Formal First Integrals Along Solutions of Differential Systems I

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ABSTRACT

We consider an analytic vector field $\dot{x} = X(x)$ and study, via a variational approach, whether it may possess analytic first integrals. We assume one solution Γ is known and we study the successive variational equations along Γ. Constructions in [MRRS07] show that Taylor expansion coefficients of first integrals appear as rational solutions of the dual linearized variational equations. We show that they also satisfy linear "filter" conditions. Using this, we adapt the algorithms from [Bar99, vHW97] to design new ones optimized to this effect and demonstrate their use. Part of this work stems from the first author's Ph.D. thesis[1] [AM10].

Categories and Subject Descriptors

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[1] https://sites.google.com/site/ainhoaaparicio/home/recherche/publications-pre-prints

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1. INTRODUCTION

Consider an analytic differential vector field

$$\dot{x} = X(x). \tag{1}$$

Let us recall that a *first integral* of (1) is a complex-valued function F defined on a domain $U \subset \mathbb{C}^n$ such that

$$D_X F = 0 \quad \text{where} \quad D_X := \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}.$$

This is equivalent to F being constant along every solution of system (1).

The existence of first integrals (meromorphic, rational, polynomial...) is relevant to the study of the integrability of complex analytic vector fields. Since the direct computation of first integrals is in general an open problem, only indirect techniques are available. Among those, the approach we suggest here is variational: assuming one solution of (1) is known, we consider the variational equations along it.Whenever the solution is an equilibrium point, a wide battery of normal form theories is available for us to characterize the behavior of (1) along the solution. No such local tools exist, though, to characterize *formal* first integrals along a *non-equilibrium* solution ϕ.

The aim of our paper is to start filling this gap by means of an algorithmic answer to the following question. Assume the vector field (1) has a first integral which is holomorphic, at least, along a solution ϕ: how would we compute, or recover, its Taylor expansion along ϕ? We first recall that the coefficients of such an expansion are rational solutions of linear differential systems (namely, the *linearized variational equations* LVE_ϕ^m). We prove they are also solutions of linear (algebraic) systems: the filter equations (see Section 3.1). We then adapt the algorithms of [Bar99, vHW97] to design an efficient algorithm for computing such Taylor expansions of (unknown) holomorphic first integrals.

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2. BACKGROUND

2.1 Taylor Expansions

The *modulus* $|i|$ of a multi-index $i = (i_1, \dots, i_n) \in \mathbb{N}^n$ is defined as the sum of its entries. Multi-index addition is defined $(i_1, \dots, i_n) + (j_1 \dots j_n) := (i_1 + j_1, \dots, i_n + j_n)$. We use the *standard lexicographic order*, denoted by $<_{\text{lex}}$, where $(i_1, \dots, i_n) <_{\text{lex}} (j_1, \dots, j_n)$ means $i_1 = j_1, \dots, i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some $k \geq 1$.

DEFINITION 1. *Let $F : U \subset \mathbb{C}^n \to \mathbb{C}$ be a complex analytic function over the open set U. We define the* lexicographically sifted differential of F of order m *as the row vector*

$$F^{(m)}(x) := \text{lex}\left(\frac{\partial^m F}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}\right)_{i_1+\dots+i_n=m},$$

where entries are ordered as per $<_{\text{lex}}$ on multi-indices.

Let $F : U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and let $\phi : I \subset \mathbb{C} \to U$ be a parametrization of a Riemann surface $\Gamma \subset U$. Then F admits a Taylor expansion along ϕ, $F(\phi + y) = F(\phi) +$

$$\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1+\dots+i_n=m} \binom{m}{i_1, \dots, i_n} \frac{\partial^m F}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}(\phi)\, y_1^{i_1} \dots y_n^{i_n}$$

where y is a vector of n formal variables. Using symmetric powers of vectors (see [AM10], Chapter 2), a compact form is

LEMMA 2. *The Taylor expansion of F along ϕ is*

$$F(\phi + y) \quad = \quad F(\phi) + \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle F^{(m)}(\phi)\, , \text{Sym}^m y \right\rangle$$

PROOF. By construction ([AM10, Ch. 2]), the entry corresponding to multi-index $(i_1, \dots, i_n)$ in $\text{Sym}^m y$ is exactly $\binom{m}{i_1,\dots,i_n} y_1^{i_1} \dots y_n^{i_n}$. Vectors $F^{(m)}(\phi)$ and $\text{Sym}^m y$ have the same dimension

$$d_{m,n} := \binom{n+m-1}{m}$$

□

2.2 Variational equations (VE_ϕ^m)

The subject matter of this section is described in [MR99, MRRS07, AM10]. Denote by $\Phi(t, z)$ the flow of (1), $(t, z) \in \mathbb{C} \times U$. Consider the Taylor expansion of $\Phi(t, z)$ with respect to the phase variables z at the point (t, x)

$$\Phi(t, z) = \Phi(t, x) + \Phi^{(1)}(t)(z-x) + \dots + \Phi^{(m)}(t)(z-x)^m + \dots$$

where $\Phi^{(m)}(t) := \frac{1}{m!} \frac{\partial^m}{\partial x^m} \Phi(t, x)$. Let Γ denote an integral curve of (1) parametrized by $\phi(t)$.

DEFINITION 3. *The* order-m variational equations *of* (1) *along an integral curve Γ are the differential system satisfied by $(\Phi^{(1)}(t), \dots, \Phi^{(m)}(t))$ when $x \in \Gamma$.*

All variational systems are non-linear except for the first-order one, $m = 1$, which we express as $\dot{\xi}_1 = A_1 \xi_1$ where

$$A_1 := \left[\frac{\partial X_i}{\partial x_j}(\phi)\right]_{i,j} \in \text{Mat}\,(n, \boldsymbol{k}),$$

with $\boldsymbol{k} := \mathbb{C}\langle\phi\rangle = \mathbb{C}\left(\phi, \dot{\phi}, \dots\right)$. By means of tensor constructions, a linear differential system $\left(\text{LVE}_\phi^m\right)$ equivalent to $\left(\text{VE}_\phi^m\right)$ can be built for each $m \in \mathbb{N}$: we call it m^{th} *order linearized variational equation* along ϕ.

We denote $\left(\text{LVE}_\phi^m\right)$, $m \geq 2$ by $\dot{Y}_m = A_m Y_m$, where

$$A_m := \begin{bmatrix} \mathfrak{sym}^m A_1 & 0 \\ B_m & A_{m-1} \end{bmatrix} \in \text{Mat}\left(\sum_{i=1}^{m} d_{i,n}, \boldsymbol{k}\right). \qquad (2)$$

and $\mathfrak{sym}^m$ stands for the m^{th} *symmetric power in the sense of Lie algebras*, implicitly defined for any given linear system $\dot{Y} = AY$, and any fundamental matrix U thereof, by

$$\frac{d}{dt}(\text{Sym}^m U) = (\mathfrak{sym}^m A)(\text{Sym}^m U),$$

$\text{Sym}^m U$ standing for the m^{th} *symmetric power in the sense of Lie groups* of U. For a precise account on symmetric powers see [AM10, Ch. 2], or [FH91].

