Recent algorithms for solving second order differential equations

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We consider a second order ordinary linear differential equation

$$\widetilde{L}y \equiv \partial^2 y + A_1(x)\partial y + A_2(x)y = 0 \tag{1}$$

with rational coefficients , $A_1, A_2 \in C(x)$, over a constant field C which is assumed to be of characteristic zero and algebraically closed. We denote $\partial = \dfrac{d}{dx}$ and $K = C(x)$.

After a change of variable $y \to ye^{\int -\frac{a}{2}}$ the equation (1) is changed into the *reduced form*

$$L_r y \equiv \partial^2 y - r(x)y = 0 \quad \text{where} \quad r(x) = \frac{A_1^2}{4} + \frac{A_1'}{2} - A_2 \tag{2}$$

Given two linearly independent solutions of (1), say y_1, y_2, either formal or actual, the differential field $K < y_1, y_2 >$ generated by K, y_1 and y_2 is called a *Picard-Vessiot extension* of (1). The group of K-differential automorphisms (i.e., of field automorphisms leaving K pointly fixed and commuting with ∂) is called the *differential Galois group of* (1) *over* K. We denote it by $G(\widetilde{L}) = \mathrm{Gal}_K(\widetilde{L})$ and by $PG(\widetilde{L}) = G(\widetilde{L})/\mathcal{Z}(G(\widetilde{L})) \simeq G(\widetilde{L})/(G(\widetilde{L}) \cap C^*)$ the corresponding projective group.

A differential Galois group is a linear algebraic group over C ; it can then be represented as a subgroup of $GL(2, C)$. In the case of an operator in reduced form L_r the differential Galois group is a special linear algebraic group over C and it can thus be represented as a subgroup of $SL(2, C)$.

The Galois correspondence states the link between properties of solutions and the form of the differential Galois group. The equation(2) has no liouvillian solutions (also called solutions in closed form) if and only if the differential Galois group $G(L)$ is isomorphic to $SL(2, C)$. At the opposite end, all solutions are algebraic if and only if the differential Galois group $G(L)$ is a finite group. In the case when $G(L) \neq SL(2, C)$ since the order is only 2, then all solutions are liouvillian.

The Kovacic algorithm ([K86]) is an algorithm to effectively determine whether or not a second order linear differential equation has liouvillian solutions with a computation of those. It can be pushed up to the calculation of the differential Galois group of the equation in reduced form. What follows applies to general second order differential equations in form (1) as well as form (2).

This talk is concerned with the case when the solutions are algebraic and an explicit direct computation of those. The idea consists in refering to a small amount of standard equations the solutions of which were computed once for all. Using a theorem of Klein each equation is seen as an adequate pullback of one of the standard equations. Our aim is to make this pullback explicit.

1. **Standard equations**

The possible projective differential Galois groups in this case are the dihedral groups $\mathbf{D}_n$ for all $n \in \mathbb{N}$, the tetrahedral group $\mathbf{A}_4$, the octahedral group $\mathbf{S}_4$ and the icosahedral group $\mathbf{A}_5$.

The standard equations in reference are the hypergeometric equations

$$St_G = \partial^2 + \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{x(x-1)}$$

where the coefficients a, b, c are related to the differences λ, μ, ν of the exponents at $0, 1$, and ∞ by the relations $a = \dfrac{1-\lambda^2}{4}$ $b = \dfrac{1-\mu^2}{4}$ and $c = \dfrac{1-\nu^2+\lambda^2+\mu^2}{4}$. More precisely, one can choose $(\lambda, \mu, \nu) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n}\right)$ for $G = \mathbf{D}_n$, $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{3}\right)$ for $G = \mathbf{A}_4$, $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ for $G = \mathbf{S}_4$ and $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{5}\right)$ for $G = \mathbf{A}_5$.

The index G refers to the differential Galois group of the equation $St_G y = 0$ corresponding to the chosen values of a, b, c. The solutions of these hypergeometric equations are Legendre functions.

2. **Klein Theorem**

Definition 1. Let $L_1 \in C(z)\left[\frac{d}{dz}\right]$ and $L_2 \in C(x)\left[\frac{d}{dx}\right]$ be linear differential operators.

(1) L_2 is a *proper pullback* of L_1 by means of $f \in C(x)$ if the change of variable $z = f(x)$ changes L_1 into L_2.

(2) L_2 is a *pullback* of L_1 by means of $f \in C(x)$ if there exists $v \in C(x)$ such that $L_2 \otimes (\partial + v)$ is a proper pullback of L_1 by means of f.

Theorem 1. *Let L be a second order linear differential operator over $C(x)$ in reduced form with projective differential Galois group PG.*

Then, $PG \in \{\mathbf{D}_4, \mathbf{A}_4, \mathbf{S}_5, \mathbf{A}_5\}$ if and only if L is a pullback of St_G.

