Application of the Morales-Ramis theorem to test the non-complete integrability of the planar three-body problem.

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Abstract. We give a proof of the non complete integrability of the planar three-body problem. We use a criterion of non complete integrability deduced from the Morales-Ramis theorem and based on a parameterized linear differential system, the variational system. We show how to apply this criterion to the three-body problem using symbolic computation tools for parameterized linear differential systems. Our strategy consists in studying locally and globally this normal variational system without transforming it into an equivalent linear differential equation.

KeyWords: Hamiltonian Systems, Symbolic Computation, Differential Galois Theory, Completely Integrable Systems, Three-Body Problems

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1 Introduction

Let us consider three bodies in a newtonian reference system and let us assume that the only forces acting on them are their mutual gravitational attraction. Each body is represented by its mass m_i, its position q_i and its moment p_i ($p_i = m_i \frac{dq_i}{dt}$). According to the Newton law and the law of gravitation the equations of the motion of these bodies can be written in the following form:

$$\begin{cases} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \\ i = 1, 2, 3 \end{cases}$$

where

$$H = \sum_{j=1}^{3} \frac{||p_j||^2}{2m_j} - \sum_{1 \le j,k \le 3} \frac{m_j m_k}{||q_j - q_k||}.$$

These equations form a differential system, called *Hamiltonian system*. It depends on the three parameters m_1, m_2 and m_3. In the sequel, we choose a normalization by assuming (without loss of generality) that $m_3 = 1$. The *Hamiltonian* H represents the mechanical energy and is conserved. One says that H is a *first integral* for the Hamiltonian system. If one can find a sufficient number of first integrals satisfying independence and involution properties then one may assume that the Hamiltonian system will have a non-chaotic behavior in the studied region and one says that it will be *completely integrable* ([M-R],

[Au1]). If we assume that the three bodies are in a plane, one can reduce the number of variables of the initial Hamiltonian ([Tsy1]). We also assume that the constant of the cinetic moment is non zero and without loss of generality we fix it equal to one. We then work with the following Hamiltonian (that we denote H again):

$$H = \frac{1}{2}\left(\frac{1}{m_1}+1\right)\left({p_1}^2 + \frac{(p_3\,q_2 - p_2\,q_3 - 1)^2}{{q_1}^2}\right) + \frac{1}{2}\left(\frac{1}{m_2}+1\right)\left({p_2}^2 + {p_3}^2\right) + p_1\,p_2$$
$$- \frac{p_3\,(p_3\,q_2 - p_2\,q_3 - 1)}{q_1} - \frac{m_2}{\sqrt{{q_2}^2 + {q_3}^2}} - \frac{m_1}{q_1} - \frac{m_1\,m_2}{\sqrt{(q_1-q_2)^2 + {q_3}^2}}.$$

The non-integrability of this Hamiltonian system will imply the non-integrability of the planar three-body problem.

In 1890, Poincaré proved that the three-body problem does not have any additional analytic first integral besides the known integrals ([Poin]). To obtain this result, he studied a *variational system* which is a *linear* differential system computed along a particular solution of the Hamiltonian system.

Definition 1. *The* variational system *along a solution* $\mathrm{x}_0(t)$ *of a Hamiltonian system is the linear differential system:*

$$Y'(t) = A(t)\,Y(t)$$

where

$$A(t) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{H}(H, \mathrm{x}_0(t))$$

$\mathcal{H}(H, \mathrm{x}_0(t))$ *is the Hessian of* H *at* $\mathrm{x}_0(t)$.

We emphasize on the special structure of this variational system (the matrix A is infinitesimally symplectic), which makes further computations much easier despite the presence of parameters in the matrix A.

During the last twenty years many significant improvements regarding complete (meromorphic) integrability of Hamiltonian systems have been obtained by Ziglin ([Zig1], [Zig2]) in 1982, by Baider, Churchill, Rod and Singer ([C-R-S]) in 1996, and Morales and Ramis ([M-R]) in 1998. They all found necessary conditions of complete (meromorphic) integrability based on the monodromy group ([Zig1], [Zig2]) or the differential Galois group ([C-R-S], [M-R]) of the *variational system*. Our study will rely on the theorem of Morales and Ramis ([M-R], [Au1]) :

Theorem 1. *Let* (S) *be a Hamiltonian system,* $\mathrm{x}_0(t)$ *a particular solution of* (S)*,* $Y'(t) = A(t)\,Y(t)$ *the variational system of* (S) *computed along the solution* $\mathrm{x}_0(t)$ *and* G *the differential Galois group of* $Y'(t) = A(t)\,Y(t)$.
If the system (S) *is completely integrable with meromorphic first integrals, then the connected component of the identity in the group* G*, denoted* G^0*, is an abelian group.*

This theorem remains true if one replaces the *variational system* with the *normal variational system*. It is obtained after a standard symplectic transformation which reduces the order of the variational system but keeps its infinitesimally symplectic structure ([Au1]).