2.3 First integrals

2.3.1 *Junior forms*

Let ϕ be a non-constant solution of (1) and F be a first integral. We henceforth normalize F by assuming $F(\phi) = 0$. Following [Aud01], we define the *valuation* of F along ϕ as the integer $\nu \geq 1$ satisfying

$$F(\phi) = 0\,, \; \dots\,, \; F^{(\nu-1)}(\phi) = 0 \quad \text{and} \quad F^{(\nu)}(\phi) \neq 0.$$

The *junior form of F* along ϕ is then defined as

$$F^\circ(y) := \frac{1}{\nu!}\left\langle F^{(\nu)}(\phi)\, , \text{Sym}^\nu y\right\rangle;$$

$F^\circ(\phi)$ is the lowest-degree homogeneous polynomial in the Taylor expansion of F, i.e

$$F(y + \phi) = F^\circ(y) + \sum_{i=\nu+1}^{\infty} \frac{1}{i!}\left\langle F^{(i)}(\phi)\, , \text{Sym}^i y\right\rangle$$

DEFINITION 4. *A first integral F of* (1) *of valuation ν along ϕ is said to be* non-degenerate along ϕ *if $\nu = 1$ (i.e. $F^\circ(y)$ is linear) and* degenerate of order ν along ϕ *if $\nu \geq 2$.*

The following Lemmae deal with classical facts about the valuation of first integrals and junior forms.

LEMMA 5. *[Aud01] Let F_1 and F_2 be first integrals of* (1) *vanishing along ϕ and having valuations ν_1 and ν_2 respectively along ϕ. We then have*

$$(F_1 F_2)^\circ(y) = F_1^\circ(y) \cdot F_2^\circ(y)$$

and the valuation of $F_1 \cdot F_2$ along ϕ is $\nu_1 + \nu_2$.

The following result (see Chapter 2 of [AM10] for symmetric products of vectors) will be useful in the sequel:

LEMMA 6. *Let ϕ be a non-constant solution of* (1) *and let $F_1, \dots, F_k$ be holomorphic first integrals of* (1)*, non-degenerate and vanishing along ϕ. Then,*

1. $(F_1 \cdot F_2)^{(2)}(\phi) = (F_1^{(1)}(\phi) \circledS F_2^{(1)}(\phi))$,
2. $(F_1^m)^{(m)}(\phi) = \text{Sym}^m(F_1^{(1)}(\phi))$,
3. $(F_1^{m_1} \cdot \dots \cdot F_k^{m_k})^{(m_1+\dots+m_k)}(\phi) =$
   $(\text{Sym}^{m_1}(F_1^{(1)}(\phi)) \circledS \dots \circledS \text{Sym}^{m_k}(F_k^{(1)}(\phi)))$

PROOF. 1. Since F_1 and F_2 are non-degenerate, their product $F_1 \cdot F_2$ is a first integral of valuation 2 along ϕ. Now $\frac{\partial^2 (F_1 \cdot F_2)}{\partial x_{i_1} \partial x_{i_2}}(\phi)$ is equal to:

$$\frac{\partial^2 F_1}{\partial x_{i_1} \partial x_{i_2}}(\phi) \cdot F_2(\phi) + \frac{\partial F_1}{\partial x_{i_1}}(\phi) \cdot \frac{\partial F_2}{\partial x_{i_2}}(\phi) +$$

$$\frac{\partial F_1}{\partial x_{i_2}}(\phi) \cdot \frac{\partial F_2}{\partial x_{i_1}}(\phi) + \frac{\partial^2 F_2}{\partial x_{i_1} \partial x_{i_2}}(\phi) \cdot F_1(\phi).$$

Since $F_1(\phi) = F_2(\phi) = 0$, for every $i_1, i_2 = 1, \ldots, n$ we have $\left((F_1 \cdot F_2)^{(2)}(\phi) \right)_{(i_1, i_2)}$ equal to

$$C_{i_1, i_2} = \frac{\partial F_1}{\partial x_{i_1}}(\phi) \cdot \frac{\partial F_2}{\partial x_{i_2}}(\phi) + \frac{\partial F_1}{\partial x_{i_2}}(\phi) \cdot \frac{\partial F_2}{\partial x_{i_1}}(\phi).$$

Computing $(F_1^{(1)}(\phi) \circledS F_2^{(1)}(\phi))$ in the canonical base $\{e_i\}$ yields exactly

$$\left(F_1^{(1)}(\phi) \circledS F_2^{(1)}(\phi) \right)_{(i_1, i_2)} = \sum_{1 \le i_1 \le i_2 \le n} C_{i_1, i_2} e_{i_1} \cdot e_{i_2},$$

hence $(F_1 \cdot F_2)^{(2)}(\phi) = F_1^{(1)}(\phi) \circledS F_2^{(1)}(\phi)$.

2. Since F_1 is non-degenerate along ϕ, F_1^m has valuation m, hence all its partial derivatives of order less than m vanish at ϕ. Thus, the entry in $(F_1^m)^{(m)}(\phi)$ corresponding to any modulus-m multi-index of exponent $(m_1, \ldots, m_n)$ is

$$\frac{\partial^m (F_1^m)}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}}(\phi) = \binom{m}{m_1, \ldots, m_n} \prod_{j=1}^{n} \left(\frac{\partial F}{\partial x_j}(\phi) \right)^{m_j},$$

equal to the entry in $\mathrm{Sym}^m(F_1^{(1)}(\phi))$ corresponding to the same modulus-m exponents multi-index.

3. Follows from the two previous items.

□

2.3.2 Holomorphic first integrals and (LVE_ϕ^m)

Let $\boldsymbol{k} := \mathbb{C}\langle \phi \rangle$ denote our base field; if X is rational, the fact $\dot{\phi} = X(\phi)$ implies $\boldsymbol{k} = \mathbb{C}(\phi)$. Let A_m be the matrix of the order-m variational equations of (1), as written in (2).

The *dual (or adjoint)* m^{th} *order variational equation along* ϕ, denoted by $(\mathrm{LVE}_\phi^m)^\star$, is defined by $\dot{V} = A_m^\star V$ with

$$A_m^\star := -{}^t A_m \in \mathrm{Mat}\left(\sum_{i=1}^{m} d_{i,n}, \boldsymbol{k} \right).$$

LEMMA 7. *If F is a holomorphic first integral of* (1) *and ϕ is a non-constant solution of* (1) *then*

$$V_m := {}^t(F^{(m)}(\phi), \ldots, F^{(1)}(\phi)) \in \boldsymbol{k}^{\sum_{i=1}^m d_{i,n}}$$

is a (rational) solution of $(\mathrm{LVE}_\phi^m)^\star$.

