Computing Closed Form Solutions of Integrable Connections

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ABSTRACT
We present algorithms for computing rational and hyperexponential solutions of linear D-finite partial differential systems written as integrable connections. We show that these types of solutions can be computed recursively by adapting existing algorithms handling ordinary linear differential systems. We provide an arithmetic complexity analysis of the algorithms that we develop. A Maple implementation is available and some examples and applications are given.

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1. INTRODUCTION
Many systems coming from various areas of science can be described by systems of partial differential equations (PDS). Apart from its theoretical interest, computing closed form solutions (*e.g.*, polynomial, rational, (hyper)exponential, liouvillian solutions) can help in studying such systems. In this paper, we are merely interested in *D-finite* linear PDS given in the form of *integrable connections* which consists in considering m first order linear PDS of the form

$$\frac{\partial Y}{\partial x_i} = A_i(x_1, \ldots, x_m)\, Y, \quad i = 1, \ldots, m, \tag{1}$$

where Y is a vector of n unknown functions of m independent variables $x_1, \ldots, x_m$ and the A_i's are $n \times n$ matrices whose coefficients are rational functions in $x_1, \ldots, x_m$ with coefficients in a field C of characteristic zero, satisfying the *integrability conditions* (3).

The main contribution of the present paper consists in two algorithms that are fully implemented in Maple[1]. The first one computes rational solutions of integrable connections and the second one is dedicated to the calculation of hyperexponential solutions (including rational solutions) of such connections. In both cases, our strategy proceeds by recursion by considering one by one each system appearing in (1). We first compute solutions of the system $\partial Y/\partial x_1 = A_1 Y$ using an appropriate algorithm for ordinary differential systems (ODS) and considering $x_2, \ldots, x_m$ as transcendental constants. We then prove that we can reduce the size n of the matrices of the connection and the number m of variables (see Proposition 6 and Lemma 3) before applying recursion and considering the second system. Rational solutions of ODS are computed using the algorithm developed in [3] and exponential solutions with the one in [21]. Note that the latter algorithms handling ODS are implemented in the Maple package ISOLDE ([7]).
This recursive approach for handling integrable connections has already been considered in slightly different contexts. In [6], we use this approach in characteristic $p > 0$ for computing rational solutions of integrable connections whose partial p-curvatures vanish. Note that the authors of [19] use a similar approach for computing hyperexponential solutions of D-finite PDS over Laurent-Ore algebras. Contrary to [19], the present paper contains a specific algorithm for rational solutions which is useful in itself for computing *eigenrings* (see Example 3) and reducing the size of the connection before computing non-rational hyperexponential solutions (see Example 4). Algorithms for computing closed form solutions of D-finite PDS are also developed in [18] where a different approach is used. In particular, the authors do not write PDS as integrable connections (see the discussion at the end of Subsection 5.1 below). Note also that algorithms for computing rational solutions of holonomic systems based on other techniques can be found in [10, 20]. Finally, another contribution of the present paper is that we provide an arithmetic complexity analysis of the algorithms that we develop.

The paper is organized as follows. In the next section, we recall basic notions on D-finite PDS and integrable connections that are needed in the sequel. In Section 3, we give an overview of the algorithms developed in [3], resp. [21],

[1] see `http://www.ensil.unilim.fr/~cluzeau/PDS.html`

for computing rational, resp. exponential, solutions of ODS, and we provide a complexity analysis of these algorithms. In Section 4, we develop our recursive algorithm for computing rational solutions of integrable connections. We also give a complexity estimate of our algorithm, illustrate our implementation with some examples and give some applications. The last section is concerned with the computation of hyperexponential solutions of integrable connections: we give an algorithm and provide an implementation.

