curves $\mathcal{Q}$ and $\mathcal{R}$ which lie on different polycurves and have nonempty intersection, any triple that is a simple triangle of $\mathcal{A}(\mathbf{B})$ is also a simple triangle of $\mathcal{A}(\mathbf{P})$ and a simple triangle of $\mathcal{A}(\mathbf{C})$. It is obvious that a simple triangle of an arrangement does not always define a unique curved triangle, and even when it does this is generally not a face of the arrangement.

To express the second condition assuring, together with (2) and (3), the equivalence of the arrangements $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$, we need to define the *equivalence of simple triangles*:

Definition 4.3. – *Let $\mathcal{T}_{\mathbf{B}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ be a simple triangle of $\mathcal{A}(\mathbf{B})$. We shall say that the simple triangle $\mathcal{T}_{\mathbf{P}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ (corresponding to it in $\mathcal{A}(\mathbf{P})$) is equivalent to it if and only if the order of the points $\widetilde{\mathcal{Q}} \cap \widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{Q}} \cap \widetilde{\mathcal{S}}$ on $\widetilde{\mathcal{Q}}$ and the order of the points $\mathcal{P}(\widetilde{\mathcal{Q}}) \cap \mathcal{P}(\widetilde{\mathcal{R}})$ and $\mathcal{P}(\widetilde{\mathcal{Q}}) \cap \mathcal{P}(\widetilde{\mathcal{S}})$ on $\mathcal{P}(\widetilde{\mathcal{Q}})$ are identical, and the same condition holds for any permutation of $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$.*

In this definition, we deal with elements of the arrangement $\mathcal{A}(\mathbf{B})$. As operations on Bézier curves are forbidden, we search for a "polygonal" criterion of equivalence between $\mathcal{T}_{\mathbf{B}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ and $\mathcal{T}_{\mathbf{P}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$. We study here the case $\mathrm{card}(\mathcal{Q} \cap \mathcal{R}) = \mathrm{card}(\mathcal{R} \cap \mathcal{S}) = \mathrm{card}(\mathcal{S} \cap \mathcal{Q}) = 1$, the other cases can be treated similarly.

In this case, the three curves determine a unique curved triangle. If we look at the position of the "convex side" of the curves regarding the interior of this curved triangle, there are four types of such triangles that can appear, as we can see in Figure 12.

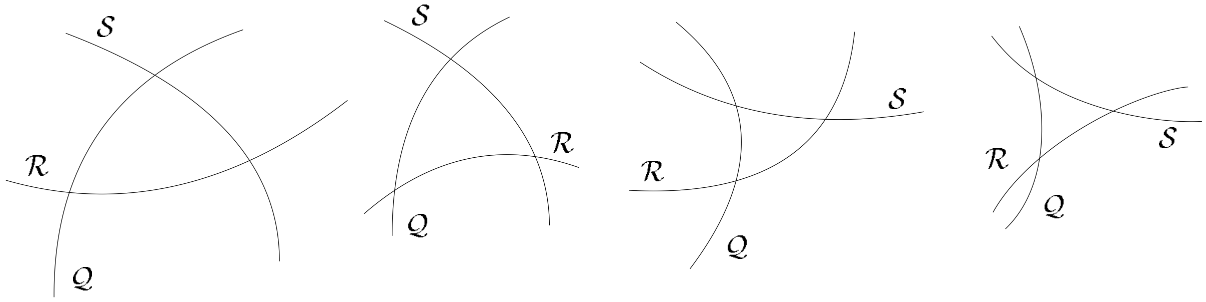


Figure 12. *The four possible types of curved triangles: 3-convex, 2-convex-1-concave, 1-convex-2-concave and 3-concave.*

It is easy to see that if the curved triangle defined by the three curves is 3-convex or 3-concave, then $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ is equivalent either to $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ or to $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$. It suffices then to test the equivalence of $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ and $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ in order to decide whether $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ and $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ are equivalent. On the other hand, the equivalence of $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ and $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ does not guarantee the equivalence of $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ and $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ for the two other possible types of curved triangles, as we can see in the following figure: If $\mathcal{Q}$, $\mathcal{P}(\mathcal{Q})$, and $\mathcal{C}(\mathcal{Q})$ are went through from left to right, then $\mathcal{P}(\mathcal{Q})$ crosses $\mathcal{P}(\mathcal{R})$ first and then $\mathcal{P}(\mathcal{S})$, $\mathcal{C}(\mathcal{Q})$ crosses $\mathcal{C}(\mathcal{R})$ first and then $\mathcal{C}(\mathcal{S})$, whereas $\mathcal{Q}$ crosses $\mathcal{S}$ first and only then $\mathcal{R}$ (and similarly for the permutations of $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$).

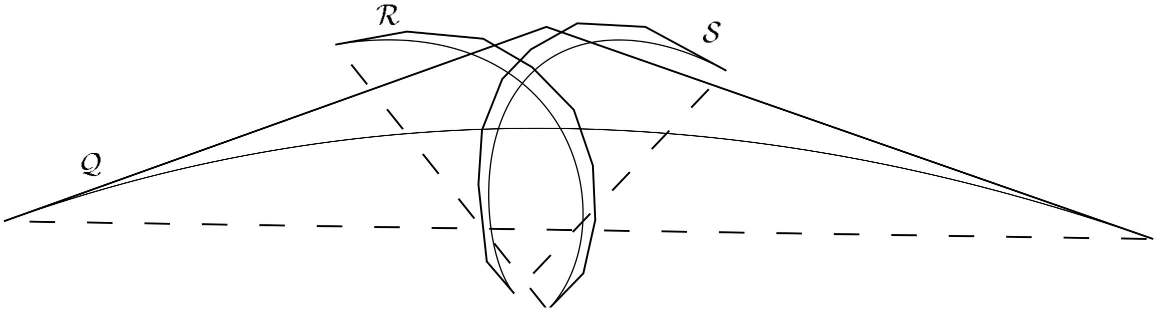


Figure 13. $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ *is equivalent to* $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$*, but it is not equivalent to* $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$*.*

To cope with this situation, we must subdivide the Bézier curves $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ until their intersection points are separated; that is, if we obtain the expressions $\mathcal{Q} = \cup\mathcal{Q}_i$, $\mathcal{R} = \cup\mathcal{R}_j$, and $\mathcal{S} = \cup\mathcal{S}_k$, then $\mathcal{Q}\cap\mathcal{R}$ and $\mathcal{Q}\cap\mathcal{S}$ must not belong to the same $\mathcal{Q}_i$, and the same condition must hold for any permutation of $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$. There exist very simple and intuitive polygonal criteria to decide whether the intersection points of the three curves are separated. Indeed, any pair of $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ satisfies conditions (2), thus any pair $(\mathcal{Q}_i,\mathcal{R}_j)$, $(\mathcal{R}_j,\mathcal{S}_k)$, or $(\mathcal{S}_k,\mathcal{Q}_i)$ either is disjoint or satisfies conditions (2).