Let L have a projective differential Galois group PG and suppose the standard equation with projective differential Galois group PG has H_1, H_2 as a C-basis of solutions. The theorem of Klein says that L is a pullback of St_{PG}. Suppose we know f and v. Then, a C-basis of solutions of $Ly = 0$ is given by $H_1(f(x))e^{\int v}$ and $H_2(f(x))e^{\int v}$.

H_1 and H_2 are known for all standard equations. To get the solutions in explicit form one should then determine the projective differential Galois group and, in case it is finite, determine the pullback f and v. The idea is to build these quantities using the semi-invariants of the equation.

3. **Invariants and semi-invariants**

Definition 2. (1) A polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is said *invariant* with respect to a differential operator L if its evaluation on a C-basis y_1, y_2 of solutions is invariant under the action of the differential Galois group $G(L)$ of L. The rational function $h(x) = I(y_1(x), y_2(x))$ is called the *value* of the invariant polynomial I.

(2) A polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is said *semi-invariant* with respect to a differential operator L if the logarithmic derivative $\dfrac{h'}{h}$ of its evaluation $h(x) = I(y_1(x), y_2(x))$ on any C-basis y_1, y_2 of solutions is rational, i.e., in $C(x)$.

The invariant polynomials (in short invariants) of degree m of a differential equation $Ly = 0$ are elements of the m^{th} symmetric power $\mathrm{Sym}^m(\mathrm{Sol}(L))$. Their values are elements of the space $\mathrm{Sol}(\mathrm{Sym}^m(L))$. An isomorphism between these two spaces preserving the Galois representations allows to identify an invariant to its value. As a consequence, determining the invariants or the semi-invariants of degree m of L is equivalent to determining the rational solutions of the m^{th} symmetric power $\mathrm{Sym}^m(L)$ of L. On another hand, we know the full set of possible m since we know the list of invariants and semi-invariants of the finite groups $\mathbf{D}_n, \mathbf{A}_4, \mathbf{S}_4, \mathbf{A}_5$.

This provides us with a perfectly effective procedure to determine the invariants or semi-invariants of L and consequently the type of its differential Galois group.

Suppose now L has a differential Galois group G with semi-invariant S of degree m and value $\sigma(x)$. And suppose the value of S with respect to the standard operator St_G with group G equals σ_0 (modulo C^*). Then, the value of S w.r.t. both the differential operator $S_G = St_G \otimes (\partial_z + \dfrac{\sigma_0'}{m\sigma_0})$ and the differential operator $L = \widetilde{L} \otimes (\partial_x + \dfrac{\sigma'}{m\sigma})$ is equal to 1 and the following property holds.

Proposition 1. *L is a proper pullback $z = f(x)$ of S_G.*

A direct examination in each case will provide the pullback f.

4. Pullback formulæ

• **Primitive case:** $PG \in \{\mathbf{A}_4, \mathbf{S}_4, \mathbf{A}_5\}$

The standard equation in reference is $St_G y = 0$ where the differences of exponents are $\lambda = \dfrac{1}{3}$ at $x = 0$, $\mu = \dfrac{1}{2}$ at $x = 1$, and $\nu = \dfrac{1}{3}$ for $\mathbf{A}_4$, $\dfrac{1}{4}$ for $\mathbf{S}_4$ and $\dfrac{1}{5}$ for $\mathbf{A}_5$ at infinity.

The differential Galois group of this equation has a semi-invariant S of degree $m = 4$ in the case of $\mathbf{A}_4$, $m = 6$ in the case of $\mathbf{S}_4$ and $m = 12$ in the case of $\mathbf{A}_5$ with value $s(x) = x^{-m/3}(x-1)^{-m/4}$. The new standard equation $S_G = St_G \otimes (\partial + \dfrac{1}{3z} + \dfrac{1}{4(z-1)})$ reads

$$S_G = \partial^2 + \frac{7z-4}{6z(z-1)}\partial - \frac{1}{144}\frac{(6\nu-1)(6\nu+1)}{z(z-1)}.$$

It has exponents $\left(0, \dfrac{1}{3}\right)$ at 0, $\left(0, \dfrac{1}{2}\right)$ at 1 and $\left(\dfrac{6\nu+1}{12}, \dfrac{-6\nu+1}{12}\right)$ at infinity where ν has the previous value in each case. The semi-invariant S of degree m has now value 1. The coefficients of the pullback equation $\partial^2 y + a_1 \partial y + a_0 y = 0$ satisfy $a_1 = \dfrac{f''}{f'} + f'\dfrac{7f-4}{6f(f-1)}$ and $a_0 = -\dfrac{(6\nu-1)(6\nu+1)f'^2}{144f(f-1)}$.

