



## Liouvillian Solutions of Linear Differential Equations of Order Three and Higher

MARK VAN HOEIJ<sup>¶</sup>, JEAN-FRANÇOIS WEIL<sup>||</sup>, FELIX ULMER<sup>§\*\*</sup> AND  
JACQUES-ARTHUR WEIL<sup>‡</sup>

<sup>¶</sup> *Florida State University, U.S.A.*

<sup>‡</sup> *Université de Limoges, France*

<sup>§</sup> *Université de Rennes I, France*

---

Singer and Ulmer (1997) gave an algorithm to compute Liouvillian (“closed-form”) solutions of homogeneous linear differential equations. However, there were several efficiency problems that made computations often not practical. In this paper we address these problems. We extend the algorithm in van Hoeij and Weil (1997) to compute semi-invariants and a theorem in Singer and Ulmer (1997) in such a way that, by computing one semi-invariant that factors into linear forms, one gets *all* coefficients of the minimal polynomial of an algebraic solution of the Riccati equation, instead of only one coefficient. These coefficients come “for free” as a byproduct of our algorithm for computing semi-invariants. We specifically detail the algorithm in the cases of equations of order three (order two equations are handled by the algorithm of Kovacic (1986), see also Ulmer and Weil (1996) or Fakler (1997)).

In the appendix, we present several methods to decide when a multivariate polynomial depending on parameters can admit linear factors, which is a necessary ingredient in the algorithm.

© 1999 Academic Press

---

### 1. Introduction

In this paper  $k$  will denote a differential field whose field of constants  $\mathcal{C}$  is algebraically closed of characteristic 0. For the computations, however (Section 2.2), we will consider more concretely the field  $\mathbb{C}(x)$  with the usual derivation  $d/dx$ . We denote by  $L = \partial^n + a_{n-1}\partial^{n-1} + \dots + a_0\partial^0$  ( $a_i \in k$ ,  $L \in k[\partial]$ ) a differential operator and by  $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y = 0$  the corresponding linear homogeneous differential equation. We now briefly recall the basic definitions from differential Galois theory that are needed later on (see Magid, 1994; Singer, 1997; van der Put, 1998, for details and references).

A *Picard–Vessiot extension* (PVE)  $K$  of  $k$  for  $L$  is a differential field extension  $K = k \langle y_1, \dots, y_n \rangle$ , where  $\{y_1, \dots, y_n\}$  a fundamental set of solutions, with no new constants in  $K$ . It is the equivalent of a splitting field for  $L(y) = 0$ . Under our assumptions a PVE exists and is unique up to differential automorphisms. We denote by  $V(L)$  the solution space  $V(L) = \{y \in K \mid L(y) = 0\}$  of an operator  $L$ . The dimension of  $V(L)$  equals the order of  $L$ . The *differential Galois group*  $G$  of  $L$  is the group of differential

<sup>¶</sup>E-mail: hoeij@math.fsu.edu

<sup>||</sup>E-mail: ragot@unilim.fr, weil@unilim.fr

<sup>\*</sup>E-mail: ulmer@univ-rennes1.fr

automorphisms of  $K/k$ . It acts faithfully on the vector space  $V(L)$ , and so  $G$  can be viewed as a subgroup of  $GL(V(L))$ ; more precisely, it is a linear algebraic group over  $\mathcal{C}$ . There is a Galois correspondence between algebraic subgroups of  $G$  and differential subfields of the PVE of  $L(y) = 0$ . The fixed field of  $G$  under this correspondence is  $k$ .

A solution of  $L(y) = 0$  in  $k$  is called a *rational solution*, a solution in an algebraic extension of  $k$  is called an *algebraic solution*, a solution whose logarithmic derivative is in  $k$  is called an *exponential solution* and a solution belonging to a differential field obtained by successive adjunctions of integrals, exponentials of integrals and algebraic extensions is called a *Liouvillian solution*. If  $L$  has a Liouvillian solution then  $L$  also has a Liouvillian solution of the form  $y = \exp(\int r)$  where  $r \in \bar{k}$  is algebraic over  $k$  (see Singer, 1981, 1997).

DEFINITION 1.1. An element  $r$  of the PVE is called a *Riccati solution* for  $L$  if  $r$  is the logarithmic derivative  $r = y'/y$  of some non-zero solution  $y$  of  $L$ . If  $r$  is an algebraic Riccati solution (a Riccati solution in  $\bar{k}$ ) then the minimum polynomial of  $r$  over  $k$  is called a *Riccati polynomial* of  $L$ .

DEFINITION 1.2. An element  $I \in \text{Sym}^m(V(L))$  is called a *semi-invariant* of the differential Galois group  $G$  of  $L$  of degree  $m$  if there is a character  $\chi : G \mapsto \mathcal{C}$  of degree 1 such that for all  $g \in G$  the action of  $g$  on  $I$  is  $g(I) = \chi(g)I$ . If  $\chi$  is the trivial character, i.e.  $\forall g \in G, g(I) = I$ , then  $I$  is called an *invariant*.

We call a polynomial or semi-invariant *completely factorable* if it is a product of linear factors. It is known (see Section 2 in Singer and Ulmer, 1997) that an algebraic Riccati solution  $r$  of degree  $m$  over  $k$  corresponds to a completely factorable semi-invariant  $I$  of degree  $m$  of the differential Galois group  $G$ . In Singer and Ulmer (1997), the Riccati polynomial of  $r$  is computed as follows:

- (1) Compute the spaces of all semi-invariants up to a certain degree. This degree can be bounded in terms of the order of the operator using group theory.
- (2) In each such space, find a completely factorable semi-invariant  $I \in \text{Sym}^m(V(L))$  where  $m$  is the degree. If such a semi-invariant is found, then  $L(y) = 0$  has an algebraic Riccati solution and the first coefficient of the associated Riccati polynomial is given by the logarithmic derivative of the value of  $I$ . The value of  $I$  is the image of  $I$  in  $K$ .
- (3) In Singer and Ulmer (1997), the factorization of  $I$  into linear forms is used to compute the remaining coefficients. This involves computing with splitting fields and so this step could be costly.

The main objective in this paper is to give an efficient method to compute (if it exists) a Riccati polynomial. The goal is a method that is efficient enough to handle operators of order three on a computer, or sometimes even higher order if the Riccati polynomial is not too big.

First we extend the algorithm in van Hoeij and Weil (1997) for computing invariants to the case of semi-invariants. If a completely factorable semi-invariant is known and given in a certain form, then Theorem 2.1 immediately gives us all coefficients of the Riccati polynomial. This central result avoids the third step above and makes the computation much more feasible in practice.

In the appendix we discuss methods for finding the linear combinations of the basis corresponding to a completely factorable semi-invariant  $I$  (second step). If the dimension of the vector space of semi-invariants for a character is high then it is a very costly step. This can be resolved by looking at semi-invariants (not always of minimal degree) that are guaranteed to be completely factorable. This is particularly interesting because the second step above, i.e. selecting a completely factorable semi-invariant which can be a very costly step, is then no longer needed. The result is a Kovacic-like algorithm for  $\text{order}(L) = 3$ , given in Section 4.

## 2. Semi-invariants of Differential Galois Groups

### 2.1. THE FORMALISM

The basis of our approach is the following useful formalism:

**PROPOSITION 2.1.** (VAN HOEIJ AND WEIL 1997, PROPOSITION 2.5) *Let  $K$  be the Picard–Vessiot extension and  $G$  the differential Galois group of  $L$ . Define the  $\mathcal{C}$ -algebra homomorphism*

$$\phi : \text{Sym}(V(L)) \rightarrow K[X_1, \dots, X_n]$$

by (it suffices to define  $\phi$  for homogeneous elements  $\bar{y}$  of degree 1)

$$\phi(\bar{y}) = \sum_{i=1}^n X_i y^{(i-1)} \quad \text{for } y \in V(L).$$

Here  $\bar{y}$  is the element of  $\text{Sym}^1(V(L)) \subset \text{Sym}(V(L))$  that corresponds to the solution  $y \in V(L) \subset K$  of  $L$ . Then  $\phi$  is an embedding (as  $\mathcal{C}$ -algebra and as  $G$ -module) of  $\text{Sym}(V(L))$  in  $K[X_1, \dots, X_n]$ , where the action of  $G$  on  $X_1, \dots, X_n$  is trivial.

If  $I \in \text{Sym}(V(L))$ , then we say that  $\phi(I)$  is the *canonical image* of  $I$  in  $K[X_1, \dots, X_n]$ .

Often the two vector spaces  $V(L)$ , which is a subset of  $K$ , and  $\text{Sym}^1(V(L))$ , which is a subset of  $\text{Sym}(V(L))$ , are identified. In this paper we cannot directly do so because multiplication in  $K$  is not the same as multiplication in  $\text{Sym}(V(L))$ .

The following extension of Theorem 3 of Singer and Ulmer (1997) illustrates the usefulness of the above formalism and is central to our approach.

**THEOREM 2.1.** *Let  $L(y) = 0$  be a homogeneous linear differential equation over  $k$ . Let  $K$  be a Picard–Vessiot extension of  $k$  for  $L$  and  $G$  be its differential Galois group. Assume that  $G$  has a semi-invariant  $I \in \text{Sym}^m(V(L))$  that is a product of linear factors and assume that the degree  $m$  of  $I$  is minimal with this property. Let  $P(X_1, \dots, X_n) = \phi(I) \in K[X_1, \dots, X_n]$  be its canonical image. Let  $a$  be the coefficient of  $X_1^m$  in  $P$ . Then:*

- (1)  $a \neq 0$
- (2) Let  $Q := \frac{1}{a}P(X, -1, 0, \dots, 0)$ . Then  $Q \in k[X]$  and  $Q$  is a Riccati polynomial.