PROOF. This is a direct consequence of proof in reference [MRRS07, pp. 859 – 862] which, though originally written in the context of Hamiltonian systems, is still valid for general complex analytic differential systems $\dot{x} = X(x)$. □

We denote the set of rational solutions of $(\mathrm{LVE}_\phi^m)^\star$ by

$$\mathrm{Sol}_{\boldsymbol{k}}((\mathrm{LVE}_\phi^m)^\star) := \{ W \in \boldsymbol{k}^{\sum_{i=1}^m d_{i,n}} : \dot{W} = A_m^\star W \}.$$

An immediate consequence of Lemma 7 is the following:

COROLLARY 8. *Let F be a holomorphic first integral of* (1) *with valuation ν along a non-constant solution ϕ, then* $F^{(\nu)}(\phi) \in \mathrm{Sol}_{\boldsymbol{k}}((\mathfrak{sym}^\nu A_1)^\star)$.

PROOF. Since F has valuation ν along ϕ, by Lemma 7, we have $V_\nu = (F^{(\nu)}(\phi), 0) \in \mathrm{Sol}_{\boldsymbol{k}}([A_\nu^\star])$. More explicitly, we have that

$$\begin{bmatrix} \frac{d}{dt}({}^t F^{(\nu)}(\phi)) \\ 0 \end{bmatrix} = \begin{bmatrix} (\mathfrak{sym}^\nu A_1)^\star & B_\nu^\star \\ 0 & A_{\nu-1}^\star \end{bmatrix} \cdot \begin{bmatrix} {}^t F^{(\nu)}(\phi) \\ 0 \end{bmatrix}$$

which implies that $\frac{d}{dt}({}^t F^{(\nu)}(\phi)) = (\mathfrak{sym}^\nu A_1)^\star \cdot {}^t F^{(\nu)}(\phi)$. □

EXAMPLE 9. *Consider the toy example of the* anharmonic oscillator *[Aud01]: it is modeled by a two-degree-of-freedom Hamiltonian*

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) + \lambda q_2^2 + (q_1^2 + q_2^2)^2$$

with $(q,p) = (q_1, q_2, p_1, p_2) \in \mathbb{C}^4$, and the Hamiltonian field X_H is given by the equations:

$$\dot{q} = p\,,\; \dot{p}_1 = -4q_1\left(q_1^2 + q_2^2\right)\,,\; \dot{p}_2 = -2q_2\left(2\lambda + q_1^2 + q_2^2\right)$$

This system is integrable, an additional first integral being

$$K = \frac{1}{2}p_1^2 + \frac{1}{2}(q_1 p_2 - q_2 p_1)^2 + q_1^2\left(q_1^2 + q_2^2\right),$$

and has a particular solution $\psi(t) = \left(\frac{\mathrm{i}\sqrt{2}}{2t}, 0, -\frac{\mathrm{i}\sqrt{2}}{2t^2}, 0 \right)$ for which $H(\psi) \equiv 0$. The second-order Taylor expansion of H along ψ is therefore given by:

$$\begin{aligned} H(\psi + y) \;=\; & \left(-\frac{\mathrm{i}\sqrt{2}}{t^3} y_1 - \frac{\mathrm{i}\sqrt{2}}{2t^2} y_3 \right) + \\ & \left(-\frac{3y_1^2}{t^2} + \frac{(\lambda t^2 - 1)y_2^2}{t^2} + \frac{y_3^2}{2} + \frac{y_4^2}{2} \right) + \mathrm{O}(y^3). \end{aligned} \tag{3}$$

which is the same as writing

$$H^{(1)}(\psi) = \left(-\frac{\mathrm{i}\sqrt{2}}{t^3}, 0, -\frac{\mathrm{i}\sqrt{2}}{2t^2}, 0 \right) \quad \textit{and}$$

$$H^{(2)}(\psi) = \left(-\frac{6}{t^2}, 0, 0, 0, \frac{2(-1 + \lambda t^2)}{t^2}, 0, 0, 1, 0, 1 \right)$$

Given the formal vector of variables $y := (y_1, y_2, y_3, y_4)$ its symmetric square $\mathrm{Sym}^2 y$ is easy to compute:

$${}^t[y_1^2,\, 2y_1y_2,\, 2y_1y_3,\, 2y_1y_4,\, y_2^2,\, 2y_2y_3,\, 2y_2y_4,\, y_3^2,\, 2y_3y_4,\, y_4^2].$$

It is easily checked that we can rewrite (3) *as*

$$H(\psi + y) = \langle H^{(1)}(\psi)\,,\, y \rangle + \frac{1}{2}\langle H^{(2)}(\psi)\,,\, \mathrm{Sym}^2 y \rangle + \mathrm{O}(y^3).$$

Along any solution $\phi = (q_1, q_2, p_1, p_2)$, the matrices for VE_ϕ^1 *and* LVE_ϕ^2 *are, respectively,*

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & d & 0 & 0 \\ d & c & 0 & 0 \end{bmatrix},$$

and $A_2 := \begin{bmatrix} \mathfrak{sym}^2 A_1 & 0_{10\times 4} \\ X^{(2)}(\psi) & A_1 \end{bmatrix}$, *explicitly written as*

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2a & d & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2d & c & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 2d & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & d & 0 & 0 & 2c & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & a & 0 & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-24q_1 & -8q_2 & 0 & 0 & -8q_1 & 0 & 0 & 0 & 0 & 0 & a & d & 0 & 0 \\
-8q_2 & -8q_1 & 0 & 0 & -24q_2 & 0 & 0 & 0 & 0 & 0 & d & c & 0 & 0
\end{bmatrix},$$

where $a = -8q_1^2 - 4\left(q_1^2 + q_2^2\right)$, $b = -8q_2^2 - 4\left(q_1^2 + q_2^2\right)$, $c = -2\lambda + b$ *and* $d = -8q_1 q_2$.

$$\begin{aligned} K^\circ(y) &= \frac{1}{2}\langle K^{(2)}(\psi)\,,\,\mathrm{Sym}^2 y\rangle \qquad (4) \\ &= \left(\frac{(2t-1)^2}{4t^4} y_2^2 + \frac{1}{2t^3} y_2 y_4 + \frac{1}{4}\frac{(1+2t^2)}{t^2} y_4^2\right) \end{aligned}$$

and $K^{(2)}(\psi) \in \mathrm{Sol}_{\boldsymbol{k}}((\mathfrak{sym}^2 A_1)^\star)$ *is equal to*

$${}^t\left(0,0,0,0,0,\frac{(2t-1)^2}{4t^4}, 0, \frac{1}{2t^3}, 0, 0, 14\frac{(1+2t^2)}{t^2}\right).$$

3. ADMISSIBLE SOLUTIONS OF $(\mathrm{LVE}_\phi^M)^\star$

3.1 A filter condition

We prove the existence of an additional set of linear conditions linking the entries of $(F^{(m)}(\phi)\,, \ldots, F^{(1)}(\phi))$ if F is a first integral of (1).