Lemma 4.4. – *Let* $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ *be a simple triangle of* $\mathcal{A}(\mathbf{B})$*. If* $\mathcal{Q}$*,* $\mathcal{R}$*, and* $\mathcal{S}$ *are expressed such that their intersection points are separated, then* $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$*,* $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$*, and* $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ *are equivalent.*

Proof – Let us suppose that the current expression of $\mathcal{Q}$ is $\mathcal{Q} = \cup\mathcal{Q}_i$. Then this lemma is an immediate consequence of the fact that the order of the curves $\mathcal{Q}_i$ in the expression of $\mathcal{Q}$ is identical to the order of $\mathcal{P}(\mathcal{Q}_i)$ in the expression of $\mathcal{P}(\mathcal{Q})$ and to the order of $\mathcal{C}(\mathcal{Q}_i)$ in the expression of $\mathcal{C}(\mathcal{Q})$.

□

4.3. Sufficiency of the two conditions

As we have announced at the beginning of this section, we state that the conditions given in the previous two subsections, which are based uniquely on computations on polygonal lines, are sufficient for the equivalence of $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$.

Proposition 4.5. – *Let $\mathcal{A}(\mathbf{B})$ be a simple arrangement such that the conditions $(2)[\mathcal{Q},\mathcal{R}]$ and $(3)[\mathcal{Q},\mathcal{R}]$ are fulfilled for any Bézier curves $\mathcal{Q}$ and $\mathcal{R}$ which lie on different polycurves and have nonempty intersection. Let us suppose that for any simple triangle $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ of $\mathcal{A}(\mathbf{B})$ the intersections of the curves $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ are separated. Then the arrangements $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ are equivalent.*

Proof– We remark that if $(3)[\mathcal{Q},\mathcal{R}]$ are fulfilled for any $\mathcal{Q}$, $\mathcal{R}$ lying on different polycurves, then $(3)[\mathcal{R},\mathcal{Q}]$ are also fulfilled. By hypothesis, the conditions of Lemma 4.4 are fulfilled and thus for any simple triangle $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ of $\mathcal{A}(\mathbf{B})$, the simple triangles $\mathcal{T}_{\mathbf{B}}(\mathcal{Q},\mathcal{R},\mathcal{S})$, $\mathcal{T}_{\mathbf{P}}(\mathcal{Q},\mathcal{R},\mathcal{S})$, and $\mathcal{T}_{\mathbf{C}}(\mathcal{Q},\mathcal{R},\mathcal{S})$ are equivalent.

To prove this proposition, we want to use Proposition 3.5 and therefore we show that hypotheses (H1), (H2) and (H3) are met.

Hypothesis (H1) immediately results from the equivalence of simple triangles and the simplicity of $\mathcal{A}(\mathbf{B})$. The fact that $(2)[\mathcal{Q},\mathcal{R}]$ is satisfied for any Bézier curves $\mathcal{Q}$ and $\mathcal{R}$ which lie on different polycurves and have nonempty intersection implies hypothesis (H2). Then it remains to prove the existence of the bijection described by (H3).

As $(2)[\mathcal{Q},\mathcal{R}]$ is met for any intersecting Bézier curves $\mathcal{Q}$, $\mathcal{R}$ lying on different polycurves, the function φ_0 that associates to the vertex $\mathcal{Q}\cap\mathcal{R}$ of $\mathcal{A}(\mathbf{B})$ the vertex $\mathcal{P}(\mathcal{Q})\cap\mathcal{P}(\mathcal{R})$ of $\mathcal{A}(\mathbf{P})$ is a bijection. We have to study if it satisfies hypothesis (H3.ii).

If we suppose that this condition is not met, there exist $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{R}$, and $\mathcal{S}$ such that:
- $\mathcal{Q}$ and $\mathcal{Q}'$ lie on the same polycurve of $\mathbf{B}$, denoted $\mathcal{B}^{(k)}$;
- the order of $\mathcal{Q}\cap\mathcal{R}$ and $\mathcal{Q}'\cap\mathcal{S}$ on $\mathcal{B}^{(k)}$ is not the same as the order of $\mathcal{P}(\mathcal{Q})\cap\mathcal{P}(\mathcal{R})$ and $\mathcal{P}(\mathcal{Q}')\cap\mathcal{P}(\mathcal{S})$ on $\mathcal{P}(\mathcal{B}^{(k)})$.

But this implies that the subscript of $\mathcal{Q}$ in the composition of $\mathcal{B}^{(k)}$ is less than or equal to the subscript of $\mathcal{Q}'$ in the composition of $\mathcal{B}^{(k)}$, and the subscript of $\mathcal{P}(\mathcal{Q}')$ in the composition of $\mathcal{P}(\mathcal{B}^{(k)})$ is less than or equal to the subscript of $\mathcal{P}(\mathcal{Q})$ in the composition of $\mathcal{P}(\mathcal{B}^{(k)})$. Thus we have $\mathcal{Q}=\mathcal{Q}'$. On the other hand, $\mathcal{R}$ and $\mathcal{S}$ cannot be disjoint: if they were, $\mathcal{P}(\mathcal{R})$ and $\mathcal{P}(\mathcal{S})$ were also disjoint and therefore separable by a line. The supposition we made on the order of vertices on $\mathcal{B}^{(k)}$ and $\mathcal{P}(\mathcal{B}^{(k)})$ would then imply that the piece of $\mathcal{Q}$ delimited by these two vertices crosses this separation line from left to right while the corresponding piece of $\mathcal{P}(\mathcal{Q})$ crosses it from right to left, which is not possible in the case of completely convex Bézier curves.