**PROOF.** The semi-invariant  $I$  is completely factorable, so  $I = \prod_{i=1}^m \bar{y}_i$  where  $\bar{y}_i \in \text{Sym}^1(V(L)) \subset \text{Sym}(V(L))$  and  $y_i \in V(L)$  are the corresponding solutions of  $L(y) = 0$ .

Thus, as  $P = \phi(I)$ , we have

$$P = \prod_{i=1}^m (y_i X_1 + y'_i X_2 + \cdots + y_i^{(n-1)} X_n).$$

If  $g \in G$  then  $g(y_i) \in \mathcal{C}y_j$  for some  $j$  because of the uniqueness of factorization in  $\text{Sym}(V(L))$  (which is isomorphic to a polynomial ring in  $n$  variables over  $\mathcal{C}$ ). Hence  $G$  permutes the set  $\left\{ \frac{y'_1}{y_1}, \dots, \frac{y'_m}{y_m} \right\}$ . The action of  $G$  on this set is transitive because  $m$  is minimal. Note that  $a = \prod y_i \neq 0$  because  $P \neq 0$ .

$$Q = \prod_{i=1}^m \frac{1}{y_i} (y_i \cdot X + y'_i \cdot (-1) + y''_i \cdot 0 + \cdots + y_i^{(n-1)} \cdot 0) = \prod_{i=1}^m \left( X - \frac{y'_i}{y_i} \right).$$

Since  $G$  permutes the  $\frac{y'_i}{y_i}$  it follows that  $Q$  is invariant under  $G$  and hence by the Galois correspondence  $Q \in k[X]$ .  $Q$  is irreducible because the action of  $G$  on the  $\frac{y'_i}{y_i}$  is transitive. The Riccati solutions  $\frac{y'_i}{y_i}$  are the roots of  $Q$  and hence they are algebraic functions with minimum polynomial  $Q$ .  $\square$

REMARK. With notations as above, if  $m$  is not minimal, then it is easily seen that the polynomial  $Q$  is either irreducible or a product of Riccati polynomials. Conversely, Theorem 3 of Singer and Ulmer (1997) (combined with our theorem) shows immediately that any Riccati polynomial is obtained that way. This is used implicitly in Section 3.

The algorithm presented in van Hoeij and Weil (1997) computes the invariants  $I$  of  $G$  by computing their images  $\phi(I)$ . The reason for using  $\phi$  in van Hoeij and Weil (1997) was to be able to compute invariants without having to compute symmetric powers  $\text{Sym}^m(L) \in k[\partial]$ . In the following, this same approach is extended to the computation of semi-invariants. The above then shows that this approach does not only produce the semi-invariant whose logarithmic derivative is the first coefficient of the Riccati polynomial, but in fact gives all the coefficients. This makes this algorithm for computing (semi)-invariants very suitable for finding Liouvillian solutions. We now proceed with the computation of semi-invariants via the above formalism. The next subsection is rather technical and will be easier if the reader knows about the paper of van Hoeij and Weil (1997) and about Beke's method for computing exponential solutions of operators (see van Hoeij, 1997; Singer, 1997; ?, for details and additional references).

Pfugel, 1998 — check  
year

## 2.2. COMPUTATION OF SEMI-INVARIANTS

LEMMA 2.1. *The image of a semi-invariant under the embedding  $\phi$  is an element of  $a \cdot k[X_1, \dots, X_n]$  where  $a = \exp(\int y)$  for some  $y \in k$ .*

PROOF. Let  $I$  be a semi-invariant of degree  $m$ , and  $P = \phi(I)$ . Because  $\phi$  is a  $G$ -homomorphism,  $g(P) = \chi(g) \cdot P$  for some character  $\chi$ . Let  $a \in K$  be one of the non-zero coefficients of  $P \in K[X_1, \dots, X_n]$ . Now  $g(\frac{1}{a}P) = g(\frac{1}{a})\chi(g)P = (ag(\frac{1}{a})\chi(g))\frac{1}{a}P$ . The action of  $G$  on the  $X_i$  is trivial, so that the coefficients of  $g(\frac{1}{a}P)$  are just the image of the corresponding coefficients under  $g$ . One of the coefficients of  $\frac{1}{a}P$  equals 1 and since  $g(1) = 1$  the corresponding coefficient of  $g(\frac{1}{a}P)$  must also be 1. Hence  $\forall g \in G, ag(\frac{1}{a})\chi(g) = 1$  and

so  $\frac{1}{a}P$  is invariant under  $G$ . By the Galois correspondence

$$\frac{1}{a}P \in k[X_1, \dots, X_n].$$

Furthermore  $\forall g \in G, g(a) = \chi(g) \cdot a$  showing that  $a'/a$  is left invariant by  $G$  and thus  $y = a'/a \in k$ .  $\square$

REMARK. Using the embedding  $\phi$ , we will often identify semi-invariants with elements of  $K[X_1, \dots, X_n]$ .

DEFINITION 2.1. For  $t \in k$ , we note  $\mathcal{S}_t^*$  the  $k$ -automorphism of  $k[\partial]$  defined by  $\mathcal{S}_t^*(\partial) = \partial + t$ .

DEFINITION 2.2. The equivalence relation  $\sim$  on  $k$  is defined as follows:  $t_1 \sim t_2$  when there exists  $y \in k$  for which  $t_1 - t_2 = y'/y$ .

Let  $V(L) \subset K$ ,  $a \in K$  and assume  $t = a'/a \in k$ . Then multiplication by  $a$  is a  $\mathcal{C}$ -linear map

$$\mu_a : V(L) \rightarrow V(\mathcal{S}_{-t}^*(L)).$$

The map  $\mu_a$  induces a 1-1 linear map

$$\mu_a : \text{Sym}^m(V(L)) \rightarrow \text{Sym}^m(V(\mathcal{S}_{-t}^*(L))).$$

Let  $S \in \text{Sym}^m(V(L))$ ,  $g \in G$ . Then  $g(\mu_a(S)) = \mu_a(g(S)) \cdot g(a^m)/a^m$  so if  $S$  is a semi-invariant then  $\mu_a(S)$  is an invariant iff  $g(S) = S \cdot a^m/g(a^m)$  for all  $g \in G$ .

If  $S$  is a semi-invariant then  $\phi(S)$  is of the form  $\phi(S) = b \cdot I$  for some  $I \in k[X_1, \dots, X_n]$ ,  $b \in K$  with  $t_1 = b'/b \in k$ . Then  $g(S) = S \cdot g(b)/b$ , so  $\mu_a(S)$  is an invariant iff  $g(b)/b = a^m/g(a^m)$  for all  $g \in G$ . This holds iff  $b \cdot a^m \in k$ , iff  $-tm \sim t_1$ .

Let  $L$  be a differential operator and  $K$  the Picard–Vessiot extension. Let  $S \in \phi(\text{Sym}^m(V(L))) \subset K[X_1, \dots, X_n]$  be a semi-invariant of degree  $m$ . So  $S = b \cdot I \in K[X_1, \dots, X_n]$  for some  $b \in K$ ,  $I \in k[X_1, \dots, X_n]$ . Let  $t_1 = b'/b \in k$ . The map  $\mu_a$  induces a map on  $\phi(\text{Sym}^m(V(L)))$  that we will denote by  $\phi_a$ . Then  $\phi_{\exp(\int -t/m)}(S)$  is an invariant of  $\mathcal{S}_{t/m}^*(L)$  iff  $t \sim t_1$ .

In order to compute all semi-invariants of degree  $m$ , we will construct from  $L$  a finite set  $B$ , such that whenever  $L$  has a semi-invariant in  $b \cdot k[X_1, \dots, X_n]$  then  $(b'/b \bmod \sim) \in B$ . Then for each  $t_1 \in B$  we take a  $t \in k$  with  $t \sim t_1$  and determine the invariants of  $\mathcal{S}_{t/m}^*(L)$ . If  $I \in k[X_1, \dots, X_n]$  is an invariant of  $\mathcal{S}_{t/m}^*(L)$ , then the corresponding semi-invariant of  $L$  is  $S = \phi_{\exp(\int t/m)}(I)$ . The remaining problem is to determine the set  $B$ , which will be done using the generalized exponents, in a way similar to the computation of the bounds in van Hoeij and Weil (1997). This leads to the algorithm below.

In the rest of this section, we assume that  $k = \mathbb{C}(x)$ .

#### Algorithm Semi-invariants.

**Input:** a differential operator  $L \in \mathbb{C}(x)[\partial]$  of order  $n$ , and an integer  $m$ .

**Output:** all semi-invariants of degree  $m$ .

**Step 1.** Compute a finite set  $B \subset \mathbb{C}(x)/\sim$  such that for all semi-invariants  $aI$  of  $L$ ,  $a \in K$ ,  $I \in \mathbb{C}(x)[X_1, \dots, X_n]$  the equivalence class of  $\frac{a'}{a}$  is an element of  $B$ .

**Step 2.** For each element of  $B$  take a representant  $t \in \mathbb{C}(x)$  and compute the invariants of  $\mathcal{S}_{t/m}^*(L)$ . These correspond to semi-invariants of  $L$ .

Note that this algorithm resembles Beke's algorithm for computing exponential solutions  $\exp(\int t)$  with  $t \in \mathbb{C}(x)$ . We give the same sketch of this algorithm as in Section 4 in van Hoeij (1997):

**Step 1.** Compute a finite set  $B \subset \mathbb{C}(x)/\sim$  such that for all exponential solutions  $a$  one has  $(\frac{a'}{a} \bmod \sim) \in B$ .