Let us begin with a very simple case. If F is a holomorphic first integral then its Taylor expansion along ϕ reads as

$$\hat{F}(y) = \left\langle F^{(1)}(\phi)\,,\, y\right\rangle + \frac{1}{2}\left\langle F^{(2)}(\phi)\,,\,\mathrm{Sym}^2 y\right\rangle + \ldots$$

and ${}^tF^{(1)}(\phi)$ is a rational solution of $\left(\mathrm{VE}^1_\phi\right)^\star$; but it satisfies a (non-differential) linear condition as well. Indeed, F being a first integral is equivalent to $\frac{d}{dt}F(\phi(t)) = 0$ for any solution ϕ of (1). Developing the expression we obtain

$$\frac{d}{dt}F(\phi(t)) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\phi(t))\,\dot{\phi}_i(t) = \left\langle {}^t\dot{\phi}\,,\, {}^tF^{(1)}(\phi)\right\rangle.$$

Thus, among the $V \in \mathrm{Sol}_{\boldsymbol{k}}((\mathrm{VE}^{\mathrm{m}}_\phi)^\star)$ only those satisfying the condition $\left\langle {}^t\dot{\phi}\,,\, V\right\rangle = 0$ may be admissible as possible gradients of some holomorphic first integral along ϕ.

Definition 10. *Let ϕ be a non-constant solution of* (1), $\boldsymbol{k} := \mathbb{C}\langle\phi\rangle = \mathbb{C}\left(\phi, \dot{\phi}, \ldots\right)$, *$V$ be a $\boldsymbol{k}$-vector space of finite dimension n, (e_i) be a basis of V and $\dot{\phi} = \left(\dot{\phi}_1, \ldots, \dot{\phi}_n\right)$ denote the expression $\dot{\phi} = \sum_{i=1}^n \dot{\phi}_i \cdot e_i$. Let $(f_{m,j})_{j=1\ldots d_{m,n}}$ be the corresponding canonical basis of* $\mathrm{Sym}^m(V)$.

1. *We define* $M_1 := {}^t\dot{\phi}$.
2. *We define* $M_m \in \mathrm{Mat}\left(d_{m-1,n} \times d_{m,n}\,,\, \mathbb{C}\left(\dot{\phi}\right)\right)$ *as the matrix whose j^{th} row is the symmetric product of $\dot{\phi}$ with the j^{th} basis vector $f_{m-1,j}$: $\dot{\phi}Ⓢf_{m-1,j}$.*
3. *We define the matrices*

$$\mathcal{M}_m \in \mathrm{Mat}\left(\sum_{i=0}^{m-1} d_{i,n} \times \sum_{i=1}^{m} d_{i,n}\,,\, \boldsymbol{k}\right)$$

inductively as follows:

$$\mathcal{M}_1 := M_1\,,\, \mathcal{M}_2 = \begin{bmatrix} M_2 & {}^tA_1 \\ 0 & M_1 \end{bmatrix},$$

and

$$\mathcal{M}_m := \left[\begin{array}{c|c|c} M_m & {}^t\left(\mathfrak{sym}^{m-1}A_1\right) & {}^tB_{m-1} \\ \hline 0 & \multicolumn{2}{c}{\mathcal{M}_{m-1}} \end{array}\right]$$

where $B_{m-1} \in \mathrm{Mat}\left(d_{m-1,n} \times \sum_{i=1}^{m-1} d_{i,n}\,,\, \boldsymbol{k}\right)$ *is the sub-diagonal block of the matrix of the order-$m-1$ variational equation of* (1),

$$A_{m-1} = \begin{bmatrix} \mathfrak{sym}^m A_1 & 0 \\ B_{m-1} & A_{m-1} \end{bmatrix}.$$

Example 11. *This provides an explicit construction of M_m, e.g.* $M_1 = \begin{bmatrix} \dot{\phi}_1 & \dot{\phi}_2 & \dot{\phi}_3 & \dot{\phi}_4 \end{bmatrix}$ *and*

$$M_2 = \begin{bmatrix} \dot{\phi}_1 & \dot{\phi}_2 & \dot{\phi}_3 & \dot{\phi}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{\phi}_1 & 0 & 0 & \dot{\phi}_2 & \dot{\phi}_3 & \dot{\phi}_4 & 0 & 0 & 0 \\ 0 & 0 & \dot{\phi}_1 & 0 & 0 & \dot{\phi}_2 & 0 & \dot{\phi}_3 & \dot{\phi}_4 & 0 \\ 0 & 0 & 0 & \dot{\phi}_1 & 0 & 0 & \dot{\phi}_2 & 0 & \dot{\phi}_3 & \dot{\phi}_4 \end{bmatrix}.$$

This has been successfully implemented in Maple for any order m.

The following Theorem characterizes the jets of derivatives of holomorphic first integrals.

Theorem 12. *Let F be a holomorphic first integral and ϕ be a non-constant solution of* (1), *then for each $m \geq 1$:*

1. *The above matrices M_m and $\mathcal{M}_m$ have full rank.*
2. *$F^{(m)}(\phi)$ and $F^{(m-1)}(\phi)$ are linked by*

$$M_m \cdot {}^tF^{(m)}(\phi) = \frac{d}{dt}\left({}^tF^{(m-1)}(\phi)\right). \qquad (5)$$

3. *$V_m := {}^t\left(F^{(m)}(\phi)\,, \ldots, F^{(1)}(\phi)\right)$ satisfies*

$$\mathcal{M}_m \cdot V_m = 0$$

Proof. 1. $\dot{\phi} \neq 0$ by hypothesis. Let $(f_{j,m-1})$ be the basis of $\mathrm{Sym}^{m-1}V$ as in Definition 10. Vectors

$$\left\{(f_{j,m-1}Ⓢ\dot{\phi}) \in \mathrm{Sym}^m V\right\}$$

are linearly independent. Thus, the rows of M_m are linearly independent, implying M_m has full rank. Differentiating $F^{(m-1)} \in \mathrm{Sym}^{m-1}V$ yields, on the basis element corresponding to multi-index j,

$$\begin{aligned} ({}^t\dot{F}^{(m-1)}(\phi))_j &= \sum_{i=1}^{n} \frac{\partial^m F}{\partial x_1^{j_1}\cdots\partial x_i^{j_i+1}\cdots\partial x_n^{j_n}}(\phi)\cdot\dot{\phi}_i \\ &= \left\langle \dot{\phi}Ⓢf_{j,m-1}\,,\, {}^tF^{(m)}(\phi)\right\rangle. \end{aligned}$$

But $\dot{\phi}Ⓢf_{j,m-1}$ corresponds to the exact expression of row j in matrix M_m, which proves (5).

2. All M_m being full-rank, $\mathcal{M}_m$ is, too, by construction. Let $v_i := F^{(i)}(\phi)$ and $V_m := (v_m, \ldots, v_1)$.

   At order 1 the result is already proved. For $m = 2$ the previous item implies $M_2 \cdot v_2 = \dot{v}_1 = A_1^\star \cdot v_1$, since F is a first integral. Therefore, we have

$$M_2 \cdot v_2 + {}^tA_1^\star \cdot v_1 = \mathcal{M}_2 \cdot V_2 = 0.$$

Assume $\mathcal{M}_{m-1} \cdot V_{m-1} = 0$ and let us prove it true for m as well. Since F is a first integral we have

$$M_m \cdot v_m = \dot{v}_{m-1} = (\mathfrak{sym}^{m-1} A_1)^\star \cdot v_{m-1} + B^\star_{m-1} \cdot V_{m-2},$$

which implies $[M_m, {}^t\mathfrak{sym}^{m-1} A_1, {}^t B_{m-1}] \cdot V_m = 0$. Since $\mathcal{M}_{m-1} \cdot V_{m-1} = 0$, the result follows.