2. INTEGRABLE CONNECTIONS

This section contains definitions and preliminaries on the notions of D-finite PDS and integrable connections. Let C be a field of characteristic zero and $\overline{C}$ its algebraic closure. We denote by $k = C(x_1, \ldots, x_m)$, resp. $K = \overline{C}(x_1, \ldots, x_m)$, the field of rational functions in m independent variables $x_1, \ldots, x_m$ with coefficients in C, resp. $\overline{C}$. Let $\partial_i = \partial/\partial x_i$. A *linear PDS* is a system of linear equations with coefficients in k in some unknown functions $y_1, \ldots, y_n$ of m variables $x_1, \ldots, x_m$ and their partial derivatives of finite order, *i.e.*, $\partial_1^{\nu_1} \cdots \partial_m^{\nu_m} y_i$, for $i = 1, \ldots, n$ with the ν_j's in $\mathbb{N}$.

In the present paper, we are merely interested in D-finite linear PDS (see [12, Definition 2.1] or [18]) and more precisely in integrable connections. There are many ways to define D-finite PDS. We give here a definition based on the existence of a universal differential extension $\mathcal{U}$ of k containing all solutions of such systems (see, *e.g.*, [17]).

Definition 1. *A linear PDS is said to be* D-finite *if its solution space in* $\mathcal{U}$ *is of finite dimension.*

Definition 2. *With the above notation, an (algebraic)* integrable connection over k of size n in m variables *is a linear PDS of the form*

$$\left\{ \begin{array}{lll} \Delta_1 Y = 0 & \text{with} & \Delta_1 := \partial_1 \mathbb{I}_n - A_1, \\ \quad\vdots & & \\ \Delta_m Y = 0 & \text{with} & \Delta_m := \partial_m \mathbb{I}_n - A_m, \end{array} \right. \tag{2}$$

where the matrices A_i's belong to $\mathbb{M}_n(k)$ and $[\Delta_i, \Delta_j] = 0$, i.e*, the following* integrability conditions *are satisfied:*

$$\partial_i(A_j) - A_i A_j = \partial_j(A_i) - A_j A_i, \quad \forall\, i, j \in \{1, \ldots, m\}. \tag{3}$$

Such a connection is also denoted by $[A_1, \ldots, A_m]$ which is convenient when one wants to refer to the matrices A_i.

There exist algorithms, based on Gröbner or Janet basis computations, testing if a given linear PDS is D-finite (see, *e.g.*, [12] and references therein). More precisely, in [12, Proposition 2.1], the authors show that one can extract from a D-finite system a so-called *rectangular system* that is a set of m non-zero univariate polynomials $P_i \in C[x_1, \ldots, x_m][\partial_i]$ for $i = 1, \ldots, m$. In particular, this proves that any D-finite linear PDS can be written as an integrable connection. Note that the input of the algorithms developed in [18] are the P_i's defined above whereas, in the present paper, we consider integrable connections in order to benefit from the algorithms for ODS studied in Section 3. A procedure writing a D-finite linear PDS as an integrable connection is included in the Maple package OreModules ([11]). For a module theoretical description of the above concepts in a slightly more general context, we refer to [19, 23].

Example 1. *Consider a system of linear partial differential equations in one unknown function y appearing in a problem of probability theory studied in [9]. In two variables x_1 and x_2, the system is given by:*

$$\left\{ \begin{array}{lll} -\frac{\beta}{2}\,\partial_2 y + \partial_1^2 y - x_2\,\partial_2^2 y & = & 0, \\ 2\,\partial_1\,\partial_2 y + x_1\,\partial_2^2 y & = & 0, \end{array} \right. \tag{4}$$

(see [9, Theorem 3.2, (i)] in the case $a^2 - 4\,b = 0$) where β is the so-called Peirce constant (see [9] and references therein). Using the procedure `Connection` *of the package* OreModules*, we can check that this system is D-finite and write it as an integrable connection of the form (2) with $m = 2$, $n = 4$, $C = \mathbb{Q}(\beta)$ and*

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & \frac{1}{2}\,\beta & 0 & x_2 \\ 0 & 0 & 0 & \frac{(-3-\beta)x_1}{{x_1}^2-4\,x_2} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2}\,x_1 \\ 0 & 0 & 0 & \frac{6+2\,\beta}{{x_1}^2-4\,x_2} \end{bmatrix}.$$

The unknown function y of System (4) and the unknown vector Y of the latter integrable connection are related by $Y = (y \quad \partial_2 y \quad \partial_1 y \quad \partial_2^2 y)^T$.