Algorithm. Input: $\widetilde{L}$ with finite primitive group.

1. For $m \in \{4, 6, 12\}$ check for a semi-invariant of degree m and call v its logarithmic derivative.
2. If yes, let $L = \widetilde{L} \otimes (\partial + \dfrac{1}{m}v)$ be a proper pullback of S_G with invariant value 1.
Denote $L = \partial^2 + a_1\partial + a_0$.
3. Let $s = \dfrac{(6\nu-1)(6\nu+1)}{144}$ $(\nu \in \{\dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}\}$ is known).
Output: • the pullback function $f = \dfrac{1}{1 + \frac{s}{a_0}(6a_1 + 3\frac{a_0'}{a_0})^2}$ and

- for St_{A_4}, the basis of solutions $H_1 = {}_2F_1\left([\frac{-1}{12}, \frac{1}{4}], [\frac{2}{3}]; x\right)$ and $H_2 = \sqrt[3]{x}\; {}_2F_1\left([\frac{1}{4}, \frac{7}{12}], [\frac{4}{3}]; x\right)$;
  for St_{S_4}, the basis of solutions $H_1 = {}_2F_1\left([\frac{-1}{24}, \frac{5}{24}], [\frac{2}{3}], x\right)$ and $H_2 = \sqrt[3]{x}\; {}_2F_1\left([\frac{7}{24}, \frac{13}{24}], [\frac{4}{3}], x\right)$;
  for St_{A_5}, the basis of solutions $H_1 = {}_2F_1\left([\frac{11}{60}, -\frac{1}{60}], [\frac{2}{3}], x\right)$ and $H_2 = \sqrt[3]{x}\; {}_2F_1\left([\frac{31}{60}, \frac{19}{60}], [\frac{4}{3}], x\right)$.

${}_2F_1$ denotes the hypergeometric function. In the case of St_{A_4} the solutions can also be given in terms of radicals or roots of an algebraic equation of degree 24.

- **Dihedral case:** $PG = D_n$ for $n \in \mathbb{N}$.

The procedure is similar however, one has to determine here the value of n.

For sake of more symmetry in the formulas, the standard equation in reference is chosen with exponent differences $\dfrac{1}{2}$ at $+1$ and -1 and $\dfrac{1}{n}$ at infinity. It has a semi-invariant $S_2 = Y_1Y_2$ of degree 2 and two semi-invariants $S_{n,a} = Y_1^n + Y_2^n$ and $S_{n,b} = Y_1^n - Y_2^n$ of degree n. The new standard equation

$$S_{D_n} = \partial^2 - \frac{z}{z^2-1}\partial - \frac{1}{4n^2}\frac{1}{z^2-1}$$

has exponents $\left(0, \dfrac{1}{2}\right)$ at $+1$ and -1 and $\left(\dfrac{-1}{2n}, \dfrac{1}{2n}\right)$ at infinity ; it has a semi-invariant of degree 2 and value 1. The operator $L = \partial^2 + a_1\partial + a_0$ is a pullback of S_{D_n} if $a_0 = -\dfrac{1}{4n^2}\dfrac{f'^2}{f^2-1}$ and $a_1 = -\dfrac{1}{2}\dfrac{a_0'}{a_0}$. The equation $Ly = 0$ admits the solutions $\exp \displaystyle\int \pm\sqrt{-a_0} = \exp \int \frac{1}{2n}\frac{f'}{\sqrt{f^2-1}}dx$. The number n can thus be determined with the algorithm of integration on algebraic curves ([Br90], [Ri70], [Tr84]) ; in fact, the authors give refinements of this part of the algorithm to compute a multiple of n.

Algorithm. Input: $\widetilde{L} = \partial^2 + A_1(x)\partial + A_2(x)$ with finite differential Galois group.

1. Check for a semi-invariant of degree 2 and call v its logarithmic derivative.

2. If yes, let $L = \widetilde{L} \otimes (\partial + \dfrac{1}{m}v)$ be a proper pullback of S_{D_n} with invariant value 1.
Denote $L = \partial^2 + a_1\partial + a_0$.

3. Determine a candidate for a multiple of n.

4. For an adequate n, the equation $L_n y \equiv \partial^2 y + a_1\partial y + n^2 a_0 y$ has solutions f and $\sqrt{f^2-1}$, hence f.

5. Let c be such that $c^2 = \dfrac{4n^2a_0}{f'^2 + 4n^2f^2a_0}$.

Output: the pullback function $\pm cf$ and the solutions $\left(cf \pm \sqrt{c^2f^2-1}\right)^{1/n}$.

The procedure appears to be more efficient than the Kovacic algorithm. In addition, it provides the pullback and the solutions in simple form.

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