We deduced from this theorem a criterion based on a local and global formal study (detection of logarithms and factorization) of the *normal variational system* ([Bou2], [BW02]).

Criterion 1. *Let* (S) *be a Hamiltonian system and let* $Y'(t) = A(t)\,Y(t)$ *be the normal variational system computed along a particular solution of* (S).
If the normal variational system has a completely reducible factor whose local solutions at a singular point contain logarithmic terms (or exhibit a non-trivial Stokes phenomenon), then the Hamiltonian system (S) *is not completely integrable (with meromorphic first integrals).*

In the next section we will apply this criterion to the planar three-body problem. Instead of transforming the normal variational system into a linear differential equation ([Bou1], [Bou2], [BW02]), we keep the structure of the system and make direct computations on it. This gives a new proof on non-integrability of the three-body problem, which we present to emphasize on the method used, which we believe should be fruitful on other problems (such studies are currently in progress).
Previous proofs of non-integrability were given by Alexei Tsygvints'ev in [Tsy1, Tsy2, Tsy3] and in parallel by the authors in [Bou1] and [BW02]. We enjoy noting that these parrallel proofs were obtained thanks to active friendly and open discussions with Tsygvints'ev: collaborating nicely led to two nice proofs. We thank Alexei for this, and Jean-Pierre Ramis for suggesting this beautiful problem to us.

2 Application of the criterion to the planar three-body problem

2.1 The normal variational system

We take the Lagrange solution as particular solution of the Hamiltonian system associated to H ([Tsy1]). The three bodies then form a configuration which is homographically equivalent to an equilateral triangle: each body describes a parabola centered on a vertex of the equilateral triangle.
After reductions, the normal variational system is a 4×4 linear differential system

$$Y'(x) = A(x)\, Y(x)$$

(see Annex 1).

2.2 Factorization of the normal variational system

Let us first define factorization. The system $Y'(x) = A(x)\, Y(x)$ is called *equivalent* to a system $Z'(x) = B(x)\, Z(x)$ if one is obtained from the other by a gauge transformation, i.e a change of variable $Z = PY$ with P an invertible square matrix with rational coefficients. The system is called *factorizable* (or reducible) if it is equivalent with a block triangular system (and *irreducible* otherwise), and it is called *decomposable* if it is equivalent with a block diagonal system with smaller blocks; it is called *completely reducible* if it is equivalent with a block diagonal system where blocks are irreducible (note that this includes the irreducible case).
To find factors of size 2×2 of our 4×4 normal variational system we use the second exterior system (see Annex B2 of [CW] for example or [PS02]) and we give some links to the factorization using the eigenring ([Bar2], [Pflu]).

- If $(m_1, m_2) = (1, 1)$ we get a linear differential system without parameter. It is equivalent to the system

  $$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

  where the blocks F_1 and F_2 are given in Annex 2A.

  Each block F_1 and F_2 is irreducible so the system is completely reducible.

- If $(m_1, m_2) \neq (1, 1)$, then we get two distinct situations that we detail below.
  The second exterior system has two exponential (in fact, rational) solutions $W_I = (0, 1, 0, 0, 1, 0)$ and W whose expression is too big to be given here. The vector $W - \lambda\, W_I$ induces a 2×2 factor for the system $Y' = AY$ for values of λ such that $W - \lambda\, W_I$ is a "pure tensor" for the second exterior system. Following Annex B2 of [CW], we see that these values of λ are the ones which annihilate

the determinant (the associated "plucker relation") of the matrix

$$M_\lambda = \begin{pmatrix} W[4] & -(W[2]-\lambda) & W[1] & 0 \\ W[5]-\lambda & -W[3] & 0 & W[1] \\ W[6] & 0 & -W[3] & W[2]-\lambda \\ 0 & W[6] & -(W[5]-\lambda) & W[4] \end{pmatrix}.$$

This gives use two values λ :

$$\lambda_1 = \frac{(m_1+1-2m_2+2r)\, r\, (2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2)}{2(m_1^2+m_2^2+1-m_1-m_2-m_1m_2)}$$

and

$$\lambda_2 = \frac{(m_1+1-2m_2-2r)\, r\, (2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2)}{2(m_1^2+m_2^2+1-m_1-m_2-m_1m_2)}$$

where r satisfies

$$r^2 = m_1^2 + m_2^2 - m_1m_2 - m_1 - m_2 + 1. \tag{1}$$

Remark 1. *The vector W_I corresponds to the identity matrix, and W corresponds to a matrix T in the eigenring $\mathcal{E}(A)$ of A (see [Sin1], [Bar2], [Pflu]). The values λ_1 and λ_2 are also the eigenvalues of the matrix T. One can even notice* $T - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} M_\lambda$.