3. THE ONE VARIABLE CASE

We denote by $k = C(x)$, resp. $K = \overline{C}(x)$, the field of rational functions in one independent variable x with coefficients in C, resp. $\overline{C}$. In this section we consider a linear differential system of order one of the form $Y' = AY$, where A is a square matrix of size n with entries in k, Y is an n-dimensional vector of unknown functions and the symbol $'$ denotes the derivation $\mathrm{d}/\mathrm{d}x$ with respect to x. In what follows, we briefly recall the principle of the algorithms developed in [3] for computing rational solutions of such ODS and in [21] for exponential solutions and we study their arithmetic complexity. To our knowledge, the only complexity results related to the computation of closed form solutions of ODS appearing in the literature are the ones of [16] where the author studies the bit complexity of reducing ODS.

3.1 Polynomial and rational solutions

Let $\prod_{i=1}^{s} q_i^{r_i+1}$ denote the factorization in $C[x]$ of the least common multiple of the denominators of the entries of A into irreducible factors q_i. Roughly speaking, the algorithm for computing rational solutions of $Y' = AY$ developed in [3] proceeds as follows: it first searches for a universal denominator $Q = \prod_{i=1}^{s} q_i^{m_i}$ of rational solutions of the system, *i.e.*, every rational solution can be written as P/Q for a certain polynomial vector P. Then, performing a change of variable, the problem is reduced to finding polynomial solutions of an auxiliary ODS.

Proposition 1. *Let $A \in \mathbb{M}_n(k)$ and suppose that the numerators and denominators of each non-zero entry of A are coprime. Let $\prod_{i=1}^{s} q_i^{r_i+1}$ denote the factorization of the least common multiple of the denominators of the entries of A into irreducible factors and $d = \sum_{i=1}^{s}(r_i+1)\,\deg(q_i)$. Then,* *a universal denominator $Q = \prod_{i=1}^{s} q_i^{m_i}$ of rational solutions of $Y' = AY$ can be computed using at most $\mathcal{O}(n^5 \max_i(r_i)\, d)$ operations in C.*

Proof. For each irreducible factor q_i, the algorithm in [3] proceeds by calculating a (local) simple form of the system at the singularity defined by q_i. Then, the corresponding exponent m_i is obtained by computing the integer roots of the so-called indicial polynomial that can be read off from a simple form. Using the algorithm developed in [15, Chap. 4], a simple form can be computed using at most $\mathcal{O}(n^5\, r_i^2\, \deg(q_i))$ operations in C (see [15, Remark 4.4.3]). This has to be done for each finite singularity of the system. Therefore, taking the sum from 1 to s, the cost of computing a universal denominator Q is bounded by $\mathcal{O}(n^5 \max_i(r_i)\, d)$. $\square$

Once a universal denominator Q has been computed, the algorithm given in [3] proceeds by performing the substitution $Y = Z/Q$ into the system and looks for polynomial solutions of the resulting system $Z' = (A + (Q'/Q)\, \mathbb{I}_n)\, Z$.

Proposition 2. *Let $A = (a_{i,j}) \in \mathbb{M}_n(k)$ and suppose that the numerators and denominators of each non-zero entry of A are coprime. Let Q be defined as above and*

$$r_\infty = \max\left(\max_{i,j}\left(1 + \deg(\mathrm{num}(a_{i,j})) - \deg(\mathrm{den}(a_{i,j}))\right), 0\right). \tag{5}$$

Then, computing a basis of polynomial solutions of the system $Z' = (A + (Q'/Q)\, \mathbb{I}_n)\, Z$ can be done using at most $\mathcal{O}(n^5\, r_\infty^2 + n^3\, N^2)$ operations in C where $N \in \mathbb{N}$ is a degree bound for the numerators of rational solutions of $Y' = AY$.