But if $\mathcal{R}$ and $\mathcal{S}$ are not disjoint, then $\mathcal{Q}=\mathcal{Q}'$, $\mathcal{R}$, and $\mathcal{S}$ define a simple triangle of $\mathcal{A}(\mathbf{B})$. Consequently, the change of order of the two vertices contradicts the equivalence of the simple triangles of the two arrangements, and the proof of the proposition is thus achieved.

□

An important advantage of the sufficient conditions we have found is that in fact the method based on them provides not one polygonal arrangement equivalent to $\mathcal{A}(\mathbf{B})$, the one of control polygons, but two:

Corollary 4.6. – *In the conditions of the previous proposition, the arrangement $\mathcal{A}(\mathbf{C})$ is* *also equivalent to* $\mathcal{A}(\mathbf{B})$.

5. The algorithm

In this section, we briefly present an algorithm computing an arrangement of control polygons equivalent to the one of polycurves, based on the sufficient conditions of equivalence given in Section 4.

5.1. Main subroutines

Let us start by describing the algorithmic transcription of the conditions introduced in Subsections 4.1 and 4.2.

Let $\widetilde{\mathcal{Q}} = \cup_1^q \mathcal{Q}_k$ and $\widetilde{\mathcal{R}} = \cup_1^r \mathcal{R}_k$ be two completely convex Bézier curves in a current expression. We are interested here in the intersection of the Bézier curves $\mathcal{Q}_i$ and $\mathcal{R}_j$.

Subroutine verify_intersections
Input: Two Bézier curves, $\mathcal{Q}_i$ and $\mathcal{R}_j$, and the corresponding unions of Bézier curves $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{R}}$.
Output: A boolean value, `OK`.
step 1. if $\overline{reg\big(\mathcal{P}(\mathcal{Q}_i), \mathcal{C}(\mathcal{Q}_i)\big)} \cap \overline{reg\big(\mathcal{P}(\mathcal{R}_j), \mathcal{C}(\mathcal{R}_j)\big)} = \emptyset$, let `OK` be `true` and return;
step 2. if $(2)[\mathcal{Q}_i, \mathcal{R}_j]$, $(3)[\mathcal{Q}_i, \mathcal{R}_j]$ and $(3)[\mathcal{R}_j, \mathcal{Q}_i]$ are met, let `OK` be `true` and return;
step 3. if there exist $a_1 < b_1 \in \{1, \ldots, q\}$ with $a_1 \leq i \leq b_1$ such that $(2)[\cup_{a_1}^{b_1} \mathcal{Q}_k, \mathcal{R}_j]$, $(3)[\cup_{a_1}^{b_1} \mathcal{Q}_k, \mathcal{R}_j]$ and $(3)[\mathcal{R}_j, \cup_{a_1}^{b_1} \mathcal{Q}_k]$ are met, let `OK` be `true` and return;
step 4. if there exist $a_2 < b_2 \in \{1, \ldots, r\}$ with $a_2 \leq j \leq b_2$ such that $(2)[\mathcal{Q}_i, \cup_{a_2}^{b_2} \mathcal{R}_k]$, $(3)[\mathcal{Q}_i, \cup_{a_2}^{b_2} \mathcal{R}_k]$ and $(3)[\cup_{a_2}^{b_2} \mathcal{R}_k, \mathcal{Q}_i]$ are met, let `OK` be `true` and return;
step 5. let `OK` be `false` and return.

This routine decides, for two Bézier curves seen as parts of two Bézier curves of the initial composition of the arrangement, whether the cardinality of their intersection can be computed. We remark that conditions (2) and (3) are not used in the way given in Proposition 4.1, but in a more general sense. It is still very easy to see that if the routine **verify_intersections** sets the boolean value `OK` to `true`, then we have

$$\operatorname{card}\left(\left(\bigcup_{k=a_1}^{b_1} \mathcal{Q}_k\right) \cap \mathcal{R}_j\right) = 1 \qquad \text{or} \qquad \operatorname{card}\left(\mathcal{Q}_i \cap \left(\bigcup_{k=a_2}^{b_2} \mathcal{R}_k\right)\right) = 1$$

This result is weaker than the one of Proposition 4.1, but it is sufficient for our algorithm. This slight extension of the required conditions for the intersection of two Bézier curves has been done to assure the convergence of the intersection searching process.

The following routine transcribes the condition of Lemma 4.4.

Subroutine simple_triangles
Input: For three Bézier curves of the current expression of the arrangement, $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$, the current composition of the initial curves from which they are obtained, $\widetilde{\mathcal{Q}} = \cup \mathcal{Q}_i$, $\widetilde{\mathcal{R}} = \cup \mathcal{R}_j$, and $\widetilde{\mathcal{S}} = \cup \mathcal{S}_k$.
Output: A new current composition of $\widetilde{\mathcal{Q}}$, $\widetilde{\mathcal{R}}$, and $\widetilde{\mathcal{S}}$ such that $\mathcal{T}_{\mathbf{B}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$, $\mathcal{T}_{\mathbf{P}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ and $\mathcal{T}_{\mathbf{C}}(\mathcal{Q}, \mathcal{R}, \mathcal{S})$ are equivalent.

step 1. make the list $\mathcal{LP}_1$ of the pairs (i,j) such that $reg\,(\mathcal{P}(\mathcal{Q}_i),\mathcal{C}(\mathcal{Q}_i)) \cap reg\,(\mathcal{P}(\mathcal{R}_j),\mathcal{C}(\mathcal{R}_j)) \neq \emptyset$, and similarly for $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{S}}$ and respectively $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{Q}}$ (the lists $\mathcal{LP}_2$ and $\mathcal{LP}_3$);
step 2. make the list $\mathcal{LT}$ of triples (i,j,k) such that (i,j) is in $\mathcal{LP}_1$ and (j,k) is in $\mathcal{LP}_2$, or (j,k) is in $\mathcal{LP}_2$ and (k,i) is in $\mathcal{LP}_3$, or (k,i) is in $\mathcal{LP}_3$ and (i,j) is in $\mathcal{LP}_1$;
step 3. if $\mathcal{LT}$ is not empty, subdivide $\mathcal{Q}_i$, $\mathcal{R}_j$, and $\mathcal{S}_k$ for all (i,j,k) in $\mathcal{LT}$, update $\mathcal{LP}_1$, $\mathcal{LP}_2$, and $\mathcal{LP}_3$, and go to step 2; if $\mathcal{LT}$ is empty, return.