**Step 2.** For each element of  $B$  take a representant  $t \in \mathbb{C}(x)$  and compute the rational solutions of  $\mathcal{S}_t^*(L)$ . Return the corresponding exponential solutions of  $L$ .

*Computation of set  $B$  in step 1 of algorithm semi-invariants:*

In van Hoeij and Weil (1997) and van Hoeij (1997) a generalization of the classical notion of exponents has been defined. A generalized exponent is an element of  $\mathbb{C}[x^{-1/r}]$  for some integer  $r$  (called the ramification index). In the regular singular case the generalized exponents are the exponents and the ramification index is 1 in this case. For  $t \in \mathbb{C}(x)$  and  $p \in \mathbb{P}^1(\mathbb{C})$  denote  $EP_p(\partial - t) \in \mathbb{C}[x^{-1}]/\mathbb{Z}$  as the generalized exponent  $e \in \mathbb{C}[x^{-1}]$  at the point  $p$  of  $\partial - t$  modulo the integers.  $EP_p(\partial - t)$  is the *exponential part* of  $\partial - t$  at  $p$ . Now  $t$  modulo  $\sim$  is determined by the  $EP_p(\partial - t)$  for all  $p$  (van Hoeij, 1997).

Let  $e \in \mathbb{C}[x^{-1}]$  be the generalized exponent of  $\partial - t$  at the point  $p$ . Then  $e/m$  is the generalized exponent of  $\partial - t/m$  at  $p$ . The generalized exponents of  $\mathcal{S}_{t/m}^*(L)$  at  $p$  equal  $-e/m$  plus the generalized exponents of  $L$  at  $p$ . So the generalized exponents of the monomials in lemma 2.8 in Section 4.1 in van Hoeij and Weil (1997) for  $\mathcal{S}_{t/m}^*(L)$  are  $-e$  plus the generalized exponents of the monomials for  $L$ . According to this lemma, at least one of those generalized exponents should be in  $\frac{1}{r}\mathbb{Z}$  (where  $r$  is the ramification index at  $p$ ) if  $\mathcal{S}_{t/m}^*(L)$  has an invariant. This leaves only finitely many possibilities for  $e$  modulo  $\mathbb{Z}$  and hence for  $EP_p(\partial - t) = e + \mathbb{Z} \in \mathbb{C}[x^{-1}]/\mathbb{Z}$ .

At regular points, there is only one possibility for  $e + \mathbb{Z}$ , namely  $e + \mathbb{Z} = \mathbb{Z}$ , because  $r = 1$  and the generalized exponents are integers at regular points.

At a singular point  $p$ , we can compute all generalized exponents of  $L$ , and all generalized exponents of the monomials in the lemma (these are sums of  $m$  generalized exponents of  $L$ ) and so we can compute a finite set of possible values of  $EP_p(\partial - t)$ . If  $p_1, \dots, p_l$  are the singularities, and we find  $N_i$  possible  $EP_{p_i}(\partial - t)$  then we have  $N = N_1 \cdot N_2 \cdots N_l$  possible combinations, so we find at most<sup>†</sup>  $N$  possible values of  $t$  modulo  $\sim$ . These  $t$  modulo  $\sim$  are the elements of the set  $B$  in the algorithm.

A comment on step 2: we need to compute invariants of a number of operators (one for each element of  $B$ ). It can be faster first to apply the heuristic algorithm in van Hoeij and Weil (1997), and to apply the complete algorithm invariants in van Hoeij and Weil (1997) only for the cases when the heuristic does not prove that no invariants exist.

Another efficiency improvement can be obtained as follows. For computing the invariants of  $\mathcal{S}_{t/m}^*(L)$  we need the generalized exponents of  $\mathcal{S}_{t/m}^*(L)$  at all singularities. However, these can be easily computed from  $t$  and the generalized exponents of  $L$ . Furthermore, we compute with formal solutions of  $\mathcal{S}_{t/m}^*(L)$  at one singularity, but these can be computed from the formal solutions of  $L$ . This way we only need to compute generalized exponents and formal solutions of  $L$ , not of each  $\mathcal{S}_{t/m}^*(L)$  for every  $t \in B$ .

Our algorithm for computing semi-invariants has the same drawback as Beke's algorithm for computing exponential solutions: if the singularities or the generalized exponents involve algebraic extensions over the coefficients field then the algorithm may need to compute in exponentially large algebraic extensions. For  $m = 1$  it is in fact the same

<sup>†</sup>Not every combination of exponential parts corresponds to a  $t$ , only those combinations that satisfy a generalized Fuchs relation (Lemma 9.2, van Hoeij, 1997).

algorithm. If the set  $B$  is very large then the computation can also be long. However, in many examples the generalized exponents are rational numbers, and the number of singularities is small (typically three) so that  $B$  will not be big. In such examples computing semi-invariants will not be so hard.

Sometimes, e.g. step [2.1] in Section 4.1, we need to compute only those semi-invariants whose  $n$ th power is rational (with  $n \in \mathbf{N}$  given). The computation is then shorter: in step 2 of algorithm semi-invariants, we need only to consider those  $t \in B$  for which  $nt \sim 0$  (i.e. the generalized exponents of  $\partial - t$  are in  $\frac{1}{n}\mathbf{Z}$ ). This will be used in Section 4.

Invariants are rational solutions of a system of differential equations, the symmetric power system  $Y' = S^m(A)Y$  in Section 2.1.2 of van Hoeij and Weil (1997). Semi-invariants are exponential solutions of this system. So invariants and semi-invariants can be obtained by applying the algorithms in Barkatou (1998) and Pflügel (1997) on the symmetric power system. However, this can be a costly computation because the dimension of the  $S^m(A)$  can be very high, and because the algorithms in Barkatou (1998) and Pflügel (1997) are general, so they do not benefit from the special structure of the symmetric power system.

### 2.3. A FOURTH-ORDER EXAMPLE WITH GROUP FP28

The following operator (constructed with the methods of van der and Ulmer (1998)) has a Galois group which is a central extension of  $S_5$  (FP28 in the notation of Hessinger (1998)).

$$L = \partial^4 + 4 \frac{(2x-1)\partial^3}{x(x-1)} + \frac{(2295x^2 - 2032x + 105)\partial^2}{160x^2(x-1)^2} + \frac{(1500x^2 - 287x + 105)\partial}{320(x-1)^2x^3} + \frac{7(148000x + 9375x^4 - 10088x^3 - 67250x^2 - 73125)}{2560000x^4(x-1)^4}.$$

Its exponents at the singularities 0, 1, and  $\infty$  are

$$E_0 = \left\{ -\frac{7}{8}, \frac{1}{8}, \frac{9}{8}, \frac{13}{8} \right\}, \quad E_1 = \left\{ \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10} \right\}, \quad \text{and} \quad E_\infty = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}.$$

The group FP28 has semi-invariants of degree five, including a completely factorable semi-invariant. Following our method, we do the following to compute all semi-invariants of degree five. Let  $E_p + E_p + E_p + E_p + E_p$  be the set of all sums of five exponents at  $x = p$ . Let  $F_p$  be this set modulo  $\frac{1}{r_p}\mathbf{Z}$  where  $r_p$  is the ramification index. All ramification indices  $r_p$  are one in this example, so  $F_p$  is just  $E_p + \dots + E_p$  modulo the integers (i.e. we take the fractional parts of the elements of  $E_p + \dots + E_p$ ). Then one finds  $F_0 = \{1/8, 5/8\}$ ,  $F_1 = \{1/10, 3/10, 1/2, 9/10, 7/10\}$  and  $F_\infty = \{1/8, 3/8, 7/8, 5/8\}$ . We need to try  $\mathcal{S}_{i/5}^*(L)$  for all  $t = c_0/x + c_1/(x-1)$  for which

- (1)  $0 \leq c_i < 1$  for  $i \in \{0, 1\}$ .
- (2)  $(c_p + \frac{1}{r_p}\mathbf{Z}) \cap F_p$  is non-empty for  $p \in \{0, 1, \infty\}$ , where  $c_\infty = -c_0 - c_1$ .

In general the  $c_i$  are taken in  $\mathbb{C}[x^{-1/r_p}]$  modulo  $\frac{1}{r_p}\mathbf{Z}$ , but since all generalized exponents are rational numbers we can take  $c_i$  to be non-negative rational numbers smaller than one. The above conditions leave only very few (namely two) cases for  $t$ , so computing all semi-invariants of  $L$  of degree five does not take much computer time here.