Proof. The first step of the algorithm developed in [3] for computing polynomial solutions consists in computing a bound N on the degrees of the polynomial solutions that we are searching for. This can be done by computing a simple form of the system at infinity from which we can read off the indicial polynomial at infinity. Using the algorithms developed in [15, Chap. 4], a simple form at infinity can be computed in at most $\mathcal{O}(n^5\, r_\infty^2)$ operations in C (see [15, Remark 4.4.3]). A degree bound N can then be obtained by computing the integer roots of the indicial polynomial. Finally, we compute polynomial solutions of degree bounded by N using the algorithm developed in [3, Section 3] which costs at most $\mathcal{O}(n^3\, N^2)$ operations in C (see [5, Lemma 6]). This ends the proof. $\square$

Corollary 1. *With the above notation, a basis of rational solutions of $Y' = AY$ can be computed using at most $\mathcal{O}(n^5\, (\max_i(r_i)\, d + r_\infty^2) + n^3\, N^2)$ operations in C.*

Remark 1. *Note that, in this section, we neglect the cost of computing the integer roots of the indicial polynomials: from [22, Theorem 15.21], the bit complexity of this task is quadratic in the degree of the indicial polynomial and linear in the bit size of its coefficients. We also neglect the cost of factoring the denominator of the matrix A.*

3.2 Exponential solutions

Definition 3 ([21]). *An* exponential solution *of the linear differential system $Y' = AY$ is a solution of the form $\exp(\int f\, \mathrm{d}x)\, z$, where $f \in K$ and $z \in K^n$.*

In this subsection, we estimate the cost of computing exponential solutions of an ODS using the strategy of the algorithm given in [21] which can be summarized as follows:

1. Compute the *non-ramified local exponential parts* at each singularity of the ODS;
2. For each combination composed of one non-ramified exponential part at each singularity, perform an appropriate change of variable in the system and compute polynomial solutions of the resulting system.

One bottleneck of this algorithm is that, in general, Step 2 provides a large number of combinations to check, namely, n^δ if $\delta = \sum_{i=1}^{s} \deg(q_i) + 1$ with the notation of Subsection 3.1, denotes the number of singularities of the system. The other important drawback of this algorithm is that the combinations are often defined over algebraic extensions of the base field C of large degree. That is the reason why some very useful methods have been developed to reduce the number of combinations to be considered (see [14]). To obtain a complexity estimate for this algorithm, the only thing that is missing is an estimation of the cost of computing non-ramified local exponential parts at a given singularity. Without loss of generality, we shall assume that the singularity is located at the origin $x = 0$. Let us write

$$A = \frac{1}{x^{r+1}}\,(A_0 + A_1\, x + A_2\, x^2 + \cdots),$$

where $r \in \mathbb{N}$ is the *Poincaré rank* of the system (see for example [2, 8]), the A_i's are matrices with constant entries in $\overline{C}$ with $A_0 \neq 0$.

Definition 4 ([21]). *A* non-ramified local exponential part *of the system $Y' = AY$ at $x = 0$ is a polynomial in $1/x$ of the form*

$$\tilde{f} = \frac{\alpha_{p+1}}{x^{p+1}} + \frac{\alpha_p}{x^p} + \cdots + \frac{\alpha_1}{x}, \tag{6}$$

where $0 \le p \le r$ and the α_i's are in $\overline{C}$ (the degree of the algebraic extension of C defined by the α_i's is bounded by n - see [1]), such that there exists a formal local solution of the system of the form $\exp(\int \tilde{f}\, \mathrm{d}x)\, \tilde{z}$, where $\tilde{z}$ is a vector of formal power series in x.