Lemma 5.1. – *The subroutine* **simple_triangles** *converges.*

Proof – Straightforward. □

5.2. Description of the algorithm

Using the subroutines described in the previous subsection, the algorithm that we propose for computing an arrangement of control polygons (and implicitly an arrangement of carrier polygons) equivalent to $\mathcal{A}(\mathbf{B})$ has the following structure:

Algorithm
Input: A set of polycurves $\mathbf{B} = \{\mathcal{B}^{(k)}\}_{1\leq k\leq n}$ in their initial form.
Output: The set $\mathbf{B}$ of polycurves in a current expression such that $\mathcal{A}(\mathbf{B})$ and $\mathcal{A}(\mathbf{P})$ (and $A(\mathbf{C})$) are equivalent.
step 1. make the list $\mathcal{LPC}$ of all the pairs of Bézier curves $(\mathcal{Q},\mathcal{R})$ not lying on the same polycurve and such that the axis-parallel embedding rectangles of $\mathcal{P}(\mathcal{Q})$ and $\mathcal{P}(\mathcal{R})$ are not disjoint;
step 2. while $\mathcal{LPC}$ is not empty: for every element $(\mathcal{Q},\mathcal{R})$ of $\mathcal{LPC}$, execute **verify_intersections**; if `OK` is `true`, erase $(\mathcal{Q},\mathcal{R})$ from $\mathcal{LPC}$; if `OK` is `false`, subdivide $\mathcal{Q}$ and $\mathcal{R}$ and replace in $\mathcal{LPC}$ $(\mathcal{Q},\mathcal{R})$ by the four pairs of curves resulted by subdivision;
step 3. make the list $\mathcal{T}$ of all the simple triangles of $\mathcal{A}(\mathbf{P})$ and execute **simple_triangles** for each of them.

The first step of the algorithm is meant to reduce the number of calls to the routine **verify_intersections**, because testing the intersection of the two embedding rectangles is cheaper than the execution of this subroutine. The correctness of the algorithm needs no more to be proved, the results of Section 4 assure that in the final expression the arrangement of control polygons, the arrangement of carrier polygons, and the arrangement of polycurves are equivalent. Together with Lemma 5.1, the following result assures also the convergence of the algorithm:

Lemma 5.2. – *Step 2 of the algorithm converges.*

Proof – To prove the convergence of step 2, we first show that for two completely convex Bézier curves $\mathcal{Q}$ and $\mathcal{R}$ crossing transversely in one point P we can obtain by subdivision the expressions $\mathcal{Q} = \cup_1^{\mu}\mathcal{Q}_\eta$ and $\mathcal{R} = \cup_1^{\nu}\mathcal{R}_\eta$, for which there exists $\kappa_1 \leq \kappa_2 \in \{1,\ldots,\mu\}$ and $\lambda_1 \leq \lambda_2 \in \{1,\ldots,\nu\}$ such that $(2)[\cup_{\kappa_1}^{\kappa_2}\mathcal{Q}_\eta, \cup_{\lambda_1}^{\lambda_2}\mathcal{R}_\eta]$, $(3)[\cup_{\kappa_1}^{\kappa_2}\mathcal{Q}_\eta, \cup_{\lambda_1}^{\lambda_2}\mathcal{R}_\eta]$, and $(3)[\cup_{\lambda_1}^{\lambda_2}\mathcal{R}_\eta, \cup_{\kappa_1}^{\kappa_2}\mathcal{Q}_\eta]$ are satisfied. We suppose that the relative positions of the endpoints of $\mathcal{Q}$ and $\mathcal{R}$ are those presented in Figure 14; the three other cases can be discussed in the same way.

This proof is up to a point similar to the one of Lemma 3.6. So, we have seen that it is possible to obtain by subdivision expressions of the two curves such that $(2)[\cup_{\kappa_1}^{\kappa_2}\mathcal{Q}_\eta, \cup_{\lambda_1}^{\lambda_2}\mathcal{R}_\eta]$

is fulfilled and we must study relations (3). We are again using the common supporting line Δ, the unique one satisfying

$$\Delta \cap \mathcal{Q} \subset \mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{R}), \mathcal{R}) \quad \text{and} \quad \Delta \cap \mathcal{R} \subset \mathrm{HP}_{opp}(\tan_{\mathrm{P}}(\mathcal{Q}), \mathcal{Q})$$

(we recall that $\tan_{\mathrm{P}}(\mathcal{Q})$ is the tangent to $\mathcal{Q}$ at $\mathrm{P} \in \mathcal{Q}$ and similarly for $\mathcal{R}$) and we make the notations $\{\mathrm{T}_{\mathcal{Q}}\} = \Delta \cap \mathcal{Q}$, $\{\mathrm{T}_{\mathcal{R}}\} = \Delta \cap \mathcal{R}$. Let A_1 be the midpoint of the arc $\mathcal{Q}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{Q}})}$ and A_2 be the midpoint of $\mathcal{R}\big|_{(\mathrm{P},\mathrm{T}_{\mathcal{R}})}$. We take $\{\mathrm{P}_1\} = \tan_{\mathrm{A}_2}(\mathcal{R}) \cap \mathcal{Q}$, $\{\mathrm{P}_2\} = \tan_{\mathrm{A}_1}(\mathcal{Q}) \cap \mathcal{R}$, and $\{\mathrm{P}_3\} = \tan_{\mathrm{A}_1}(\mathcal{Q}) \cap \tan_{\mathrm{A}_2}(\mathcal{R})$, as shown in Figure 14.