□

Corollary 13. *For $m \in \mathbb{N}$ in the usual notations:*

1. $\dim_{\boldsymbol{k}} (\ker (\mathcal{M}_m)) = d_{m,n} - 1$.
2. $\dim_{\boldsymbol{k}} (\ker (M_m)) = d_{m,n-1}$.

Proof. 1. By Theorem 12 $\mathcal{M}_m$ has full rank, whence

$$\dim_{\boldsymbol{k}} (\ker (\mathcal{M}_m)) = \sum_{i=1}^{m} d_{i,n} - \sum_{i=0}^{m-1} d_{i,n} = d_{m,n} - 1.$$

2. Again in virtue of the same Theorem, M_m is full-rank, implying $\dim_{\boldsymbol{k}} (\ker (M_m)) = d_{m,n} - d_{m-1,n} = d_{m,n-1}$.

□

In virtue of Theorem 12, given a holomorphic first integral F and a non-constant solution ϕ of (1), the jet of derivatives V_m along ϕ satisfies $V_m \in \mathrm{Sol}_{\boldsymbol{k}} \left((\mathrm{LVE}^m_\phi)^\star \right) \cap \ker(\mathcal{M}_m)$ for every $m \geq 1$. This motivates the following:

Definition 14. *The* set of admissible solutions of the system $(\mathrm{LVE}^m_\phi)^\star$ *is defined as*

$$\mathrm{Sol}_{\mathrm{adm}} \left((\mathrm{LVE}^m_\phi)^\star \right) := \mathrm{Sol}_{\boldsymbol{k}} \left((\mathrm{LVE}^m_\phi)^\star \right) \cap \ker(\mathcal{M}_m), \quad m \geq 1.$$

Example 15. *Let us illustrate Theorem 12 by showing how these filters work. As in Example 9 we consider the anharmonic oscillator $H = \frac{1}{2}(p_1^2 + p_1^2) + \lambda \cdot q_2^2 + (q_1^2 + q_2^2)^2$ as well as, once again, the solution $\psi(t) = \left(\frac{\mathrm{i}\sqrt{2}}{2t}, 0, -\frac{\mathrm{i}\sqrt{2}}{2t^2}, 0 \right)$, and define the base differential field $\boldsymbol{k} := \mathbb{C}\left(\psi, \dot{\psi}\right) = \mathbb{C}(t)$. Recall that $H(\psi) = 0$.*

$(\mathrm{VE}^1_\psi)^\star$ **:** *We apply our algorithm from the next Section and compute the rational solutions of $(\mathrm{VE}^1_\psi)^\star$ belonging to $\boldsymbol{k}^4$ and obtain $\mathrm{Sol}_{\boldsymbol{k}} (\mathrm{LVE}^1_\psi)^\star$ equal to*

$$\left\{ {}^t\left(-3c_1 t^2 - c_2 \frac{2}{t^3}, 0, c_1 t^3 + c_2 \frac{1}{t^2}, 0 \right) : (c_1, c_2) \in \mathbb{C}^2 \right\},$$

having dimension 2. Let $V_1 \in \mathrm{Sol}_{\boldsymbol{k}}((\mathrm{VE}^1_\psi)^\star)$ be any such vector. Applying filter condition $\mathcal{M}_1 \cdot V_1 = 0$ yields $V_1 = {}^t(-c_2 \frac{2}{t^3}, 0, c_2 \frac{1}{t^2}, 0)$ which is proportional to $H^{(1)}(\psi)$. Thus, $\mathrm{Sol}_{\mathrm{adm}}(\mathrm{VE}^1_\psi)^\star = \mathrm{span}_{\mathbb{C}}(H^{(1)}(\psi))$; this proves that H is the only holomorphic first integral which is not degenerate along ψ.

$(\mathrm{LVE}^2_\psi)^\star$ **:** *Since $H(\psi) = 0$, H^2 has valuation 2 along ψ. We know there is another first integral degenerate of order 2 along ψ,*

$$K = \frac{1}{2} p_1^2 + \frac{1}{2}(q_1 p_2 - q_2 p_1)^2 + q_1^2(q_1^2 + q_2^2).$$

Therefore, we expect there will be at least 3 admissible solutions for $(\mathrm{LVE}^2_\psi)^\star$:

$$(H^{(2)}(\psi), H^{(1)}(\psi)), \quad ((H^2)^{(2)}(\psi), 0), \quad (K^{(2)}(\psi), 0),$$

stemming respectively from H, H^2 and K. We compute $\mathrm{Sol}_{\boldsymbol{k}} \left((\mathrm{LVE}^2_\psi)^\star \right)$ and its dimension happens to be 6. After due application of the filter condition $\mathcal{M}_2$, $\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}}((\mathrm{LVE}^2_\psi)^\star)$ is exactly 3 and we have

$$\mathrm{Sol}_{\mathrm{adm}}((\mathrm{LVE}^2_\psi)^\star) = \mathrm{span}_{\mathbb{C}}\{W_2(c_1, c_2, c_3) : c_i \in \mathbb{C}\},$$

for a three-parametric vector W_2. Hence, the admissible solutions of $(\mathrm{LVE}^2_\psi)^\star$ correspond to the three first integrals of valuation $\nu \leq 2$ along ψ: H, K and H^2 [2]*.*

Since the valuation of H^2 and K along ψ is 2, both $((H^2)^{(2)}(\psi))$ and $(K^{(2)}(\psi))$ must be admissible solutions of $(\mathfrak{sym}^2 A_1)^\star$. We compute $\mathrm{Sol}_{\boldsymbol{k}}((\mathfrak{sym}^2 A_1)^\star)$; its dimension is 4. Applying the filter M_2 we discard 2 out of 4 solutions. We obtain

$$\mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^2 A_1)^\star) = \{V_2(c_1, c_3) : c_i \in \mathbb{C}\},$$

thus proving that H^2 and K are the only two holomorphic first integrals degenerate of order 2 along ψ.

$(\mathrm{LVE}^3_\psi)^\star$ **:** *We perform the same computations at order 3 and obtain that*

$$\dim_{\mathbb{C}} \mathrm{Sol}_{\boldsymbol{k}}((\mathrm{LVE}^3_\psi)^\star) = 12.$$

We expect to get at least 5 admissible solutions: those stemming from

$$H,\ H^2,\ K,\ H^3 \quad \text{and} \quad H \cdot K.$$

After applying the filter conditions we conclude that such solutions are indeed the only admissible ones since

$$\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}}(\mathrm{LVE}^3_\psi) = 5.$$

We obtain that the admissible solutions of $(\mathfrak{sym}^3 A_1)^\star$ are exactly those stemming from the first integrals degenerate of order 3 along ψ: H^3 and $H \cdot K$ discarding thanks to the filter M_3, 4 solutions out of 6. Therefore, the only first integrals degenerate of order 3 along ψ are H^3 and $H \cdot K$.