Proposition 3. *With the above notation, computing the non-ramified exponential parts of $Y' = AY$ at a singularity defined by an irreducible polynomial q can be done using at most $\mathcal{O}(n^5\, r^3\, \min(n, r)\, \deg(q))$ operations in an algebraic extension of the field C of degree $\le n$.*

Proof. The exponential parts that we are looking for can be computed by proceeding as follows: we start by computing a super-irreducible form of the system which can be done using at most $\mathcal{O}(n^5\, r^2\, \min(n, r)\, \deg(q))$ operations in

C (see [8, Proposition 4.3] with $\nu = n\,r$). This yields s subsystems of size n_i (so that $n = \sum_{i=1}^{s} n_i$) and Poincaré rank $0 \leq r_s < r_{s-1} < \cdots < r_2 < r_1 \leq r$. To each system is associated a Newton polynomial. The possible values of p in (6) are among the r_i's and the possible values for α_{p+1} are non-zero roots of the Newton polynomial associated with the subsystem of Poincaré rank p. We thus factorize the latter Newton polynomial and consider one by one its irreducible factors. We then proceed recursively by computing a super-irreducible form of each subsystem and the possible values of α_p are non-zero roots of the new Newton polynomials that we get. As $r_i \leq r$ for all i and $\sum_{i=1}^{s} n_i^k \leq (\sum_{i=1}^{s} n_i)^k = n^k$ for $k \in \mathbb{N}^*$, the cost of the second step is thus bounded by $\sum_{i=1}^{s} \mathcal{O}(n_i^5\, r^2\, \min(n_i, r)) \leq \mathcal{O}(n^5\, r^2\, \min(n, r))$ operations in the algebraic extension of C defined by the singularity q and the irreducible polynomial defining α_{p+1}. Finally, we have to apply recursion at most $r+1$ times and the algebraic extension of C containing the α_i's is of degree $\leq n$ (see [1]) which ends the proof. $\square$

Proposition 4. *With the above notation, exponential solutions of the system $Y' = A\,Y$ can be computed using at most $\mathcal{O}(n^5\,(\max_i(r_i)^2\, d \sum_i \min(n, r_i) + r_\infty^3\, \min(n, r_\infty)))$ operations in an algebraic extension of C of degree $\leq n$ and at most $\mathcal{O}(n^{\delta+3}\, N^2)$ operations in an algebraic extension of C of degree $\leq n^\delta\, \delta!$, where the r_i's, d and r_∞ are defined as in Subsection 3.1, δ is the number of singularities and N is a degree bound for all the computed polynomial solutions.*

Proof. The first part of the result comes from taking the sum of the estimate of Proposition 3 over all singularities. The second part follows from Proposition 2 applied to each combination (at most n^δ) of exponential parts. See [14] for the degree of the algebraic extension of C needed. $\square$

4. RATIONAL SOLUTIONS OF PDSs

In this section, we give an efficient algorithm for computing rational solutions of integrable connections. The idea is similar to the one used in [6, Subsection 2.1] where we proceed by recursion for computing rational solutions of integrable connections, defined over a field of characteristic $p > 0$, whose partial p-curvatures vanish. In [19], the authors also proceed by recursion for computing hyperexponential solutions of systems over Laurent-Ore algebras. Note however that, in [19], the problem of computing rational solutions is not directly considered. In Example 4 below, we shall show how starting by computing rational solutions can be useful to reduce the size of the connection considered before computing non-rational hyperexponential solutions.

Definition 5. *A* rational solution of the integrable connection (2) *is a vector $Y \in K^n$ such that $\Delta_i(Y) = 0$ for all $i \in \{1, \ldots, m\}$.*

4.1 Description of the recursive process

Let $K_1 := \overline{C}(x_2, \ldots, x_m)$ so that $K = K_1(x_1)$ and consider $\mathcal{V} := \{Y \in K^n\,;\, \Delta_1(Y) = 0\}$. A basis of the K_1-vector space $\mathcal{V}$ can be computed with the algorithm developed in [3] viewing $x_2, \ldots, x_m$ as transcendental constants. We assume that non-zero rational solutions exist otherwise (2) does not admit any non-trivial rational solution and we are done.