Let  $\tilde{L} = \mathcal{S}_{t/5}^*(L)$  where  $t = 5/(8x) + 1/(2(x-1))$ . With the algorithm from van Hoeij and Weil (1997), we compute the two-dimensional space of invariants of degree five of  $\tilde{L}$ , and determine an invariant that factors into linear factors using the appendix. We then apply Theorem 2.1 and obtain the following Riccati polynomial:

$$\begin{aligned}
P = & X^5(122683392x^3 + 246481521x^2 + 184373150x - 557519375) \\
& + X^4(245366784x^3 + 739444563x^2 + 737492600x - 2787596875)/x \\
& + X^3(966131712x^4 + 3750651762x^3 + 2746408210x^2 \\
& \quad - 33407163250x + 27875968750)/(5(x-1)x^2) \\
& + X^2(1878589440x^5 + 10760213334x^4 + 11619977980x^3 \\
& \quad - 183778983600x^2 + 297197002500x - 139379843750)/(25(x-1)^2x^3) \\
& + X(1808142336x^6 + 15408360513x^5 + 43238642915x^4 \\
& \quad - 522670373950x^3 + 1182921498750x^2 - 1068395471875x + \\
& \quad \quad 348449609375)/(125(x-1)^3x^4) \\
& + X^0(3451908096x^7 + 41967774147x^6 + 302211588068x^5 \\
& \quad - 3042266659625x^4 + 8490891950000x^3 - 11023761109375x^2 \\
& \quad + 6968992187500x - 1742248046875)/(3125(x-1)^4x^5).
\end{aligned}$$

Let  $r \in \overline{\mathbb{C}(x)}$  be a root of  $P$ . Then  $\exp(\int r dx)$  is a Liouvillian solution (in fact: algebraic solution, but we have not computed its minimum polynomial) of  $\tilde{L}$ . To obtain a solution of  $L$  we have to multiply by  $\exp(\int t/5 dx)$ , so we find the following Liouvillian solution of  $L$ :

$$\exp\left(\int \left(r + \frac{t}{5}\right) dx\right) = x^{1/8}(x-1)^{1/10} \cdot \exp\left(\int r dx\right) \in V(L).$$

We could also compute a semi-invariant of  $L$  from the invariant of  $\tilde{L}$ , and apply Theorem 2.1 on that. The Riccati polynomial obtained that way has  $r + \frac{t}{5}$  as its solution, so it can be also be obtained from  $P$  by a substitution.

In most examples we prefer to compute a Riccati polynomial through invariants, not semi-invariants, because invariants are easier to compute. However, FP28 has a five-dimensional space of invariants of degree two, and no invariants of lower degree. So computing a completely factorable invariant will be very difficult because it has to be selected from a high-dimensional space. So for FP28 it is easier to use semi-invariants.

#### 2.4. CONSTRUCTORS OF SEMI-INVARIANTS

In this part, we list some results from Weil (1995b) that will be used in Section 4. Let  $X$  denote the transposed vector  $(X_1, \dots, X_n)^t$ . Let  $D$  be the derivation on  $K[X_1, \dots, X_n]$  defined by  $D(X) = -A^t X$  and  $D$  is the usual derivation on  $K$ . Then,  $\phi(\text{Sym}(V(L))) = \{P \in K[X_1, \dots, X_n] : D(P) = 0\}$  (Weil, 1995a).<sup>†</sup>

Let  $S$  be a semi-invariant of degree  $m$ , and  $P = \phi(S)$  be its canonical image in  $K[X_1, \dots, X_n]$ . In this setting, it is shown in Weil (1995b, Proposition 50, Section 6, p. 57) that the Hessian of  $P$  (with respect to  $X_1, \dots, X_n$ ) is again, up to multiples, the

<sup>†</sup>In fact, this derivation turns  $K[X_1, \dots, X_n]$  into a differential module isomorphic to  $\text{Sym}(D/DL)$ .



canonical image of a semi-invariant (same results hold for the bordered Hessian and the Jacobian). Thus, once a semi-invariant is known, we can directly apply Hessians and related constructors on its canonical image Weil (see Weil 1995b, Section 6.3, p. 65 for a fairly detailed example). In view of Theorem 2.1, this will allow us (in Section 4) to construct Riccati polynomials of high degrees directly from invariants of small degree.

### 3. Computing Liouvillian Solutions of Equations of Order $n$

LEMMA 3.1. *If  $L$  is reducible and  $L$  has a Liouvillian solution then  $L$  has an irreducible right-hand factor  $R \in \mathbb{C}(x)[\partial]$  that has a Liouvillian solution.*

PROOF. The Liouvillian solutions of  $L$  form a subspace of  $V(L)$  that is invariant under the differential Galois group. Hence there exists a right-hand factor  $R'$  having this subspace as solution space. Now take for  $R$  any irreducible right-hand factor of  $R'$ .  $\square$

From the fact that an operator  $M$  that has a Liouvillian solution must have a solution of the form  $\exp(\int r)$  for some algebraic function  $r \in \overline{\mathbb{C}(x)}$ , i.e.  $M$  has a right-hand factor of order one in  $\overline{\mathbb{C}(x)[\partial]}$ , it follows by induction that if all solutions are Liouvillian then  $M$  is completely factorable in  $\overline{\mathbb{C}(x)[\partial]}$  (i.e.  $M$  is a product of operators of order one). Conversely, if  $M$  is completely factorable then all solutions are Liouvillian. Hence the operator  $R'$  in the lemma is the highest-order right-hand factor of  $L$  that is completely factorable in  $\overline{\mathbb{C}(x)[\partial]}$ .

- (1) Reducible case. If  $L$  is reducible and has a Liouvillian solution then according to the lemma there exists an irreducible right-hand factor that has a Liouvillian solution. Though there may be infinitely many different right-hand factors, there are only finitely many *types* (Tsarev, 1996; Singer, 1995) of irreducible right-hand factors. For each type we need to compute one right-hand factor and compute the Liouvillian solutions of that factor. This way we find a Liouvillian solution of  $L$ , if such solution exists.
- (2) Irreducible case. In Singer and Ulmer (1993b) and references therein, the authors give an upper bound for the lowest degree of (if it exists) a Riccati polynomial of  $L$ . So, by checking a finite number of degrees, one can decide if a Riccati polynomial exists. For each possible degree  $m$ , compute the semi-invariants of degree  $m$  and check (see Appendix A) if there is a completely factorable semi-invariant. If so, return the corresponding Riccati polynomial.

In the algorithms the irreducible case is usually split into the imprimitive and the primitive case (containing only a finite list of possible groups); this is explained in Singer and Ulmer (1993b).

Several ideas from Section 4 below can be re-used when handling equations of higher order (e.g. the fact that knowing in advance which groups to check for allows us to check only a small number of degrees for the semi-invariants). The important classification work of Hessinger (1998) should pave the way to a better algorithm for computing Liouvillian solutions of equations of order four.

#### 4. Computing Liouvillian Solutions of Equations of Order Three

The general algorithm can be improved in several ways. First of all, the computation of semi-invariants can often be avoided. For example, in the case that  $L$  is irreducible of order two, we know from Ulmer and Weil (1996) that it is sufficient to compute only invariants, which are easier to compute than semi-invariants.

The factorization of a semi-invariant (cf. Appendix A) becomes a problematic step if the number of parameters is not small. For second-order equations this problem does not occur because any invariant will factor into linear forms. For third-order equations we will use Section 2.4 to find invariants that are guaranteed to factor, so that Appendix A can be avoided in most cases.

Furthermore we can reduce the size and number of different degrees  $m$  that need to be checked by studying the possible Galois groups.

In this section, we develop along these guidelines a Kovacic-like algorithm for computing Liouvillian solutions of linear differential equations of order three. For the possible Galois groups, we follow the notations from Singer and Ulmer (1993a,b).

##### 4.1. THE ALGORITHM

Let  $L = \partial^3 + a_2\partial^2 + a_1\partial + a_0$  where the  $a_i \in \mathbb{C}(x)$ . If there exists an  $f \in k$  such that  $f'/f = a_2$ , then  $G \subset SL_3$ ; if not, then we can first apply  $\mathcal{S}_{-a_2/3}^*$  to  $L$  so that the coefficient  $a_2$  will vanish. Hence, we may assume in all that follows that the Galois group is unimodular, i.e.  $G \subset SL_3$ .

REMARK. In the algorithm below, the sentences “there are  $s$  invariants of degree  $m$ ” should be understood as “the vector space of all invariants of degree  $m$  has dimension  $s$ ”.

(1) Reducible case.

[1.1] If  $L$  can be written as  $L_1 \cdot (\partial - r)$  for some  $r \in \mathbb{C}(x)$  then let  $a = \exp(\int r)$  and compute a basis  $y_1, y_2$  of solutions of  $L_1$ . Return  $a, a \int (y_1/a), a \int (y_2/a)$ .

[1.2] If  $L$  allows only a factorization  $L = L_1L_2$ , where  $L_2$  has order two. Then apply the Kovacic algorithm on  $L_2$ . If a basis of Liouvillian solutions is found, then (assume that  $L$  is monic)  $L$  can be factored as  $(\partial - r)(\partial - a)(\partial - b)$  for some algebraic functions  $a$  and  $b$  and rational function  $r$ . Let  $z_1 = \exp(\int (a - b)dx)$  and  $z_2 = \int \exp(\int (r - a)dx)dx$ . Then  $y_1 = \exp(\int bdx)$ ,  $y_2 = y_1 \int z_1 dx$  and  $y_3 = y_1 \int z_1 z_2 dx$  is a basis of Liouvillian solutions of  $L$ .

(2) Test if the group is imprimitive by doing one of the following steps, [2.1] or [2.2]:

[2.1] Compute all semi-invariants of degree three whose square is invariant (see the remark at the end of Section 2.2). Decide if one of these semi-invariants has a linear factor (use Appendix A if there is a semi-invariant linearly depending on parameters).<sup>†</sup> If this is the case then the group is imprimitive; return the Riccati polynomial  $Q$  using Theorem 2.1. Otherwise the group is not imprimitive; proceed with step three.

[2.2] Compute invariants of degree six and decide which of those is a square (see Appendix A.4). The corresponding square roots are all semi-invariants of degree three whose square is invariant. Now proceed as in [2.1].

<sup>†</sup>I.e. a vector space of semi-invariants corresponding to the same character.