$(\mathrm{LVE}^4_\psi)^\star$ **:** *Similarly at order 4 we obtain that*

$$\dim_{\mathbb{C}} \mathrm{Sol}_{\boldsymbol{k}}((\mathrm{LVE}^4_\psi)^\star) = 21.$$

We expect to get at least 8 admissible solutions: those stemming from

$$H,\ H^2,\ H^3,\ H^4,\ K,\ H \cdot K,\ H^2 \cdot K \quad \text{and} \quad K^2.$$

After applying the filter conditions we see that such solutions are indeed the only admissible ones since

$$\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}} (\mathrm{LVE}^4_\psi) = 8.$$

Consequently, the admissible solutions of $(\mathfrak{sym}^4 A_1)^\star$ are exactly those stemming from the first integrals degenerate of order 4 along ψ: H^4, $H^2 \cdot K$ and K^2 discarding thanks to the filter M_4, 6 solutions out of nine. Therefore, the only order-four-degenerate first integrals along ψ are H^4, $H^2 \cdot K$ and K^2.

Even assuming no prior knowledge about the existence of K or the integrability of the system, this filter procedure leads us as far as:

[2]The values $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of (c_1, c_2, c_3) correspond to K, H^2 and H respectively.

1. *computing the germ of a valuation-2 formal first integral along ψ;*
2. *and proving there is no other holomorphic first integral of valuation at most 4 along ψ, which is a strong hint on the non-existence of any other holomorphic first integral, as they would have to be of valuation at least 5 along ψ.*

3.2 Bounds on $\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^m A_1)^\star)$

The number of analytic first integrals which are degenerate at order-m along a non-constant solution ϕ has a lower bound depending on the number of analytic algebraically independent first integrals of the system:

PROPOSITION 16. *Let ϕ be a non-constant solution of (1) and let $\mathcal{F} = \{F_1, \dots, F_N\}$ with $N \geq 1$ be a family of holomorphic first integrals of (1) satisfying Ziglin's Lemma (i.e their junior forms are algebraically independent) with valuations $\nu_1, \dots, \nu_N$ along ϕ. Then, for every $m \geq 1$, we have the inequalities*

$$\operatorname{card}\left\{r \in \mathbb{N}^N \;:\; \sum_{i=1}^N r_i \nu_i = m\right\} \leq$$
$$\dim_{\boldsymbol{k}}(\mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^m A_1)^\star)) \quad \leq \quad d_{m-1,n-1}.$$

PROOF. The upper-bound condition is easy:

$$\begin{aligned}\dim_{\boldsymbol{k}} \mathrm{Sol}_{\mathrm{adm}}(\mathfrak{sym}^m A_1)^\star &\leq \dim_{\boldsymbol{k}} \ker M_m \\ &= d_{m,n} - d_{m-1,n} = d_{m-1,n-1}.\end{aligned}$$

Now let $m \in \mathbb{N}$ and consider

$$\mathcal{G}_m := \left\{ G = \prod_{i=1}^N F_i^{m_i} \;:\; \nu(G) = m \right\},$$

the family of first integrals of valuation m which are monomials of degree m in $F_1, \dots, F_N$. For $G \in \mathcal{G}_m$, we have $G^\circ = \prod_{i=1}^N (F_i^\circ)^{m_i}$. And $\prod_{i=1}^N (F_i^\circ)^{m_i}$ are linearly independent (otherwise the F_i° would be algebraically dependent). Therefore, $\dim_{\boldsymbol{k}}(\mathrm{span}_{\mathbb{C}}(\prod_{i=1}^N (F_i^\circ)^{m_i})_{\sum m_i \nu_i = m})$ is equal to $\operatorname{card}\{r \in \mathbb{N}^N \;:\; r_1\nu_1 + \ldots + r_N \nu_N = m\}$. Now,

$$\mathrm{span}_{\mathbb{C}}(\prod_{i=1}^N (F_i^\circ))^{m_i} \subset \mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^m A_1)^\star),$$

which proves our point. $\square$

EXAMPLE 17. *Back to the anharmonic oscillator example. The table below summarizes for m ranging from 1 to 4 (first column) the dimension of $\mathrm{Sol}_{\mathrm{adm}}\left((\mathrm{LVE}_\phi^m)^\star\right)$ (second column) as well as the generators of the latter (third column). For each m we find out that the admissible solutions of the $[A_m^\star]$ are generated by*

1	1	$\mathrm{span}_{\mathbb{C}}\{H^{(1)}(\phi)\}$
2	3	$\mathrm{span}_{\mathbb{C}}\{H^{(2)}(\phi), K^{(2)}(\phi), (H^2)^{(2)}(\phi)\}$
3	5	$\mathrm{span}_{\mathbb{C}}\left\{\begin{array}{c} H^{(3)}(\phi), K^{(3)}(\phi), (H^2)^{(3)}(\phi) \\ (H^3)^{(3)}(\phi), (H \cdot K)^{(3)}(\phi) \end{array}\right\}$
4	8	$\mathrm{span}_{\mathbb{C}}\left\{\begin{array}{c} H^{(4)}(\phi), K^{(4)}(\phi), (H^2)^{(3)}(\phi) \\ (H^3)^{(4)}(\phi), (HK)^{(4)}(\phi) \\ (H^4)^{(4)}(\phi), (H^2K)^{(4)}(\phi), (K^2)^{(4)}(\phi) \end{array}\right\}$

In light of Proposition 16 consider the table below which summarizes for each value of m ranging from 1 to 4 (first column) the dimension of $\mathrm{Sol}_{\mathrm{adm}}(\mathfrak{sym}^m A_1)^\star)$ (second column) as well as the generators of the latter (third column)

1	1	$\mathrm{span}_{\mathbb{C}}\{H^{(1)}(\phi)\}$
2	2	$\mathrm{span}_{\mathbb{C}}\{K^{(2)}(\phi), (H^2)^{(2)}(\phi)\}$
3	2	$\mathrm{span}_{\mathbb{C}}\{(H^3)^{(3)}(\phi), (HK)^{(3)}(\phi)\}$
4	3	$\mathrm{span}_{\mathbb{C}}\{(H^4)^{(4)}(\phi), (H^2K)^{(4)}(\phi), (K^2)^{(4)}(\phi)\}$

Had the dimension of $\mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^3 A_1)^\star)$ been less than 2 and that of $\mathrm{Sol}_{\mathrm{adm}}((\mathfrak{sym}^4 A_1)^\star)$ less than 3, we could have discarded the possibility of there existing any holomorphic degenerate first integral of order 2 along ψ other than H^2. In this sense, Proposition 16 acts as a non-integrability indicator.

4. GENERAL ALGORITHM

To summarize, we study (1) along an integral curve Γ parametrized by $\phi(t)$. We wish to detect holomorphic (or formal) first integrals and compute their Taylor expansions. The above Theorems show that we should compute admissible solutions of the (LVE_ϕ^m). This is done as follows.