There are now two cases,whether the λ_i are distinct or not :

Decomposable case If $2\,m_1^2 + 2\,m_2^2 + 2 - 5\,m_1 - 5\,m_2 - 5\,m_1\,m_2 \neq 0$, then λ_1 and λ_2 are distinct and we have two pure tensors $W - \lambda_1 W_I$ and $W - \lambda_2 W_I$ *(i.e. two matrices of rank* 2 *in $\mathcal{E}(A)$, $T - \lambda_1 I$ and $T - \lambda_2 I$).* We construct the matrix P with two vectors generating the kernel of M_{λ_1} and two vectors generating the kernel of M_{λ_2}. After the gauge transformation $Y = P\,Z$ we get the following equivalent linear differential system:

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z.$$

The coefficients of F_1 are given in Annex 2B, and the factor F_2 is obtained from the factor F_1 after replacing r with $-r$.
Let us now prove that the factors F_1 and F_2 cannot be simultaneously reducible.
We first focus on the block F_1. It is irreducible if and only if it has no exponential solution.

The singular points of the system are ∞, i, $-i$ and 0.
It turns out that our system is locally fuchsian (at each point). So, after Moser reduction (see [Mos, Bar1]), we can reduce this system to one with simple poles. For such a system with a simple pole p (i.e of the form $Y'(x) = \frac{B(x)}{x-p}\,Y(x)$ with B analytic at p), the *indicial equation* at p is the characteristic polynomial of $B(p)$ and the *exponents at* p are the eigenvalues of $B(p)$. This will always be the case in the sequel.
If $m_1 \neq 1$, the indicial equation at infinity is

$$(m_1+m_2+1)\,(m_1^2+m_2^2-m_1\,m_2-m_1-m_2+1)\,(m_1-1)2\,m_2\,m_1^2$$
$$((m_1+m_2+1)\,(\alpha^2-3\,\alpha-1)+3r) = 0$$

If $m_1 = 1$ ($r = m_2 - 1$), the indicial equation at infinity is:

$$(m_2+2)\,(\alpha^2-3\,\alpha-1)+3\,m_2-3 = 0$$

So the indicial equation at infinity for the factor F_1 is:

$$(m_1+m_2+1)\,(\alpha^2-3\,\alpha-1)+3r = 0.$$

The exponents at i and $-i$ are -2 and -1.
The exponents at 0 are -1 and 0.

As the exponents at i, $-i$ and 0 are integers, the factor F_1 (resp. F_2) is reducible only if the indicial equation at infinity has an integer solution n_1 (resp. n_2), see [Pflu, Bou2].
But the equations

$$\begin{cases} (m_1+m_2+1)\,(n_1^2-3\,n_1-1)+3r=0 \text{ (equation for } F_1) \\ (m_1+m_2+1)\,(n_2^2-3\,n_2-1)-3r=0 \text{ (equation for } F_2) \end{cases}$$

imply

$$(m_1+m_2+1)\,(n_1^2+n_2^2-3\,n_1-3\,n_2-2)=0$$

and

$$(2\,n_1-3)^2+(2\,n_2-3)^2=26.$$

The solutions of this equation in $\mathbb{Z}\times\mathbb{Z}$ are $(2,4),(2,-1),(1,4),(1,-1),(4,2),$ $(-1,2),(4,1)$ and $(-1,1)$.
But

$$\begin{array}{l} n_1\in\{2,1\}\Rightarrow m_1+m_2+1-r=0 \\ n_1\in\{4,-1\}\Rightarrow m_1+m_2+1-r=0 \end{array}$$

so

$$n_1\in\{4,2,1,-1\}\Rightarrow r^2=(m_1+m_2+1)^2.$$

According to (1), one gets $m_1+m_2+m_1\,m_2=0$, which is excluded as m_1 and m_2 are positive (recall that they represent masses in the 3-body problem, so they must be *positive real number*).
To conclude, the factors F_1 and F_2 cannot be simultaneously reducible.