- (3) Compute the invariants of degree two. If there is one (there cannot be more) then compute the invariants of degree six and one of the following two cases applies; otherwise proceed with step four.
- [3.1] Either there is ONE invariant of degree six; then  $G \simeq PSL_2^{SL_3}$  and there is no Liouvillian solution.
- [3.2] Or there are TWO invariants of degree six; then  $G \simeq A_5^{SL_3}$ . Return either the unique Riccati polynomial of degree six (the corresponding completely factorable invariant must be found in a space of dimension two) or the unique Riccati polynomial of degree 15 obtained from Section 4.3.
- (4) Compute invariants of degree four. If there is one (there cannot be more), then  $G \simeq G_{168}^{SL_3}$ . Return the Riccati polynomial of degree 21 obtained in Section 4.3
- (5) Compute invariants of degree six:
- [5.1] If there are NO such invariants then one of the following two cases applies; otherwise proceed with step [5.2].
- [5.1.1] If there is one invariant of degree nine (there cannot be more), then  $G \simeq H_{216}^{SL_3}$ . Return the corresponding Riccati polynomial of degree nine.
- [5.1.2] Else, the group is  $SL_3$  and there are no Liouvillian solutions.
- [5.2] If there is ONE invariant of degree six, then one of the following four cases applies; if there are more invariants of degree six then proceed with step [5.3].
- [5.2.1] If the invariant is a cube (Appendix A.4), then  $G \simeq PSL_2^{SL_3} \times C_3$  and there are no Liouvillian solutions.
- [5.2.2] If there is one invariant of degree nine (there cannot be more), then  $G \simeq H_{72}^{SL_3}$ . Return the corresponding Riccati polynomial of degree nine.
- [5.2.3] If there is an invariant<sup>†</sup> of degree 12 which is the cube of a semi-invariant  $S$  (Appendix A.4), then  $G \simeq G_{168}^{SL_3} \times C_3$ . Return the Riccati polynomial of degree 21 corresponding to the unique invariant of degree 21 obtained in Section 4.3.
- [5.2.4] Else  $G \simeq A_6^{SL_3}$ . Return the Riccati polynomial of degree 45 corresponding to the unique invariant of degree 45 obtained in Section 4.3.
- [5.3] If there are TWO invariants of degree six, then one of the following two cases applies:
- [5.3.1] If there is one invariant of degree nine (there cannot be more), then  $G \simeq F_{36}^{SL_3}$ ; Return the corresponding Riccati polynomial of degree nine.
- [5.3.2] Else,  $G \simeq A_5^{SL_3} \times C_3$ . Return either the unique Riccati polynomial of degree six (the corresponding completely factorable invariant must be found in a space of dimension two) or the Riccati polynomial of degree 15 obtained in Section 4.3.

#### 4.2. PRACTICAL REMARKS ON THE ALGORITHM

We note that, in practice, some cases from the algorithm can be quickly excluded by the necessary conditions from Singer and Ulmer (1995) and/or the heuristic from Section 4.1 in van Hoeij and Weil (1997). For example, if the equation is not Fuchsian, then there are Liouvillian solutions iff step (1) or step (2) succeeds.

One consequence of the above algorithm is that we can decide whether an irreducible third-order equation has Liouvillian solutions by using only invariants of degrees two, four, six, and nine.

<sup>†</sup>Alternatively, at this step, one could directly search for a (unique) semi-invariant of degree four whose cube is rational. Or, to avoid factorization, one could compute the dimension of the space of invariants of degree 18 (dimension two for  $G_{168}^{SL_3} \times C_3$  and dimension three for  $A_6^{SL_3}$ ).

Another remark concerns the rationality problem. The algorithm we use for computing invariants, unlike for semi-invariants, does not introduce algebraic extensions in its output  $\phi(I)$ . The only case of an irreducible operator  $L$  of order three where we compute a Riccati polynomial that may require an extension of the constant field is the imprimitive case Hendriks and van der Put (there may be an extension of degree at most four, see 1995); Ulmer (there may be an extension of degree at most four, see 1994), because this is the only case where semi-invariants and Appendix A need to be used.

*Check year for Hendriks and Van der Put, 1994*

Choosing step [2.2] instead of [2.1] to handle the imprimitive case has several extra advantages. For example, in steps (3) and (5), we need not recompute the invariants of degree six (note, however, that one cannot safely perform step (3) before step (2)). Also, if the dimension of the space of invariants of degree six is bigger than two, then we know automatically that the group is imprimitive (this follows from the properties of the primitive subgroups of  $SL_3(\mathbb{C})$ ). Furthermore we do not need to compute semi-invariants this way.

#### 4.3. CORRECTNESS OF THE ALGORITHM

The fact that the successions of tests proposed in the algorithms yield the correct groups follow from the decomposition of the symmetrisations of the characters (Singer and Ulmer, 1993a) for all primitive subgroups of  $SL_3$ . To prove that the corresponding degrees of algebraic Riccati solutions are correct, we proceed by inspection of all imprimitive and finite primitive subgroups of  $SL_3$ .

##### IMPRIMITIVE GROUPS

Assume that  $G$  is irreducible. If there is a semi-invariant of degree three, then the group is either imprimitive or  $F_{36}^{SL_3}$  (Singer and Ulmer, 1993a, Table 2 and Proposition 3.6). In both cases there is a Liouvillian solution. If the group is  $F_{36}^{SL_3}$ , then there is no semi-invariant of degree three corresponding to a character of order two (i.e. whose square is an invariant). Thus, if there is a semi-invariant of degree three corresponding to a character of order two then the group must be imprimitive. An imprimitive group of degree three is monomial, i.e. there exists a basis  $\{y_1, y_2, y_3\}$  such that an element of  $G$  has only one non-zero entry in row and column. Since  $G \subset SL(3, \mathbb{C})$ , we have  $\forall g \in G$ ,  $g(y_1 y_2 y_3) = \pm y_1 y_2 y_3$  (Singer and Ulmer, 1993a, Proposition 3.6). There are at least one and at most four Riccati polynomials of degree three in this case (Hendriks and van der Put, 1995; Ulmer, 1994).

##### $A_5$

LEMMA 4.1. *If  $G \simeq A_5$  then there is exactly one Riccati polynomial of degree six and exactly one of degree 15. The Riccati polynomial of degree six can be obtained by applying Appendix A on the two-dimensional space of invariants of degree six.*

*The canonical image of the invariant of degree 15 is obtained the following way: compute the images  $P_2$  and  $P_6$  of the invariants of degrees two and six. Let  $P_{10}$  be their bordered Hessian, and  $P_{15}$  be the Jacobian of these three ( $P_{15}$  is unique and does not depend on the choice of  $P_6$ ).*

PROOF. That there exists a Riccati polynomial of degree six is proven in Singer and Ulmer (1993b). Its uniqueness follows from the fact that there are six conjugate non-Abelian subgroups of index six having each one common eigenvector. So we turn to degree 15.

There are five conjugate subgroups of order four in  $A_5$  which are all Abelian of exponent two. Using the matrix representation of  $A_5$  given in Singer and Ulmer (1993b), we get that one such group, say  $H$ , is generated by the matrices  $\{E_2E_1E_2E_3E_2E_1, E_3E_1\}$  which have  $(-\zeta_5^3 + \zeta_5 + 1, \zeta_5, 1)^t$  as a common eigenvector, where  $\zeta_5 = \exp(2\pi i/5)$ . In order to find the corresponding semi-invariant that factors into linear forms, we look at a set

$$\mathcal{S} = \{id, E_1^{-1}, E_2^{-1}, E_3^{-1}, E_2^{-1}E_1^{-1}, E_1^{-1}E_2^{-1}, E_2^{-1}E_3^{-1}, E_1^{-1}E_2^{-1}E_1^{-1}, E_3^{-1}E_2^{-1}E_1^{-1}, \\ E_1^{-2}E_2^{-1}, E_2^{-1}E_1^{-1}E_2^{-1}, E_1^{-1}E_2^{-1}E_3^{-1}, E_1^{-2}E_2^{-1}E_1^{-1}, E_3^{-1}E_1^{-1}E_2^{-1}E_1^{-1}, E_1^{-1}E_3^{-1}E_2^{-1}E_1^{-1}\}$$

of left coset representatives of  $H$  in  $A_5$ . The resulting polynomial

$$\prod_{g \in \mathcal{S}} g((-\zeta_5^3 + \zeta_5 + 1)y_1 + \zeta_5 y_2 + y_3)$$

which by construction factors into linear forms over the complex numbers, is a semi-invariant and thus an invariant<sup>†</sup> of the simple group  $A_5$ . The length of the orbit of the logarithmic derivative of the eigenvector, i.e. the degree of its minimal polynomial, divides  $15 = [A_5 : H]$ . From Singer and Ulmer (1993b), we know that this degree must be  $\geq 6$ , so the degree must be 15 and that the associated polynomial must be a Riccati polynomial of degree 15. Note that there is, up to multiple, a unique invariant of degree 15 which thus must factor into linear forms.

The statement on the construction of the invariant of degree 15 from the invariants of degree two and six follows from the uniqueness and from a simple direct computation on classical invariants (Miller *et al.*, 1938, paragraph 125, pp. 253–255).□

$G_{168}$

LEMMA 4.2. *If  $G \simeq G_{168}$ , then there is a unique Riccati polynomial of degree 21.*

*Its canonical image  $P_{21}$  is obtained the following way. Let  $P_4$  be the image of the invariant of degree four, let  $P_6$  be its Hessian, and let  $P_{14}$  be the bordered Hessian of  $P_4$  and  $P_6$ ; then  $P_{21}$  is obtained from the Jacobian of  $P_4, P_6, P_{14}$ .*

PROOF. From Singer and Ulmer (1993b) we get that there exists a Riccati polynomial of degree 21. Since the group is simple, any semi-invariant must be an invariant. There is a unique trivial summand in the decomposition of the character of the 21st symmetric power of the faithful irreducible unimodular three-dimensional characters, we get that the semi-invariant must correspond to the, up to multiple, unique invariant of degree 21 and thus factor into linear forms. The corresponding group  $H$  and the coset representatives are given in Singer and Ulmer (1993b).