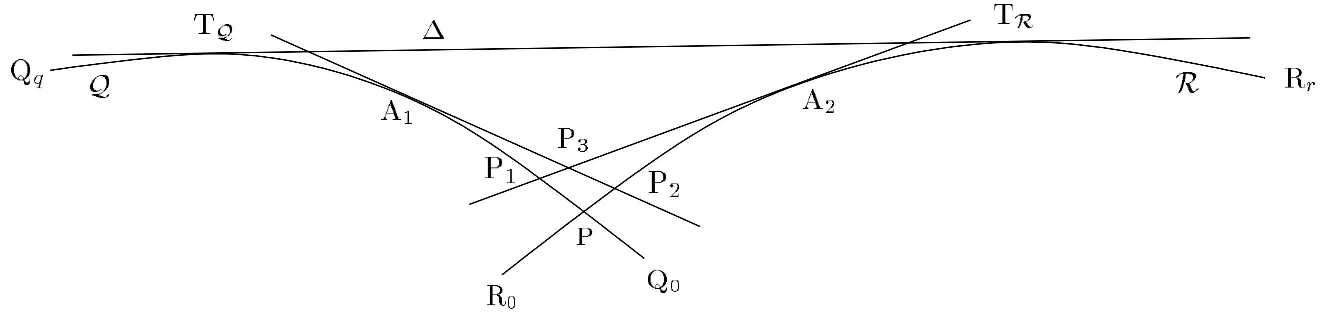


Figure 14. *Any other relative position of the endpoints of $\mathcal{Q}$ and $\mathcal{R}$ is discussed similarly.*

We subdivide $\mathcal{Q}$ until an endpoint of one of the Bézier curves composing it lies on $\mathcal{Q}\big|_{(\mathrm{PA}_1)}$, and similarly for $\mathcal{R}$. Let us denote by $\Diamond\mathrm{PP}_1\mathrm{P}_3\mathrm{P}_2$ the curved rhombus delimited by $\mathcal{Q}$, $\mathcal{R}$, and the lines $\mathrm{P}_1\mathrm{P}_3$ and $\mathrm{P}_2\mathrm{P}_3$. Then we have $\mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R}) \subset \Diamond\mathrm{PP}_1\mathrm{P}_3\mathrm{P}_2$. If we take $\mathrm{Q}_i, \mathrm{Q}_{i+1}, \mathrm{R}_j$, and R_{j+1} the vertices of $\mathcal{P}(\mathcal{Q})$ and respectively $\mathcal{P}(\mathcal{R})$ that satisfy $[\mathrm{Q}_i\mathrm{Q}_{i+1}] \cap [\mathrm{R}_j\mathrm{R}_{j+1}] = \mathcal{P}(\mathcal{Q}) \cap \mathcal{P}(\mathcal{R})$, we obtain

$$\mathcal{Q}\big|_{(\mathrm{P}_1,\mathrm{A}_1)} \subset \mathrm{HP}_{opp}(\mathrm{R}_j\mathrm{R}_{j+1}, \mathcal{R}) \qquad \text{and} \qquad \mathcal{R}\big|_{(\mathrm{P}_2,\mathrm{A}_2)} \subset \mathrm{HP}_{opp}(\mathrm{Q}_i\mathrm{Q}_{i+1}, \mathcal{Q})$$

To finish the demonstration of the statement we made at the beginning of the proof of this lemma, it is now enough to take $\kappa_1 = 1$ and $\lambda_1 = 1$.

The proof of the lemma will be achieved by remarking that we cannot have simultaneously an endpoint of one of the Bézier curves currently composing $\mathcal{Q}$ belonging to a region $reg\big(\mathcal{P}(\mathcal{R}_\beta), \mathcal{C}(\mathcal{R}_\beta)\big)$ and an endpoint of one of the Bézier curves currently composing $\mathcal{R}$ belonging to a region $reg\big(\mathcal{P}(\mathcal{Q}_\alpha), \mathcal{C}(\mathcal{Q}_\alpha)\big)$, where $\mathcal{Q}_\alpha$ (respectively $\mathcal{R}_\beta$) is a Bézier curve from the current composition of $\mathcal{Q}$ (respectively $\mathcal{R}$). For this reason, we can take in the formulae above either $\kappa_2 = \kappa_1$ or $\lambda_2 = \lambda_1$ and the convergence of step 2 of the algorithm is proven.

□

For clarity, we have presented here an algorithm that works on simple arrangements of polycurves. The algorithm we have implemented calls a supplementary subroutine to prevent infinite subdivision when the arrangement is degenerate. After an user-given number of subdivisions, this routine tests if the arrangement is degenerate up to a user-given tolerance. If it is not, the necessary subdivisions are performed. If it is, the algorithm points out the pairs or the triples of Bézier curves that are in a degenerate position (up to the given tolerance). The test of degeneracy is based on the Hausdorff distance between the control and the carrier polygons.

As we have announced in the introduction, no complete theoretical study of complexity can be done. Indeed, we cannot estimate the number of iterations needed in step 2 as a function of the size of the arrangement. We can intuitively see in the proof of Lemma 5.2

that the smaller is the angle between two intersecting curves (the closer they are to admit a common tangent at their common point), the larger can be the number of subdivisions the two curves have to pass through to fulfill conditions (2) and (3). So, to evaluate the number of iterations, a "form factor" of the arrangement should be taken into account. Moreover, the number of subdivisions required for a given simple triangle in step 3 is also impossible to evaluate, for the same reasons. Yet, an experimental evaluation of the cost of this algorithm is presented in the following section.

6. Numerical results and examples

To test the proposed algorithm, we run its implementation on a representative sample of sets of polycurves, that in the following we will call (abusively) arrangements. Each such arrangement has in its composition about 100 Bézier curves. So, there are arrangements of 100 polycurves of 1 Bézier curve each, arrangements of 50 polycurves of 2 Bézier curves each, ..., arrangements of 10 polycurves of 10 Bézier curves each. The average number of polycurves per arrangement is 33.5 and the average degree of the Bézier curves is about 3.5, which means that in the initial form the average number of control points per arrangement is about 350. We remark that the average degree (that remains almost unchanged after the processing of the arrangement) represents the ratio between the average size of the control polygon and the average size of the carrier polygon, showing the relevance of Corollary 4.6.