Factorizable case If $2\,m_1^2+2\,m_2^2+2-5\,m_1-5\,m_2-5\,m_1\,m_2=0$, we introduce a new parameter s such that:

$$m_1=\frac{(5+3\sqrt{3})\,(s-2+\sqrt{3})\,(s-7+4\sqrt{3})}{(1-s)(1+s)}$$

$$m_2=\frac{(5+3\sqrt{3})\,(s+2-\sqrt{3})\,(s+7-4\sqrt{3})}{(1-s)(1+s)}$$

with

$$2-\sqrt{3}<|s|<1 \text{ or } |s|<7-4\sqrt{3}.$$

The vector W is again solution of the second exterior system and it is a pure tensor ($W-\lambda_1\,W_I=W-\lambda_2\,W_I=W$). The kernel of the matrix $M_{\lambda_1}=M_{\lambda_2}=M_0$ is generated by two vectors. One completes these two vectors into a basis to get the gauge transformation matrix P.

Remark 2. *One finds an element T in the eigenring $\mathcal{E}(A)$ of A associated to the solution W and it has one single eigenvalue,* 0.

One gets the equivalent linear differential system:

$$Z'=\begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z.$$

The coefficients of F are given in Annex 2C.

The exponents of the singular points ∞, $-i$, i and 0 are :
at infinity : the roots of $\alpha^2 - 2\alpha - 1 = 0$,
at 0: -1 and 1,
at i and $-i$: -2 and -1.
As the equation $\alpha^2 - 2\alpha - 1 = 0$ has no integer solution, the system $Y' = FY$ has no exponential solution and the factor F is irreducible.

2.3 Formal solutions with logarithmic terms

We prove that each factor F_1, F_2 and F has formal solutions with logarithmic terms at the point i. We study each of the previous cases

- $(m_1, m_2) = (1, 1)$
  We get two formal solutions with logarithmic terms which are linearly independant (computations made using the package Isolde in the computer algebra system MAPLE, see [Pflu]).

- $(m_1, m_2) \neq (1, 1)$

  - $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 \neq 0$
    Let us prove that the factors F_1 and F_2 both have one formal solution at the point i with logarithmic terms. The factor F_1 has two exponents $\rho_0 = -1$ and $\rho_1 = 1$ at the point i. As they differ from an integer there may be formal solutions with logarithmic tems. After moving the singularity i to the point 0, one can compute an equivalent 2×2 linear system:

$$x\,Y'(x) = (N_0 + N_1\,x + \cdots + N_k\,x^k + \cdots)\,Y(x)$$

    We want to find the number of formal solutions $Y(x) = x^{\rho_0}\,(Y_0 + Y_1\,x + \cdots)$ which are linearly independent and without logarithmic term. The coefficients Y_k satisfy the following recurrence relation:

$$((k + \rho_0)\,I - N_0)\,Y_k = \sum_{j=1}^{k} N_j\,Y_{k-j},\, k \in \mathbb{N}$$

    The coefficient Y_k is uniquely determined when $k + \rho_0 > \rho_1$. The $\rho_1 - \rho_0 + 1 = 3$ first equations can be written

$$\mathcal{M}\,\mathcal{Y} = 0$$

    where $\mathcal{Y}$ is the vector ${}^t(Y_0, \ldots, Y_{\rho_1 - \rho_0})$ and where $\mathcal{M}$ is the following 6×6 matrix:

$$\mathcal{M} = \begin{pmatrix} -I - N_0 & 0 & 0 \\ -N_1 & -N_0 & 0 \\ -N_2 & -N_1 & I - N_0 \end{pmatrix}$$

    Its determinant is zero and the 5×5 matrix obtained from the rows $2, 3, 4, 5, 6$ and the columns $1, 2, 3, 4, 5$ has the following determinant:

$$1152\,i\,(m_1 + m_2 + m_1\,m_2)\,(m_2^2 - m_1\,m_2 - m_2 - m_1 + m_1^2 + 1)^5$$

    It is non zero for each (m_1, m_2) in $\mathbb{R}_+^* \times \mathbb{R}_+^* - \{(1, 1)\}$.
    The kernel of the matrix $\mathcal{M}$ has one single element and the 2×2 system $Y' = F_1\,Y$ (resp. $Y' = F_2\,Y$) has $2 - 1$ formal solution with a logarithmic term.