The statement on  $P_{21}$  follows from the uniqueness and from direct computation on classical invariants (Miller *et al.*, 1938, paragraph 125, pp. 253–255).□

<sup>†</sup>This invariant does not correspond to the invariants given in Singer and Ulmer (1993b) where another basis of the solution space was chosen in order to express the invariants.

$H_{216}^{SL_3}$  AND  $H_{72}^{SL_3}$ 

From Singer and Ulmer (1993b), we get that there exists a Riccati polynomial of degree nine. Since there is no non-trivial character of degree one and a unique trivial summand in the decomposition of the character of the ninth symmetric power of the faithful irreducible unimodular three-dimensional characters, we get that the semi-invariant must correspond to the, up to multiple, unique invariant of degree nine which thus must factor into linear forms. The corresponding group  $H$  and the coset representatives are given in (Singer and Ulmer, 1993b).

 $G_{168} \times C_3$ 

The, up to multiple, unique invariant of degree 21 of  $G_{168}$  factors into linear forms (see Section 4.3). Since its order is divisible by three it is also an invariant of  $C_3$  and thus  $a$ , up to multiple, unique invariant of  $G_{168} \times C_3$  that factors into linear forms. Since 21 is the minimal degree of a Riccati polynomial in this case (Singer and Ulmer, 1993b), it must correspond to a Riccati polynomial. This Riccati polynomial can be constructed from the semi-invariant of degree four (whose cube is rational) as in Section 4.3.

 $A_6^{SL_3}$ 

LEMMA 4.3. *If  $G \simeq A_6^{SL_3}$ , then there is exactly one Riccati polynomial of degree 45, and it is obtained from an invariant, which can be computed in the following way: let  $P_6$  be the invariant of degree six, let  $P_{12}$  be the Hessian of  $P_6$ , and let  $P_{30}$  be the bordered Hessian of  $P_6$  and  $P_{12}$ . Then  $P_{45}$  is obtained from the Jacobian of  $P_6, P_{12}, P_{30}$ .*

We note that there is also a Riccati polynomial of degree 36 (Singer and Ulmer, 1993b) but we know of no simple formula to obtain it; furthermore, the space of invariants of degree 36 for  $A_6^{SL_3}$  has dimension five so selecting the completely factorable invariant would be a very hard task.

PROOF. There are three types of non-conjugate subgroups of order 24 in  $A_6^{SL_3}$ . Only one type consisting of 45 conjugate non-Abelian subgroups of order 24 is 1-reducible (the 24 elements have a common eigenvector). Using the matrix representation of  $A_6^{SL_3}$  given in Singer and Ulmer (1993b), we get that one such group, say  $H$ , is generated by the matrices  $\{E_4E_1E_2E_3E_1E_2E_1^{-1}E_4, (E_4E_1^{-1})^2, E_3E_2E_4E_3, E_3E_4E_3\}$  which have a common eigenvector. In order to find the corresponding semi-invariant that factors into linear forms, we look at a set  $\mathcal{S}$  of left coset representatives of  $H$  in  $A_6^{SL_3}$  (containing 45 elements). Since the group is perfect and has no non-trivial character of degree one, the semi-invariant must be an invariant. There is, up to multiple, only one invariant of degree 45 for this group which thus must factor into linear forms by the above. The length of the orbit of the logarithmic derivative of the eigenvector divides  $45 = [A_6^{SL_3} : H]$ . From Singer and Ulmer (1993b), we get that this degree must be  $\geq 36$  and thus that the degree must be 45 and that the associated polynomial must be a Riccati polynomial of degree 45. The statement on  $P_{45}$  follows from the uniqueness and from direct computation on classical invariants (Miller *et al.*, 1938, paragraph 125, pp. 253–255).  $\square$

$F_{36}^{SL_3}$

There are nine conjugate subgroups of order 12 in  $F_{36}^{SL_3}$  which are all Abelian. Using the matrix representation of  $F_{36}^{SL_3}$  given in Singer and Ulmer (1993b), we get that one such group, say  $H$ , is generated by the matrices  $\{V, U^3\}$  which have  $(0, -1, 1)^t$  (see Singer and Ulmer, 1993b, for the notations) as a common eigenvector. Note that  $U^3$  is  $\omega$  times the identity where  $\omega$  is a third root of unity, and that  $U^3$  is in  $F_{36}^{SL_3}$ . In order to find the corresponding semi-invariant that factors into linear forms, we look at a set  $\mathcal{S} = \{id, S, T, S^{-1}, T^{-1}, T^{-1}S^{-1}, TS^{-1}, T^{-1}S, TS\}$  of left coset representatives of  $H$  in  $F_{36}^{SL_3}$ . The resulting polynomial

$$\prod_{g \in \mathcal{S}} (g(-y_2 + y_3)) = -y_3^3 y_2^6 + y_1^3 y_2^6 - y_1^6 y_2^3 + y_3^6 y_2^3 - y_3^6 y_1^3 + y_1^6 y_3^3$$

which by construction factors into linear forms over the complex numbers, is in fact an invariant<sup>†</sup> of  $F_{36}^{SL_3}$  (check using the generators). The length of the orbit of the logarithmic derivative of the eigenvector, i.e. the degree of its minimal polynomial, divides  $9 = [F_{36}^{SL_3} : H]$ . From Singer and Ulmer (1993b) we know that this degree must be  $\geq 6$ , so the degree must be nine and the associated minimal polynomial must be an irreducible Riccati polynomial of degree nine.

$A_5^{SL_3} \times C_3$

The, up to multiple, unique invariant of degree 15 of  $A_5$  which, by Section 4.3, factors into linear forms is, since its order is divisible by three, also an invariant of  $C_3$  and thus an, up to multiple, unique invariant of  $A_5 \times C_3$  which factors into linear forms. Since six is the minimal degree of a Riccati polynomial in this case (Singer and Ulmer, 1993b), it must correspond to such a Riccati polynomial.

#### 4.4. EXAMPLE

$$L = \partial^3 + \frac{21(x^2 - x + 1)}{25x^2(x-1)^2} \partial - \frac{21(2x-1)(x^2 - x + 2)}{50x^3(x-1)^3}.$$

The operator is irreducible. There is a two-dimensional space of invariants of degree six. None of them is a square so we proceed to step (3). There is one invariant of degree two, so we know that  $G/Z(G) \simeq A_5^{SL_3}$ .

The invariants of degree six (variables  $\bar{y}_i$ , parameters  $a_1, a_2$ ) are:

$$\begin{aligned} & (2560a_1 - 30a_2)\bar{y}_1^3 \bar{y}_2^3 + 6\bar{y}_3 \bar{y}_1^5 a_1 \\ & + (-1024a_1 + 30a_2)\bar{y}_3^6 - 393216\bar{y}_3 \bar{y}_2^5 a_1 \\ & + (15360a_1 - 90a_2)\bar{y}_1 \bar{y}_2 \bar{y}_3^4 + 90\bar{y}_3^2 \bar{y}_2^2 \bar{y}_1^2 a_2. \end{aligned}$$

Using a complete factorization approach, we find that for  $256a_1 = 3a_2$  we have the following factor (up to conjugation):

$$\bar{y}_1 - \alpha^2 \bar{y}_2 + \alpha \bar{y}_3, \quad \text{where } \alpha^5 - 256 = 0$$

<sup>†</sup>Note that if the result would be a semi invariant, one could find the corresponding one-dimensional character using this construction.

and the remaining factor is simply  $\bar{y}_3$ .

The coefficients of the corresponding canonical image are:

$$\begin{aligned} & \left[ -4608x^4(x-1)^4, -3072(-1+2x)x^3(x-1)^3, \right. \\ & -1536/5x^2(7x^2-7x-3)(x-1)^2, \\ & -\frac{1536}{25}x^2(133x^2-133x+33)(x-1)^2, \\ & -\frac{2304}{125}x(x-1)(-1+2x)(77x^2-77x-34), -\frac{110592}{625} \\ & +\frac{1204224}{625}x^3 - \frac{602112}{625}x^4 - \frac{562176}{625}x - \frac{39936}{625}x^2, \\ & -\frac{13824}{125}x(x-1)(7x-4)(-1+2x)(7x-3), \frac{262656}{625} \\ & +\frac{4666368}{625}x^3 - \frac{2333184}{625}x^4 - \frac{479232}{625}x - \frac{1853952}{625}x^2, \\ & \left. -\frac{3072(-1+2x)(637x^4-1274x^3+25x^2+612x+126)}{3125x(x-1)}, \dots \right]. \end{aligned}$$

From this, we read the coefficients of the minimum polynomial of an algebraic solution of the corresponding Riccati equation using Theorem 2.1. We use rows that correspond to the monomials  $X_1^l X_2^{6-l}$ ,  $l = 0, 1, \dots, 6$ . The way we numbered the monomials  $X_1^{l_1} X_2^{l_2} X_3^{6-l_1-l_2}$ , this corresponds to rows 1, 2, 4, 7, 11, 16, 22. We obtain:

$$\begin{aligned} X^6 & - 4\frac{(-1+2x)X^5}{x(x-1)} + \frac{(133x^2-133x+33)X^4}{5x^2(x-1)^2} \\ & - \frac{12(7x-4)(-1+2x)(7x-3)X^3}{25x^3(x-1)^3} \\ & + \frac{(351-11662x^3+5831x^4-2862x+8693x^2)X^2}{125x^4(x-1)^4} \\ & - \frac{4(-1+2x)(9604x^4-19208x^3+14275x^2-4671x+567)X}{3125x^5(x-1)^5} \\ & + \frac{(16807x^4-33614x^3+24907x^2-8100x+972)(-1+2x)^2}{12500x^6(x-1)^6}. \end{aligned}$$