Now let us present the values, related to the final form of the arrangements, that we consider significant for the complexity of the algorithm:

- N, the total number of Bézier curves appearing in the (final) expression of the arrangement; this value also represents the total size of the carrier polygons.
- K_1, the number of iterations performed in step 2 of the algorithm;
- τ, the number of simple triangles found in the arrangement;
- τ_n, the number of simple triangles not satisfying the equivalence conditions;
- K_2, the number of subdivisions needed to make all the simple triangles equivalent.

The following table gives the average and maximal values of all these parameters.

	N	K_1	τ	τ_n	K_2
Average value	1202.38	6.69	5287.7	77.15	1.285
Maximal value	3296	10	47913	278	10

If we suppose that all the Bézier curves appearing in an arrangement are subdivided at each of the iterations performed, the growth of the number of Bézier curves should be 2 power the average value of K_1 plus the average value of K_2. For our sample, the theoretical growth is about 256. The table of numerical results shows us that in practice this value is about 12, which represents a reasonable growth ratio.

The maximal possible number of simple triangles (that is, triples of curves each pair of which have a nonempty intersection) in an arrangement containing N_0 Bézier curves is $\mathcal{O}(N_0^3)$. In our case, this represents a six digits number, one order of magnitude more than the worst case we found and two orders of magnitude more than our average result. Moreover, only about 1/70 of the simple triangles detected do not satisfy the equivalence conditions, and for most of them one subdivision is enough to solve this problem.

The maximal number of subdivisions applied to the Bézier curves of the whole sample is 16 (that means of course that the maximal K_1 and the maximal K_2 do not concern the same curve). Studying the proportion of Bézier curves having passed through i subdivisions, for $i = 1, \ldots, 16$, we have obtained the following graph (the y-axis represents the percents of the total number of curves):

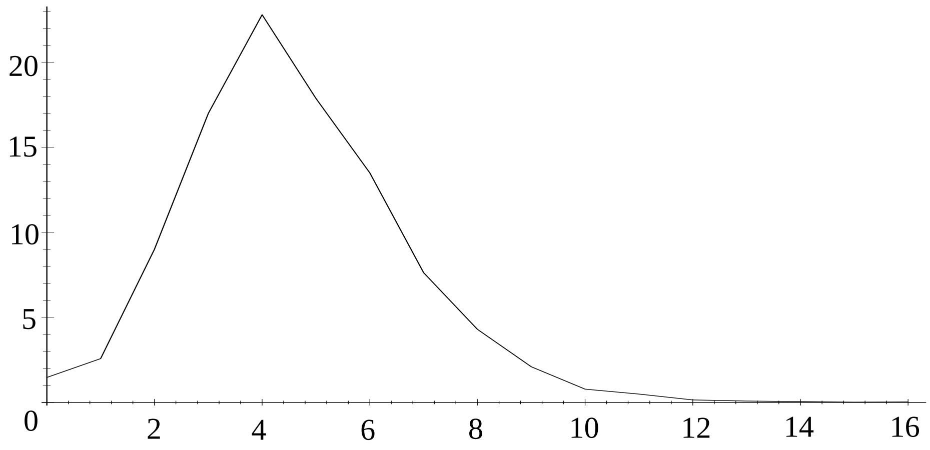


Figure 15. *Repartition of the Bézier curves on the number of applied subdivisions.*

The general shape of this graph is, as expected, the one of a Gaussian. The maximal value is 22.8% and it corresponds to 4 iterations. The values corresponding to 13 or more subdivisions are less than 0.1%.

We can also remark that the average number of control points in the final arrangement of control polygons is 4208.6, while the average size of the carrier polygons arrangements is 1232.4. As we specified, the average degree of the Bézier curves, representing the ratio of these two values, remains about 3.5. So, the use of the arrangement of carrier polygons simplifies the computation of the incidence graph.

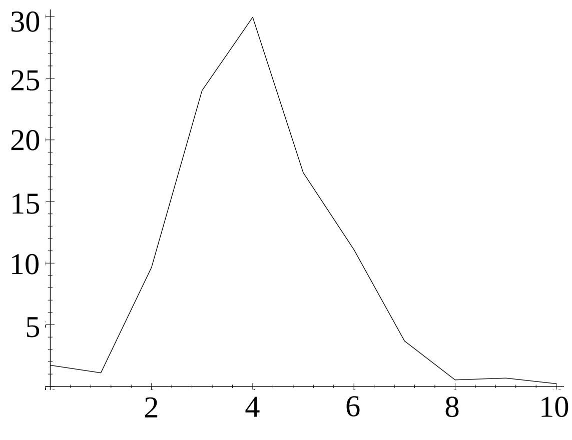
To take a better look at the complexity of the algorithm, let us present the numerical results that we have obtained for some subsamples of arrangements. For representativity, we have chosen the subsamples consisting respectively in arrangements of 10 polycurves of 10 Bézier curves each, in arrangements of 20 polycurves of 5 Bézier curves each, and in arrangements of 100 polycurves of 1 Bézier curve each. The following table gives the maximal and average values of the parametres described above for these subsamples.

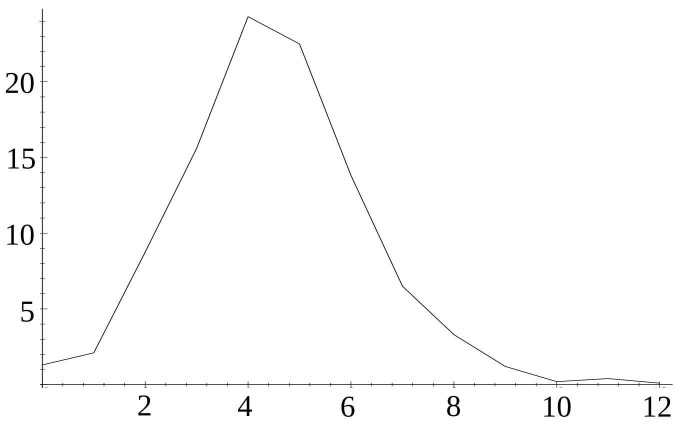
		N	K_1	τ	τ_n	K_2
10x10	Average value	858.44	6	1361.56	44.9	1.31
	Maximal value	1184	7	5401	103	6
20x5	Average value	1102.7	6.6	2620.5	63.8	1.3
	Maximal value	1729	8	3824	124	6
100x1	Average value	2474.6	7.7	16709.6	179.8	1.31
	Maximal value	3296	10	47913	278	10

We can remark that the smaller is the number of Bézier curves initially composing each polycurve, the larger will be the cost of computing the required polygonal arrangements. As a matter of fact, the only subroutine which is not sensitive to the number of Bézier curves per polycurve is the subdivision subroutine, called by all the routines that we have presented.