  - $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 = 0$
    We study the factor F. We again have two exponents $\rho_0 = -1$ and $\rho_1 = 1$ at the point i. We construct the matrix $\mathcal{M}$ from which we extract the matrix with the rows $2, 3, 4, 5, 6$ and the columns $1, 2, 3, 4, 5$. The determinant of this submatrix is

$$(3 + i\,\sqrt{3})\,(s + i\,\sqrt{3} - 2\,i)4\,(s + 2i - i\,\sqrt{3})$$

and it is non zero. So again there is a formal solution with logarithmic terms for the 2×2 system $Y' = FY$.

2.4 Conclusion

We recall that both the parameters m_1 and m_2 are positive real. We have three situations:

- If $(m_1, m_2) = (1, 1)$ then the normal variationnal system has no parameter, it is completely reducible and has formal solutions with logarithmic terms at the point i.

- If $(m_1, m_2) \neq (1, 1)$ and $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 \neq 0$ then the normal variational system is *decomposable* i.e. equivalent to a block diagonal 4×4 system of the form
$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z.$$
The factors F_1 and F_2 both depend on the parameters m_1 and m_2. They cannot be simultaneously reducible and they both have formal solutions with logarithmic terms at the point i.

- If $(m_1, m_2) \neq (1, 1)$ and $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 = 0$ then the normal variational system is *factorizable* i.e. equivalent to a system
$$Z' = \begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z.$$
The factor F depends on the parameters, it is irreducible and has formal solutions with logarithmic terms at the point i.

We thus obtain the following theorem, which derives immediately from Criterion 1. and the above results :

Theorem 2. *The planar three-body problem is not completely integrable with meromorphic first integrals.*

The tools we have used overall were: quite big symbolic computations (but relatively easy for a computer) directed from mathematical insight of the problem, a small but crucial physical hypothesis (masses are positive real numbers), and relatively simple mathematical criteria. It turns out that many families of systems provided by Newtonian mechanics exhibit the features that we have heavily used (the variational equation is reducible, is relatively easy to factor for a computer directed by an expert hand, and there are logarithms in formal solutions). We thus believe that other expert hands than ours might fruitfully use this scheme of thought to prove other non-integrability results.

ANNEX 1

Normal variational system along Lagrange' solution.

$$Y'(x) = \begin{pmatrix} s_1 & s_2 & s_5 & 0 \\ s_3 & s_4 & 0 & s_5 \\ & & & \\ s_6 & s_7 & -s_1 & -s_3 \\ s_7 & s_8 & -s_2 & -s_4 \end{pmatrix} Y(x)$$

with

- $s_1 = \frac{m_2 \left(\left(-5\, {m_1}^2 - 5 - 14\, m_1\right)x^2 + 4\sqrt{3}(m_1-1)(m_1+1)x - (m_1-1)^2\right)}{4\,(m_1+m_2+1)\, m_1\, x\,(x2+1)}$

- $s_2 = \frac{-3\sqrt{3}(m_1-1)(m_1+1)m_2\, x^2 - 4\,(m_1+1)(m_1\, m_2 - 2\, m_1 + m_2)x + \sqrt{3}(m_1-1)(m_1+1)m_2}{4\,(m_1+m_2+1)\, m_1\, x\,(x2+1)}$

- $s_3 = \frac{-3\sqrt{3}(m_1-1)(m_1+1)m_2\, x^2 + \left(-4\, {m_1}^2 m_2 - 8\, m_1 - 4\, m_2 - 24\, m_1\, m_2 - 8\, {m_1}^2\right)x + \sqrt{3}(m_1-1)(m_1+1)m_2}{4\,(m_1+m_2+1)\, m_1\, x\,(x2+1)}$

- $s_4 = \frac{m_2 \left(\left({m_1}^2 + 1 + 10\, m_1\right) x^2 - 4\sqrt{3}(m_1-1)(m_1+1)\, x - 3\,(m_1+1)^2\right)}{4\,(m_1+m_2+1)\, m_1\, x\,(x2+1)}$

- $s_5 = \frac{(m_1+m_2+1)\,(x2+1)}{2\, m_1\, m_2\,(m_1+m_2+m_1\, m_2)2}$

- $s_6 = \Delta \frac{(m_1+1)(\left(-13\, {m_1}^2 m_2 - 2\, {m_1}^2 - 24\, m_1\, m_2 - 2\, m_1 - 13\, m_2\right)x^2 + 4\sqrt{3}(m_1-1)(m_1+1)m_2\, x - m_2\,(m_1-1)^2)}{(1+x^2)^2 x^2}$