Alternatively, we can avoid the factoring by using Sections 2.4 and Section 4.3 to construct directly the (unique) Riccati polynomial of degree 15 via bordered Hessian and Jacobian applied on the invariants of degree two and six. The result is the following irreducible Riccati polynomial:

$$\begin{aligned} X^{15} & - 6\frac{(7x^4-14x^3-4x^2+11x-3)}{x(x-1)(-2+x)(-1+2x)(x+1)}X^{14} \\ & + \frac{21(49x^4-98x^3-20x^2+69x-18)}{5x^2(x+1)(-2+x)(x-1)^2}X^{13} \\ & - \frac{13(9604x^6-28812x^5+19119x^4+9782x^3-15741x^2+6048x-756)}{50x^3(-2+x)(-1+2x)(x+1)(x-1)^3}X^{12} \end{aligned}$$



$$+ \frac{13(50421x^6 - 151263x^5 + 107203x^4 + 37699x^3 - 71843x^2 + 27783x - 3402)}{125x^4(x+1)(-2+x)(x-1)^4} X^{11}$$

$$+ \dots$$

### Acknowledgements

We thank Anne Bellido, Marius van der Put and Michael F. Singer for fruitful conversations during the preparation of this paper. J.A. Weil especially thanks Marius van der Put and the NWO AIDA project for their material and intellectual support. This research was partially supported by the European Cathode II working group and, for the third author, also by the CALIFE project (INTAS-RBFR 95-0412). Most computations were realized thanks to the French CNRS-UMS MEDICIS.

### References

- Barkatou, M. A. (1998). On rational solutions of systems of linear differential equations, this volume.
- Barkatou, M. A., Pflügel, E. (1998). An algorithm computing the regular formal solutions of a system of linear differential equations, this volume.
- Brill, A. (1998). Über die zerfällung einer ternärform in linearfactoren. *Math. Ann.*, **50**, 157–182.
- Cox, D., Little, J., O’Shea, D. (1992). *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics, Berlin, Springer.
- Fakler, W. (1997). On second order homogeneous linear differential equations with Liouvillian solutions. *Theor. Comput. Sci.*, **187**, 27–48.
- Gelfan’ d, I. M., Kapranov, M. M., Zelevinski, A. V. (1994). *Discriminants, Resultants, and Multidimensional Determinants*. Basel, Birkhäuser.
- Hendriks, P., van der Put, M. (1995). Galois action on solutions of a differential equation. *J. Symb. Comput.*, **19**, 559–576.
- Hessinger, S. (1998). Galois groups of fourth order linear differential equations. Ph.D. Dissertation, North Carolina State University.
- Hoeij, M. van (1997). Factorization of differential operators with rational functions coefficients. *J. Symb. Comput.*, **24**, 1–30.
- Hoeij, M. van, Weil, J.-A (1997). An algorithm for computing invariants of differential Galois groups. *J. Pure Appl. Algebra*, **117 & 118**, 353–379.
- Hohl, J.-C. (1995). Massively parallel search for linear factors in polynomials with many variables. *ICM*, preprint.
- Kobayashi, H., Fujise, T., Furukawa, A. (1988). Solving systems of algebraic equations by a general elimination method. *J. Symb. Comput.*, **5**, 303–320.
- Kovacic, J. (1986). An algorithm for solving second order linear homogeneous differential equations. *J. Symb. Comput.*, **2**, 3–43.
- Lang, S. (1992). *Algebra*. 3<sup>rd</sup> edn, Addison-Wesley.
- Magid, A. (1994). *Lectures on Differential Galois Theory*. University Lecture Series. Providence, RI, American Mathematical Society.
- Miller, G. A., Blichfeldt, H. F., Dickson, L. E. (1938). *Theory and Applications of Finite Groups*. New York, G. G. Stechert and Co.
- Pflügel, E. (1997). An algorithm for computing exponential solutions of first order linear differential systems. In *Proceedings of ISSAC’97*, New York, ACM Press.
- Put, M. van der (1998). Galois theory of differential equations, algebraic groups and Lie algebras, this volume.
- Put, M. van der, Ulmer, F. (1998). Differential equations and finite groups. MSRI Preprint #1998-058.
- Ragot, J.-F. (1997). Sur la factorisation absolue des polynômes. Ph.D. Dissertation, Université de Limoges.
- Singer, M. F. (1981). Liouvillian solutions of  $n^{th}$  order linear differential equations. *Am. J. Math.*, **103**, 661–682.
- Singer, M. F. (1995). Testing reducibility of linear differential operators: a group theoretic perspective. *J. Appl. Algebra Eng. Comm. Comp.*, **7**, 77–104.

- Singer, M. F. (1997). Direct and inverse problems in differential Galois theory. To appear in: *Collected Works of Ellis R. Kolchin*.
- Singer, M. F., Ulmer, F. (1993a). Galois groups for second and third order linear differential equations. *J. Symb. Comput.*, **16**, 1–36.
- Singer, M. F., Ulmer, F. (1993b). Liouvillian and algebraic solutions of second and third order linear differential equations. *J. Symb. Comput.*, **16**, 37–73.
- Singer, M. F., Ulmer, F. (1995). Necessary conditions for Liouvillian solutions of (third order) linear differential equations. *J. Appl. Algebra Eng. Comm. Comp.*, **6**, 1–22.
- Singer, M. F., Ulmer, F. (1997). Linear differential equations and products of linear forms. *J. Pure Appl. Algebra*, **117** & **118**, 549–564.
- Sit, W. (1992). An algorithm for solving parametric linear systems. *J. Symb. Comput.*, **13**, 353–413.
- Trager, B. M. (1976). Algebraic factoring and rational function integration. In *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computation*.
- Tsarev, S. P. (1996). An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator. In *Proceedings of ISSAC'96*. New York, ACM Press.
- Ulmer, F. (1994). Irreducible linear differential equations of prime order. *J. Symb. Comput.*, **18**, 385–401.
- Ulmer, F., Weil, J.-A. (1996). Note on Kovacic's algorithm. *J. Symb. Comput.*, **22**, 179–200.
- Weil, J.-A. (1995a). First integrals and Darboux polynomials of homogeneous linear differential systems. *Proceedings of AAECC 11*, In Giusti, M., Mora, T., eds, **948**, Springer.
- Weil, J.-A. (1995b). Constantes et polynômes de Darboux en algèbre différentielle: application aux systèmes différentiels linéaires. Ph.D. Dissertation, École Polytechnique.

## Appendix A. Deciding When a Homogeneous Polynomial Admits Linear Factors

If  $A$  is a ring, we denote by  $A[X_1, \dots, X_n]_m$  the space of homogeneous polynomials of total degree  $m$  in the variables  $X_i$ . Let  $P \in \mathbb{C}[t_1, \dots, t_s][X_1, \dots, X_n]_m$  be homogeneous of degree  $m$  in the variables  $X_i$ . In this appendix, we show how to compute potential values of the parameters  $t_i$  such that  $P$  (then viewed as a homogeneous polynomial in  $\mathbb{C}[X_1, \dots, X_n]$ ) admits a linear factor. The subproblem relevant in this paper is the following: let  $P = \sum t_i P_i$  where  $P_i \in \mathbb{C}[X_1, \dots, X_n]_m$ . Decide for which values of the  $t_i$  the polynomial  $P$  admits a linear factor.

This problem is part of the general problem of absolute factorization of polynomials. For the latter, the reader may consult Ragot (1997) where several methods and abundant references are given.

### APPENDIX A.1. BRILL'S EQUATIONS

Completely factorable polynomials of degree  $m$  form a Zariski closed set. The defining equations, the Brill equations, are homogeneous equations of degree  $m$  that can be computed (Brill, 1998; Gelfan'd *et al.* 1994). This can be used (Singer and Ulmer, 1997) to select the completely factorable semi-invariants.

### APPENDIX A.2. COMPLETE FACTORIZATION OF NON-PARAMETRIZED POLYNOMIALS

If there are no parameters in  $P$ , then many methods exist (Hohl, 1995; Kobayashi, Fujisem Furukawa, 1988; Ragot, 1997, for another method and references). We outline below a method that seems, from our experience, to be the most interesting. The tools and results are similar to the ones that are used in the three papers mentioned above.

Let  $C$  be a perfect field (usually  $C$  is  $\mathbb{Q}$  or a finite extension of  $\mathbb{Q}$ ).

DEFINITION A1. Let  $P$  be a polynomial of  $C[X_1, \dots, X_n]_m$  of degree  $d > 0$  in  $X_1$ . A  $n-1$ -tuple  $(a_2, \dots, a_n)$  of  $C^{n-1}$  is called a non critical value for  $P$  in  $X_1$  if the polynomial  $P(T, a_2, \dots, a_n)$  is square-free and of degree  $d$ . This implies in particular that all roots of  $P(T, a_2, \dots, a_n)$  are simple.

REMARK. The existence of such a non-critical value for  $P$  implies that  $P$  is square-free.

From now on, let  $\underline{Z}$  denote the multi-variable  $X_3, \dots, X_n$ .