1. Compute rational solutions of (VE_ϕ^1). Apply filter condition from Theorem 12.
2. Order m. Assume we know a parametrized admissible solution $V_{m-1} = \sum c_i V_{i,m-1}$ in $\mathrm{Sol}_{\mathrm{adm}}(\mathrm{LVE}_\phi^{m-1})$.
   (a) Compute rational solutions of $\dot{Y} = \mathfrak{sym}^m(A) Y$ (and filter via Theorem 12); this gives junior forms of first integrals of valuation m along ϕ.
   (b) Compute rational solutions of the inhomogeneous system (with parametrized right-hand side)
   $$\dot{Y} = \mathfrak{sym}^m(A)^\star Y + B_m^\star V_{m-1}.$$

Details on how to optimize this in the case $\boldsymbol{k} = \mathbb{C}(t)$ are given in the next Section.

5. AN ALGORITHM FOR THE RATIONAL FUNCTION BASE FIELD CASE

Assume now $\boldsymbol{k} = \mathbb{C}(t)$. We recall Barkatou's algorithm in [Bar99] to compute rational solutions of linear systems and adapt it to a variant of [vHW97] tailored to our context.

DEFINITION 18. *Given $A \in \mathrm{Mat}(n, \boldsymbol{k})$ and $P \in \mathrm{GL}_n(\boldsymbol{k})$, $Y = PZ$ transforms system $\dot{Y} = AY$ into $\dot{Z} = P[A]Z$ where*

$$P[A] := P^{-1}(AP - \dot{P}).$$

Such a change of variables is usually called a gauge transformation.

A *universal denominator* (UD in short) is a rational function $r(t)$ such that any rational solution is $Y = r(t)Z$, Z being a vector of polynomials. We briefly recall how such a UD is computed. Consider a system $\dot{Y} = A(t)Y$, with $A \in \mathcal{M}_n(\mathbb{C}(t))$. If Y is a rational solution and $p \in \mathbb{C}$ is a finite pole of Y then p is a pole of A. Moreover for any finite pole p of A one can compute an integer ℓ_p such that for any rational solution Y the function $(t-p)^{-\ell_p} Y$ has no pole at p. A UD is obtained by taking the product

$$\prod_{p \text{ pole of } A} (t-p)^{\ell_p}.$$

In order to compute the bound ℓ_p one must first compute a gauge-equivalent system in a suitable so-called *simple form* from which the indicial polynomial at p can be immediately obtained. ℓ_p is then the smallest integer root of this indicial equation. Simple forms are computed by using an adapted version of the super-reduction algorithm ([HW87, BP09]).

Once a UD $r(t)$ is known, we have rational solutions if, and only if, system $\dot{Z} = (A - \frac{\dot{r}}{r}\mathrm{Id}_n)Z$ has polynomial solutions. To achieve this, [Bar99] computes a gauge transformation P_∞ polynomial in t^{-1} (with P_∞^{-1} polynomial in t) such that $P_\infty[A]$ is in simple form at infinity (note that $P_\infty[A] - \dot{r}/rId_n = P_\infty[A - \dot{r}/rId_n]$) and computes the coefficients of Z from regular solutions at ∞.

In the parametrized right-hand side case $\dot{Y} = AY + \sum c_i V_i$, this algorithm returns the values of the c_i for which the system admits a rational solution, hence it is adapted to part (2.b) of our general algorithm.

5.1 Symmetric Powers of Differential Systems

Recall [AM10] that Sym is a group morphism, i.e.

$$\mathrm{Sym}^m(UV) = \mathrm{Sym}^m(U)\,\mathrm{Sym}^m(V),$$

and $\mathfrak{sym}$ is a vector space morphism:

$$\mathfrak{sym}^m(A + \lambda B) = \mathfrak{sym}^m(A) + \lambda\, \mathfrak{sym}^m(B).$$

The following Lemmae, valid for all complex matrices, summarize properties which will be used below.

Lemma 19. • *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of M. The eigenvalues of $\mathfrak{sym}^m(M)$ (resp. $\mathrm{Sym}^m(M)$) are of the form $\sum_{|i|=m} i_j\lambda_j$ (resp. $\prod_{|i|=m} \lambda_j^{i_j}$).*

- *If $M \in \mathcal{M}_n(\boldsymbol{k})$ is such that M and $\dot{M}$ commute, then $\mathrm{Sym}^m(e^M) = e^{\mathfrak{sym}^m(M)}$.*

Lemma 20. *Given gauge transformation $B = P[A]$,*

- *$\mathfrak{sym}^m(P[A]) = \mathrm{Sym}^m(P)[\mathfrak{sym}^m(A)]$.*
- *If A has a regular singularity at $t = 0$ and matrix $P[A] = \frac{A_0}{t} + \cdots$ has a pole of order one, we then have $\mathrm{Sym}^m(P)[\mathfrak{sym}^m(A)]) = \frac{\mathfrak{sym}^m(A_0)}{t} + \cdots$.*

5.2 Case in which all singularities are regular

In this Section we consider the regular singular case, as it often occurs in examples and simplifies the exposition.

Any system $\dot{Y} = AY$ with a regular singularity, say at $t = 0$, can be transformed, e.g. using Moser's algorithm (see [Mos60], [BP09]), to a gauge-equivalent system with a first-kind singularity, i.e. $\dot{Z} = BZ$, B having at most a simple pole at 0: $B(t) = \frac{1}{t}B_0 + B_1 + \cdots$. The indicial polynomial coincides with the characteristic polynomial of B_0.

It is well-known that, using a suitable polynomial gauge transformation, one can assume that the eigenvalues of B_0 do not differ by non-zero integers. In this case the system $\dot{Z} = BZ$ has a formal solution matrix at $t = 0$ of the form $\hat{U} = \hat{\Phi}(t) \cdot t^\Lambda$, where $\hat{\Phi} \in \mathcal{M}_n(\mathbb{C}[[t]])$ satisfies $\det \Phi(0) \neq 0$ and $\Lambda \in \mathcal{M}_n(\mathbb{C})$ is the normal Jordan form of B_0. The eigenvalues of B_0 are called the *(local) exponents* (or local data) at $t = 0$.