In this table, the values of K_2 are almost identical and the variation of K_1 does not suffice to explain the variation of N. We deduce that the number of Bézier curves to be subdivided at each iteration is sensibly larger in the case of polycurves composed of one Bézier curve each with respect to the case of polycurves composed of 10 Bézier curves each. A similar remark holds for the simple triangles of the arrangements, which is illustrated by the values of τ and τ_n.

The following graphs present the repartition of Bézier curves on the number of applied subdivisions respectively for the three mentioned subsamples. For the arrangements of polycurves composed of 10 Bézier curves each, the maximal number of applied subdivisions is 10; for the arrangements of polycurves composed of 5 Bézier curves each, this value is 12; finally, the arrangements of monocurve polycurves give the maximal number of applied subdivisions of the whole sample, that is, the value we discuss is 16 in this case.





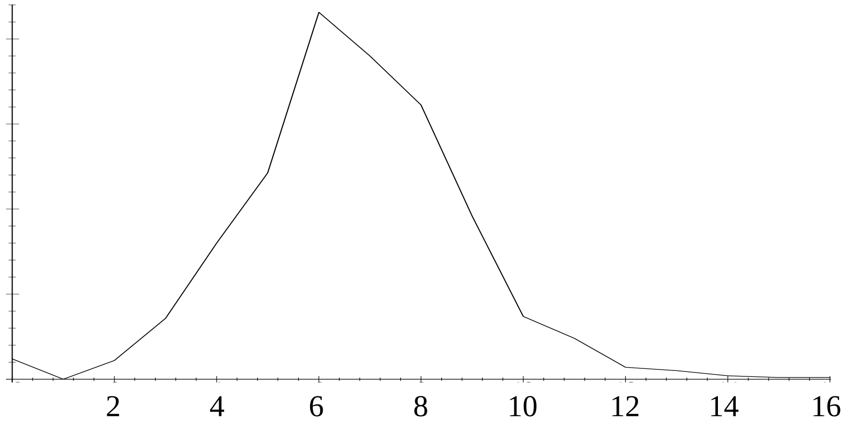


Figure 16 *Repartition of the Bézier curves on the number of applied subdivision for the mentioned subsamples.*

These three graphs are also Gaussians. The first one corresponds to the arrangements of polycurves of 10 Bézier curves each and its maximal value is equal to 29.95% and reached for 4 subdivisions (the value corresponding to 3 subdivisions is equal to 24%). The second graph is obtained for the arrangements of polycurves of 5 Bézier curves each, its maximal value is 24.35%, corresponding to 4 subdivisions, and the second maximal value is 22.46%, for 5 subdivisions. Finally, the third graph is associated to the arrangements of monocurve polycurves, its maximal value is 21.57%, achieved for 6 subdivisions, and the second maximal value is 19%, for 7 subdivisions.

We do not present here examples of the set of arrangements used for the numerical results, because the large number of Bézier curves makes it difficult to distinguish the faces of the arrangement. The following two examples are therefore simpler, and we can observe the results that the algorithm provides.