- $s_7 = \Delta \frac{(-3\sqrt{3}\left({m_1}^2 m_2 + 2\, {m_1}^2 + 4\, m_1\, m_2 + 2\, m_1 + m_2\right)(m_1-1)x^2 - 4\, m_2\,(m_1+1)\left({m_1}^2 + 4\, m_1 + 1\right)x + \sqrt{3}(m_1-1)m_2\,(m_1+1)^2)}{(1+x^2)^2 x^2}$

- $s_8 = \Delta \frac{\left(-7\, {m_1}^2 m_2 + 10\, {m_1}^2 + 12\, m_1\, m_2 + 10\, m_1 - 7\, m_2\right)x^2 - 4\sqrt{3}(m_1-1)(m_1+1)m_2\, x - 3\, m_2\,(m_1+1)^2}{(1+x^2)^2 x^2}$

where $\Delta = \frac{m_2\,(m_1\, m_2 + m_2 + m_1)^3}{2\, m_1\,(m_1+m_2+1)^3}$.

ANNEX 2

Factorization of the normal variational system

- ANNEX 2A

$$\boxed{\text{Case } m_1 = m_2 = 1}$$

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

with

$$F_1 = \begin{pmatrix} -\frac{(3x^2+2x+1)(2x^4-5x^2+6x-1)}{x(3x^4-6x^2-1)(x^2+1)} & -\frac{(x^2-2x-1)(3x^4-2x^3+3x^2+2)}{x(3x^4-6x^2-1)(x^2+1)} \\ -\frac{x(9+20x^3+12x+9x^4+18x^2)}{(3x^4-6x^2-1)(x^2+1)} & -\frac{-4+3x^5-26x^3-5x-4x^4-8x^2}{(3x^4-6x^2-1)(x^2+1)} \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -\frac{(3x^2-2x+1)(2x^4-5x^2-6x-1)}{x(3x^4-6x^2-1)(x^2+1)} & \frac{(-1+2x+x^2)(3x^4+2x^3+3x^2+2)}{x(3x^4-6x^2-1)(x^2+1)} \\ \frac{x(9-20x^3-12x+9x^4+18x^2)}{(3x^4-6x^2-1)(x^2+1)} & -\frac{4+3x^5-26x^3-5x+4x^4+8x^2}{(3x^4-6x^2-1)(x^2+1)} \end{pmatrix}$$

- ANNEX 2B

$$\boxed{\begin{array}{c}\text{Case } \ (m_1, m_2) \neq (1,1) \text{ and} \\ 2\,m_1^2 + 2\,m_2^2 + 2 - 5\,m_1 - 5\,m_2 - 5\,m_1\,m_2 \neq 0\end{array}}$$

$$r^2 = m_1^2 + m_2^2 - m_1 m_2 - m_1 - m_2 + 1$$

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

$$F_2(m_1, m_2, r) = F_1(m_1, m_2, -r)$$

and the entries of the matrix F are given by

$$\begin{aligned}
&\Delta\,\mathbf{F_1}[\mathbf{1},\mathbf{1}] = \\
&-r\sqrt{3}\,(2 - 5\,m_2 + 2\,{m_2}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,{m_1}^2)\,(m_1 - 1)\,x^4 \\
&- 12\,r\,(m_1 + m_2 + 1)\,(m_1\,m_2 + m_2 - {m_1}^2 - 1)\,x^3 \\
&- 6\,r\sqrt{3}\,(2\,{m_1}^2 + m_1 + 2 + 2\,{m_2}^2 + m_1\,m_2 + m_2)\,(m_1 - 1)\,x^2 \\
&+ 4\,r\,(m_1 + m_2 + 1)\,(4\,{m_2}^2 - m_2 - m_1\,m_2 - 4\,m_1 + {m_1}^2 + 1)\,x \\
&- r\sqrt{3}\,(2 - 5\,m_2 + 2\,{m_2}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,{m_1}^2)\,(m_1 - 1)
\end{aligned}$$