LEMMA A1. Let  $P$  be a homogeneous square-free polynomial of  $C[X_1, X_2, \underline{Z}]_m$ , of degree  $d > 0$  in  $X_1$ . Assume that  $P$  is of degree  $m$  in  $X_2$ , and let  $(1, 0, \dots, 0)$  be a non-critical value for  $P$  in  $X_1$ . Denote by  $p(T)$  the polynomial  $P(T, 1, \underline{0})$  and let  $\alpha$  be a root of  $p(T)$ . Then  $P(X_1 + \alpha X_2, X_2, \underline{Z})$  is of degree  $m - 1$  in  $X_2$ .

PROOF. Let  $\bar{P}(X_1, X_2) = P(X_1, X_2, \underline{0})$ . It is a homogeneous polynomial of  $C[X_1, X_2]$ . Write

$$\bar{P} = X_2^{m-d} \sum_{i=0}^d b_i X_1^i X_2^{d-i}.$$

It factors in  $\bar{C}[X_1, X_2]$  as

$$bX_2^{m-d} \prod_{i=1}^d (X_1 - \alpha_i X_2)$$

where the  $\alpha_i$ , with  $i$  from 1 to  $d$ , are the roots of  $p(T)$ . Hence

$$\bar{P}(X_1 + \alpha X_2, X_2) = bX_2^{m-d} \prod_{i=1}^d (X_1 + (\alpha - \alpha_i) X_2).$$

The roots of  $p(T)$  are simple, so  $\alpha - \alpha_i = 0$  for one unique  $i$ , which proves that  $\bar{P}(X_1 + \alpha X_2, X_2)$  is of degree  $m - 1$  in  $X_2$ . It is also true for  $P(X_1 + \alpha X_2, X_2, \underline{Z})$ , since  $\bar{P}(X_1 + \alpha X_2, X_2)$  is its evaluation in  $\underline{Z} = \underline{0}$ .  $\square$

PROPOSITION A1. Let notations and assumptions be as above, and let  $\tilde{P}_T(X_1, X_2, \underline{Z}) = P(X_1 + TX_2, X_2, \underline{Z})$ ; then  $P$  admits a linear factor in  $\bar{C}[X_1, X_2, \underline{Z}]$  iff there exists a root  $\alpha$  of  $p$  such that  $\frac{\partial^{m-1} \tilde{P}_\alpha}{\partial X_2^{m-1}}$  is a linear factor of  $\tilde{P}_\alpha$  in  $C(\alpha)[X_1, \underline{Z}]$ .

PROOF. Let  $L = X_1 + aX_2 + \sum_{i=3}^n c_i X_i$  be a linear factor of  $P$  and let  $C'$  be the field of its coefficients. Denote by  $r$  the degree of  $C'$  over  $C$ .

The polynomial  $L$  is absolutely irreducible and by Theorem 1.2.2 of Ragot (1997),  $F = \text{Norm}_{C'/C}(L)$  is an irreducible polynomial of  $C[X_1, X_2, \underline{Z}]$ . It is clearly of degree  $r$ , and it is a factor of  $P$ . The polynomial  $F(T, 1, 0, \dots, 0)$  is a factor of  $p(T)$ , it is hence square-free.

On the other hand,  $F(T, 1, 0, \dots, 0) = \text{Norm}_{C'/C}(L(T, 1, 0, \dots, 0)) = \text{Norm}_{C'/C}(T + a)$ , and by Lemma 1.2.1 of Ragot (1997) Trager (or Theorem 2.1 of 1976, extended to perfect fields),  $\text{Norm}_{C'/C}(T + a)$  is a power of an irreducible polynomial of  $C[T]$ .

This implies that  $F(T, 1, 0, \dots, 0)$  is an irreducible polynomial of degree  $r$ . Let  $\alpha = -a$ ; then  $\alpha$  is of degree  $r$  and  $C' = C(\alpha)$ .

Hence,

$$L = X_1 - \alpha X_2 + \sum_{i=3}^n c_i(\alpha) X_i$$

where the  $c_i$ 's are polynomials with coefficients in  $C$ .

Let  $P = L \cdot Q$ , then  $\tilde{P}_\alpha = \tilde{L}_\alpha \cdot \tilde{Q}_\alpha = (X_1 + \sum_{i=3}^n c_i(\alpha) X_i) \cdot \tilde{Q}_\alpha$ . Now  $\tilde{P}_\alpha$ , and then  $\tilde{Q}_\alpha$ , are of degree  $m - 1$  in  $X_2$  (Lemma A1) and so

$$\frac{\partial^{m-1} \tilde{P}_\alpha}{\partial X_2^{m-1}} = c(\alpha) \cdot \left( X_1 + \sum_{i=3}^n c_i(\alpha) X_i \right).$$

Conversely, if  $X_1 + \sum_{i=3}^n c_i(\alpha) X_i$  is a factor of  $\tilde{P}_\alpha$  then  $X_1 - \alpha X_2 + \sum_{i=3}^n c_i(\alpha) X_i$  is a factor of  $P$ .  $\square$

This suggests the following fast algorithm to check whether a given square-free polynomial has linear factors (and compute them as a byproduct).

After maybe a linear change of variables<sup>†</sup> we may assume that  $P$  satisfies the hypotheses of proposition A1. Form  $P_\alpha$  and  $L_\alpha$  as in Proposition A1. Then the corresponding  $L$  is a factor of  $P$  iff  $L_\alpha$  divides all coefficients of powers of  $X_2$  in  $P_\alpha$ ; the latter conditions are easily stated via resultants or even Euclidean divisions and, together with the relation  $p(\alpha) = 0$ , they yield a system of polynomial equations for  $\alpha$ : our polynomial  $P$  has a linear factor iff the latter system is consistent.

### APPENDIX A.3. COMPLETE FACTORIZATION OF PARAMETRIZED POLYNOMIALS

The above method can be adapted to polynomials depending on parameters, although the conditions that  $P$  be square-free and that  $(1, 0, \dots, 0)$  be non-critical may induce some nuisance (non-linear branchings); if one avoids the branching, then the algorithm is incomplete but yields an interesting heuristic.

The following straightforward method can also be used, without any restrictions on the parameters. Write  $L = X_1 + \sum_{i=2}^n c_i X_i$ . We want conditions for  $L$  to be a factor of  $P$ . Let  $R := \text{resultant}(P, L, X_1) = P(-\sum_{i=2}^n c_i X_i)$  (note that  $R = P(-\sum_{i=2}^n c_n X^n, X_2, \dots, X_n)$ ). Of course, there is a linear factor iff  $R$  is the zero polynomial. The coefficients of  $R$  depend on the  $t_i$  and the  $c_i$ ; equating all these to zero yields a polynomial system that we solve with any available method (see e.g. Cox *et al.*, 1992, for an introduction to these).

In our specific problem, the system is linear in the  $t_i$  and non-linear in the  $c_i$  so we can also apply the methods of Sit (1992) for solving it.

Surprisingly, this simple method is quite satisfactory in practice and can be faster than (our implementation of) the Brill equations. We may interpret this experimental fact as follows. Using the Brill equations is like solving the system that we would obtain by first eliminating the  $c_i$ ; however, here, it may be that another solving strategy is better (e.g. because of the linearity in the  $t_i$ ) so it could compensate for the fact of adding the extra variables  $c_i$ .

<sup>†</sup>The condition on a change of variables to be such that  $P$  satisfies the hypothesis of Proposition A1 are easily seen to form a Zariski closed set so “almost any” change of variables will do.

APPENDIX A.4. DECIDING WHEN A POLYNOMIAL IS A  $k$ TH POWER

Let  $P$  be of the form  $P = \sum t_i P_i$  where the  $P_i \in \mathbb{C}[X_1, \dots, X_n]$  are homogeneous of degree  $m$ . After a linear transformation we may assume that the degree of  $P$  in  $X_1$  equals  $m$ .

Assume that  $P = Q^k$ . Let  $L$  be the coefficient of  $X_1^m$ . We need to distinguish two cases: after substituting values in  $\mathbb{C}$  for the  $t_i$  either  $L$  vanishes or  $L$  remains non-zero. First we compute under the assumption that  $L$  vanishes. So we have a linear relation  $L = 0$  between the parameters  $t_i$  ( $P$  is linear in the  $t_i$ ). We use this relation to eliminate one of the  $t_i$  and then apply recursion.

Afterwards we may assume that  $L$  does not vanish. Because  $P$  is homogeneous in the  $t_i$  we may then assume that  $L = 1$ . We use this relation to eliminate one of the  $t_i$ . Then  $P$  is a monic polynomial in  $X_1$  with coefficients in  $\mathbb{C}[t_1, \dots, t_s, X_2, \dots, X_n]$ , and the degree of  $Q$  in  $X_1$  is  $d = m/k$ . Write  $Q = \sum_{i=0}^d q_i X_1^i$  where  $q_d = 1$ . Taking the coefficient of  $X_1^{m-1}$  on both sides of the equation  $P = Q^k$  (the coefficient on the right-hand side is  $kq_{d-1}$ ) we find  $q_{d-1} \in \mathbb{C}[t_1, \dots, t_s, X_2, \dots, X_n]$ . Repeating this for the next coefficients  $X_1^{m-2}, X_1^{m-3}, \dots, X_1^{m-d}$  we determine  $q_{d-2}, q_{d-3}, \dots, q_0 \in \mathbb{C}[t_1, \dots, t_s, X_2, \dots, X_n]$ . Then we can compute  $P - Q^k$  and equate all coefficients with respect to  $X_1, \dots, X_n$  to 0. This gives a set of polynomial equations in the parameters  $t_i$ . Solving it gives the values of  $t_i$  for which  $P$  is a  $k$ th power.

*Originally Received xx July 1998*  
*Accepted xx Month 1998*