For each finite singularity p, one can compute, with the Moser algorithm, a polynomial gauge transformation P_p such that $B = P_p[A] = \frac{1}{t-p}B_{0,p} + \cdots$ and the order of the UD at $t = p$ is the least integer eigenvalue of $B_{0,p}$. We thus have $\mathrm{Sym}^m(P_p)[\mathfrak{sym}^m(A)]) = \frac{\mathfrak{sym}^m(B_{0,p})}{t-p} + \cdots$ hence the eigenvalues of $\mathfrak{sym}^m(B_{0,p})$ are $\sum_{|i|=m} i_j\lambda_j$, $\lambda_1, \ldots, \lambda_n$ being the eigenvalues of $B_{0,p}$. We may thus compute a UD r_m for system $\dot{Y} = \mathfrak{sym}^m(A).Y$ from the local data computed on $\dot{Y} = AY$. Since ∞ is regular, $P_\infty[A]$ has simple pole at infinity; hence $\mathrm{Sym}(P_\infty)[\mathfrak{sym}^m(A)]$ also has a simple pole at infinity. It follows that the Barkatou algorithm for polynomial solutions may be applied directly to

$$\mathrm{Sym}^m(P_\infty)[\mathfrak{sym}^m(A)^\star] - \frac{\dot{r}_m}{r_m}Id.$$

To summarize, bounds at singularities (for denominators) are computed from local data of (VE^1_ϕ), the transformation to simple form at ∞ is lifted to $\mathfrak{sym}^m(A)^\star$ and polynomial solutions of the latter are then as in [Bar99] or [vHW97]. This gives junior forms of first integrals of valuation m along ϕ. Note that, as in [vHW97], we may reduce the size of computations by means of suitable formal solutions from (VE^1_ϕ) at this stage.

Part 2.b of the algorithm is adapted similarly. The only thing that changes is that we need a min between the "universal bound" computed above (at each singularity) and the valuations of the right-hand side $B_m^\star V_{m-1}$.

5.3 Irregular Singularities

For a singularity, say $t = 0$, which is not regular singular, $\hat{U} = \hat{\Phi}(t)t^\Lambda e^Q$ with notations as above and $Q = \mathrm{diag}\left(q_1(1/t^{\frac{1}{r}}), \ldots, q_n(1/t^{\frac{1}{r}})\right)$ with $q_i \in \mathbb{C}[1/t^{1/r}]$. These exponential parts of $A_1^\star$ may be computed from [Bar97]. Once exponents are computed at p (using [Bar99, §4.3] as above), the corresponding exponents of $\mathfrak{sym}^m(A)$ may be computed from these exponents and the exponential parts using (a small adaptation of) procedure `global-bounds` in [vHW97, §3 (p. 368)]. Regarding bounds at infinity, the symmetric power of a super-reduced system at infinity may not be simple. However, it will be very close. So if we let $SR(A)$ denote a super-reduced form of A at infinity (see [Bar99, App. A.1, A.2]), then $SR(\mathfrak{sym}^m(SR(A)))$ involves a small calculation once $SR(A)$ is known.

5.4 Further Reduction Strategies

The first part of Lemma 20 shows that whenever we have a gauge transformation $Y = PZ$ which "simplifies" the system, then $\mathrm{Sym}^m(P)$ provides the same simplification on $\mathfrak{sym}^m(A)$. If, say, r linearly independent admissible solutions are computed for (VE^1_ϕ), and an invertible P is built whose first columns are the said admissible solutions, the first r columns of $P[A^\star]$ will vanish and $\mathrm{Sym}^m(P)$ will yield $\binom{r+m-1}{m-1}$ columns of zeroes in $\mathfrak{sym}^m(A^\star)$.

If (VE^1_ϕ) has been put in *reduced form* [AMW09, AMW10, AM10] then, as shown in [AMCW11] (also [AM10, Ch. 3]), rational solutions of all $\mathfrak{sym}^m(A^\star)$ will have *constant* coefficients, notably simplifying part 2.a of the algorithm, namely kernel computation on a $\mathfrak{sym}$ power.

6. CONCLUSION

Given a holomorphic vector field X, the vector of the derivatives up to order m of a holomorphic first integral of X appears as a rational solution of $(\mathrm{LVE}^m)^\star$ [MRRS07]. In this work, following [AM10] we have proved that those germs satisfy an additional set of linear conditions and we have introduced the notion of admissible solution to this effect. This construction provides a method allowing us to retrieve those admissible solutions, germs of holomorphic first integrals among them [AM10]. Combining the latter with Barkatou's use of local simple forms for rational solutions, we have introduced an algorithm allowing us to efficiently compute (whenever the base field is $\mathbb{C}(t)$) those admissible solutions.

EXAMPLE 21. *Consider the one-parameter family of classical two-degrees-of-freedom Hamiltonian systems*

$$H_\epsilon(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + V_\epsilon(q_1\,,\, q_2),$$

where $V_\epsilon(q_1, q_2) = \frac{1}{4}(q_1^4 + q_2^4) + \frac{\epsilon}{2}(q_1 q_2)^2$ *with* $\epsilon \in \mathbb{C}$. *This family has been proven to be integrable only for the values* 0, 1 *and* 3 *of the parameter* ϵ *[Yos88]. These systems admit*

$$\phi = \left(\frac{c_1}{t}\,,\,\frac{c_2}{t}\,,\,-\frac{c_1}{t^2}\,,\,-\frac{c_2}{t^2}\right)\quad i = 1, 2,$$

as solution curves[3]. *We pick two particular cases, one integrable* $(\epsilon = 3)$ *and one non-integrable* $(\epsilon = 2)$.

The table below summarizes, for m *ranging from* 1 *to* 3, $\dim \mathrm{Sol}_{\boldsymbol{k}}((\mathrm{LVE}_\phi^{\mathrm{m}})^\star)$ *(second column for* $\epsilon = 3$ *fourth column for* $\epsilon = 2$*) and the dimension of* $\mathrm{Sol}_{\mathrm{adm}}((\mathrm{LVE}_\phi^{\mathrm{m}})^\star)$ *(third column for* $\epsilon = 3$ *and fifth column for* $\epsilon = 2$*):*

1	4	3	2	1
2	14	9	6	3
3	34	19	9	5

For $\epsilon = 3$ *the potential is completely integrable. Indeed, the system admits another polynomial first integral* $F = p_1 p_2 + q_1 q_2 (q_1^2 + q_2^2)$ *which, same as* H, *is non degenerate along all particular solutions* ϕ.

The dimension of $\mathrm{Sol}_{\mathrm{adm}}((\mathrm{LVE}_\phi^{\mathrm{m}})^\star)$ *is maximal for all* m *considered; this suggests* H *could be superintegrable (the third first integral being non degenerate along* ϕ*). This is consistent with the superintegrability necessary condition in [MPY08]. In fact a necessary condition for superintegrability for a general Hamiltonian system of dimension 2 is that the set of admissible solutions be of maximal dimension (possibly from a certain order on).*

For $\epsilon = 2$, *the potential is not meromorphically integrable. In particular, we can affirm that* H *is the only first integral which is not degenerate along* ϕ. *At order* 2, *in addition to the solutions corresponding to* H *and* H^2, *there is an additional admissible solution and at order* 3 *there are* 2 *additional admissible solutions. Applying the necessary condition given above it is clear that the results obtained for* $\epsilon = 2$ *do not hint at superintegrability – as expected.*

[AM10] conjectured that for each m, $\mathrm{Sol}_{\mathrm{adm}}((\mathrm{LVE}_\phi^{\mathrm{m}})^\star)$ are but the solutions of a differential submodule of $\nabla_m^\star = \frac{d}{dt} - A_m^\star$. Once proven, this conjecture, together with the algorithm exposed in this work, will pave the way towards an effective theory of formal first integrals of differential systems along a non-constant solution.

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[3]Where c_1 and c_2 are defined by algebraic relations.