$$\begin{aligned}
&\Delta\,\mathbf{F_1}[\mathbf{1},\mathbf{2}] = - \\
&(2 - 5\,m_2 + 2\,{m_2}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,{m_1}^2)(-2\,{m_2}^2 + 2\,r\,m_2 + 2\,m_2 + 2\,m_1\,m_2 - 2\,{m_1}^2 - r - m_1\,r + 2\,m_1 - 2)\,x^4 \\
&+ 4\sqrt{3}\,(m_1 - 1)\,r\,(m_1 + m_2 + 1)^2\,x^3 \\
&+(12\,m_1\,m_2 + 28\,{m_1}^3\,r - 28\,{m_2}^3 + 8 - 18\,r\,m_2 + 18\,m_1\,r - 28\,{m_1}^3\,m_2 - 28\,{m_1}^3 + 12\,{m_2}^2\,m_1 - 28\,m_1 - 18\,{m_1}^2\,r\,m_2 \\
&+ 36\,{m_2}^2 + 36\,{m_1}^2 + 36\,{m_2}^2\,{m_1}^2 + 18\,{m_1}^2\,r - 48\,m_1\,r\,m_2 + 28\,r + 8\,{m_2}^4 + 8\,{m_1}^4 \\
&- 28\,{m_2}^3\,m_1 - 28\,m_2 + 12\,{m_1}^2\,m_2 - 8\,r\,{m_2}^3)x^2 \\
&- 4\sqrt{3}\,(m_1 - 1)\,r\,(m_1 + m_2 + 1)^2\,x \\
&-48\,m_1\,r\,m_2 + 12\,m_1\,r\,{m_2}^2 - 9\,{m_1}^2\,r\,m_2 + 6\,m_1\,m_2 + 4 - 14\,{m_2}^3\,m_1 - 3\,{m_1}^2\,r + 18\,{m_2}^2\,{m_1}^2 + 6\,{m_2}^2\,m_1 - 3\,m_1\,r \\
&+ 18\,{m_1}^3\,r + 12\,r\,{m_2}^2 + 12\,r\,{m_2}^3 - 9\,r\,m_2 + 6\,{m_1}^2\,m_2 - 14\,{m_1}^3\,m_2 + 18\,r \\
&- 14\,m_2 - 14\,m_1 + 18\,{m_2}^2 + 18\,{m_1}^2 - 14\,{m_1}^3 - 14\,{m_2}^3 + 4\,{m_2}^4 + 4\,{m_1}^4
\end{aligned}$$

$$
\begin{aligned}
&\Delta\, \mathbf{F_1}[\mathbf{2},\mathbf{1}] = - \\
&(2 - 5\,m_2 + 2\,m_2{}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,m_1{}^2)(2\,m_2{}^2 + 2\,r\,m_2 - 2\,m_2 - 2\,m_1\,m_2 + 2\,m_1{}^2 - r - m_1\,r - 2\,m_1 + 2)\,x^4 \\
&+ 4\,\sqrt{3}\,(m_1 - 1)\,r\,(m_1 + m_2 + 1)^2\,x^3 \\
&+(-12\,m_1\,m_2 - 4\,m_1{}^3\,r + 28\,m_2{}^3 - 8 - 18\,r\,m_2 + 18\,m_1\,r + 28\,m_1{}^3\,m_2 + 28\,m_1{}^3 - 12\,m_2{}^2\,m_1 + 28\,m_1 - 18\,m_1{}^2\,r\,m_2 \\
&- 36\,m_2{}^2 - 36\,m_1{}^2 - 36\,m_2{}^2\,m_1{}^2 + 18\,m_1{}^2\,r + 48\,m_1\,r\,m_2 - 4\,r - 8\,m_2{}^4 - 8\,m_1{}^4 \\
&+ 28\,m_2{}^3\,m_1 + 28\,m_2 - 12\,m_1{}^2\,m_2 - 40\,r\,m_2{}^3)x^2 \\
&- 4\,\sqrt{3}\,(m_1 - 1)\,r\,(m_1 + m_2 + 1)^2\,x \\
&-4 + 48\,m_1\,r\,m_2 + 12\,m_1\,r\,m_2{}^2 - 9\,m_1{}^2\,r\,m_2 + 14\,m_2{}^3\,m_1 - 3\,m_1{}^2\,r - 18\,m_2{}^2\,m_1{}^2 - 6\,m_1\,m_2 - 3\,m_1\,r - 14\,m_1{}^3\,r + 12\,r\,m_2{}^2 \\
&- 6\,m_2{}^2\,m_1 - 9\,r\,m_2 - 6\,m_1{}^2\,m_2 + 14\,m_1{}^3\,m_2 - 20\,r\,m_2{}^3 - 14\,r + 14\,m_2 + 14\,m_1 \\
&- 18\,m_2{}^2 - 18\,m_1{}^2 + 14\,m_1{}^3 + 14\,m_2{}^3 - 4\,m_2{}^4 - 4\,m_1{}^4
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\, \mathbf{F_1}[\mathbf{2},\mathbf{2}] = \\
&r\,\sqrt{3}\,(2 - 5\,m_2 + 2\,m_2{}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,m_1{}^2)\,(m_1 - 1)\,x^4 \\
&+ 4\,r\,(m_1 + m_2 + 1)\,(4\,m_2{}^2 - m_2 - m_1\,m_2 - 4\,m_1 + m_1{}^2 + 1)\,x^3 \\
&+ 6\,r\,\sqrt{3}\,(2\,m_1{}^2 + m_1 + 2 + 2\,m_2{}^2 + m_1\,m_2 + m_2)\,(m_1 - 1)\,x^2 \\
&- 12\,r\,(m_1 + m_2 + 1)\,(m_1\,m_2 + m_2 - m_1{}^2 - 1)\,x \\
&+ r\,\sqrt{3}\,(2 - 5\,m_2 + 2\,m_2{}^2 - 5\,m_1 - 5\,m_1\,m_2 + 2\,m_1{}^2)\,(m_1 - 1)
\end{aligned}
$$