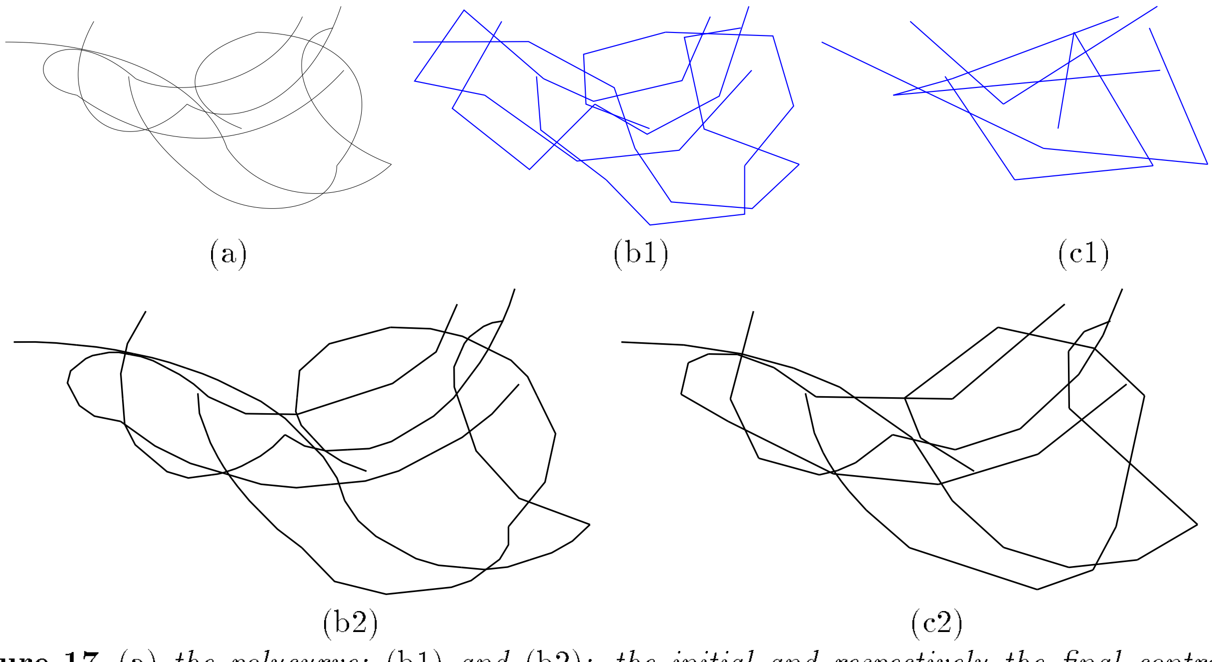
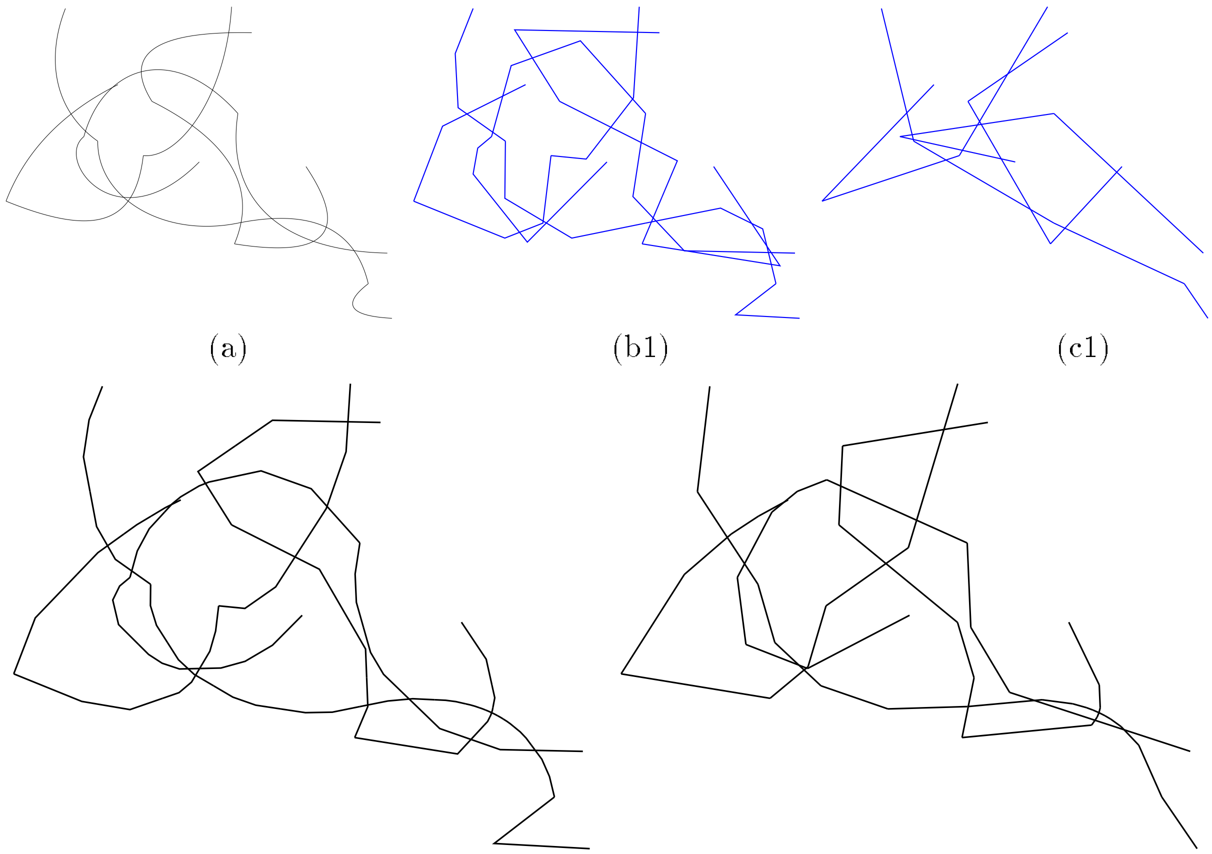


Figure 17 (a) *the polycurve;* (b1) *and* (b2)*: the initial and respectively the final control polygon;* (c1) *and* (c2)*: the initial and respectively the final carrier polygon. In step 2 of the algorithm, 4 iterations have been performed. In step 3, all the simple*

triangles have been found equivalent.



(a) (b1) (c1)

(b2) (c2)

Figure 18 (a) *the polycurve;* (b1) *and* (b2)*: the initial and respectively the final control polygon;* (c1) *and* (c2)*: the initial and respectively the final carrier polygon. In step 2 of the algorithm, 5 iterations have been performed. In step 3, one simple triangle not satisfying the conditions of equivalence has been found. Its composing curves had to be subdivided once.*

7. Conclusions

In this paper, we have presented a method for constructing an arrangement of polygonal lines equivalent to a given arrangement of composite Bézier curves, named polycurves. The novelty of the approach consists in a complete elimination of the algebraic equations solving usually required by computations on curves. Our objective has been doubly reached, for we have found not one arrangement of polygonal chains, but two, the arrangement of control polygons and the arrangement of carrier polygons.

On the one hand, these two arrangements have the same combinatorial structure as the arrangement of polycurves, and therefore make its incidence graph easier and numerically safer to compute. Indeed, computations on polygons can be considered, under certain conditions, accurate even in floating point arithmetics, supposition that is never realistic in the case of algebraic curves. Moreover, the algorithm we described needs only minor modifications to perform the computation of the incidence graph, modifications that will make the object of future work.

On the other hand, the arrangements of control and carrier polygons provide approximations, in terms of distance, for the arrangement of polycurves. So, every face of $\mathcal{A}(\mathbf{B})$, the arrangement of polycurves, has a polygonal approximation in $\mathcal{A}(\mathbf{P})$, the arrangement of control polygons, and another one in $\mathcal{A}(\mathbf{C})$, the arrangement of carrier polygons. Moreover, good properties of inclusion hold for these three corresponding faces. Thus, problems on the arrangement of polycurves (point location problems, for example) can be solved dealing uniquely with the polygonal arrangements.

As we have mentioned in the introduction, our method can be applied to arrangements that are composed of curves more general than composite Bézier curves. These curves must be simple, piecewise convex, and any convex piece must have associated with it converging series of carrier and bounding polygons which are convex. These two series of polygons can be associated with any convex Jordan arc whenever we can perform on it two types of computation: Construction of a tangent at a given point and "picking a midpoint", that is, selecting a point which splits the arc in two roughly equal parts [16].

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