where $\Delta = 8(x^2+1)^2 r(m_2^2 - m_1m_2 - m_2 - m_1 + m_1^2 + 1)(m_1 + m_2 + 1)$.

- ANNEX 2C

$$
\boxed{\begin{array}{c}\text{Case } \ (m_1, m_2) \neq (1,1) \text{ and} \\ 2\,m_1^2 + 2\,m_2^2 + 2 - 5\,m_1 - 5\,m_2 - 5\,m_1\,m_2 = 0\end{array}}
$$

$$
\begin{array}{c}
m_1 = \frac{(5+3\sqrt{3})\,(s-2+\sqrt{3})\,(s-7+4\sqrt{3})}{(1-s)(1+s)} \\
m_2 = \frac{(5+3\sqrt{3})\,(s+2-\sqrt{3})\,(s+7-4\sqrt{3})}{(1-s)(1+s)} \\
2 - \sqrt{3} < |s| < 1 \text{ or } |s| < 7 - 4\sqrt{3}.
\end{array}
$$

$$
Z' = \begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z
$$

$$
\begin{aligned}
&-16\,(-s^2 - 7 + 4\,\sqrt{3})\,(x^2+1)^2\,F[1,\,1] = \\
&-8\,(1 - 6\,s + s^2)\,(-2 + \sqrt{3})\,x^3 \\
&+ 16\,(-s - 3 + 2\,\sqrt{3})\,(3\,s - 3 + 2\,\sqrt{3})\,x^2 \\
&- 8\,(\sqrt{3} + 2)\,(-s^2 + 56\,\sqrt{3} + 24\,s\,\sqrt{3} - 97 - 42\,s)\,x
\end{aligned}
$$

$$
\begin{aligned}
&-16\,(-s^2 - 7 + 4\,\sqrt{3})\,(x^2+1)^2\,F[1,\,2] = \\
&-8\,(-s - 3 + 2\,\sqrt{3})\,(3\,s - 3 + 2\,\sqrt{3})\,x^3 \\
&- 16\,(1 - 6\,s + s^2)\,(-2 + \sqrt{3})\,x^2 \\
&+8\,(-s - 3 + 2\,\sqrt{3})\,(3\,s - 3 + 2\,\sqrt{3})\,x \\
&+32\,s^2 + 224 - 128\,\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
&-16\,(-s^2-7+4\,\sqrt{3})\,(x^2+1)^2\,F[2,\,1]=\\
&-8\,(-s-3+2\,\sqrt{3})\,(3\,s-3+2\,\sqrt{3})\,x^3\\
&+16\,(\sqrt{3}+2)\,(-s^2+56\,\sqrt{3}+24\,s\,\sqrt{3}-97-42\,s)\,x^2\\
&+8\,(-s-3+2\,\sqrt{3})\,(3\,s-3+2\,\sqrt{3})\,x\\
&-32\,s^2-224+128\,\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
&-16\,(-s^2-7+4\,\sqrt{3})\,(x^2+1)^2\,F[2,\,2]=\\
&-8\,(\sqrt{3}+2)\,(-s^2+56\,\sqrt{3}+24\,s\,\sqrt{3}-97-42\,s)\,x^3\\
&-16\,(-s-3+2\,\sqrt{3})\,(3\,s-3+2\,\sqrt{3})\,x^2\\
&-8\,(1-6\,s+s^2)\,(-2+\sqrt{3})\,x
\end{aligned}
$$

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