Lemma 1. *The K_1-vector space $\mathcal{V} = \ker_{K^n}(\Delta_1)$ is stable under the action of each Δ_i for $i = 2, \ldots, m$.*

Proof. Let $i \in \{2, \ldots, m\}$. From the integrability conditions (3), we have $\Delta_1\, \Delta_i = \Delta_i\, \Delta_1$. So $\Delta_1\, \mathcal{V} = 0$ implies $\Delta_i\, \Delta_1\, \mathcal{V} = 0$ and thus $\Delta_1\, \Delta_i\, \mathcal{V} = 0$ so that $\Delta_i\, \mathcal{V} \subseteq \mathcal{V}$. $\square$

Proposition 5. *With the above notation, there exists a non-singular matrix $P \in \mathbb{M}_n(K)$ such that the matrices $B_i := P^{-1}\,(A_i\, P - \partial_i(P))$, for $i = 1, \ldots, m$, are of the form:*

$$B_i = \begin{pmatrix} B_i^{11} & B_i^{12} \\ 0 & B_i^{22} \end{pmatrix}, \quad B_i^{11} \in \mathbb{M}_s(K).$$

Moreover, $B_1^{11} = 0$ and for all $i = 2, \ldots, m$, B_i^{11} does not involve the variable x_1, i.e., *$B_i^{11} \in \mathbb{M}_s(K_1)$.*

Proof. Let $v_1, \ldots, v_s$ be a basis of $\mathcal{V}$ with $0 < s \leq n$ and $P \in \mathbb{M}_n(K)$ a non-singular matrix having $v_1, \ldots, v_s$ as first columns. Performing the change of variable $Y = P\,Z$ in (2) yields a new connection $[B_1, \ldots, B_m]$ with $B_i = P^{-1}\,(A_i\, P - \partial_i(P))$. The form of the B_i's and the fact that $B_1^{11} = 0$ come directly from the definition of P and Lemma 1. Now, the assertion $B_i^{11} \in \mathbb{M}_s(K_1)$ follows from the integrability conditions (3) applied to the restrictions $\Delta_{i|\mathcal{V}}$ of the Δ_i's to $\mathcal{V}$: indeed, $\forall\, i \in \{2, \ldots, m\}$, we have $\Delta_{i|\mathcal{V}}\, \Delta_{1|\mathcal{V}} = \Delta_{1|\mathcal{V}}\, \Delta_{i|\mathcal{V}}$ that is $(\partial_i - B_i^{11})\, \partial_1 = \partial_1\, (\partial_i - B_i^{11})$ which implies $\partial_i\, \partial_1 - B_i^{11}\, \partial_1 = \partial_1\, \partial_i - B_i^{11}\, \partial_1 - \partial_1(B_i^{11})$ and then $\partial_1(B_i^{11}) = 0$. $\square$

Now, with the above notation, as $v_1, \ldots, v_s$ is a K_1-basis of $\mathcal{V} = \ker_{K^n}(\Delta_1)$, we know that every rational solution of (2) can be written $Y = \sum_{i=1}^{s} \gamma_i\, v_i$ with $\gamma_i \in K_1$ for $1 \leq i \leq s$. In matrix notation, this can be written $Y = V\,\Gamma$ with $V = (v_1\ \ldots\ v_s) \in \mathbb{M}_{n \times s}(K)$ and $\Gamma = (\gamma_1\ \ldots\ \gamma_s)^T \in K_1^s$. We are then reduced to computing $\Gamma \in K_1^s$.

Proposition 6. *With the above notation, we have that $Y = V\,\Gamma \in K^n$ is a rational solution of $[A_1, \ldots, A_m]$ if and only if $\Gamma \in K_1^s$ is a rational solution of $[B_2^{11}, \ldots, B_m^{11}]$,* i.e.,

$$\begin{cases} \tilde{\Delta}_2\, \Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_2 := \partial_2\, \mathbb{I}_s - B_2^{11}, \\ \quad \vdots \\ \tilde{\Delta}_m\, \Gamma = 0 \quad \text{with} \quad \tilde{\Delta}_m := \partial_m\, \mathbb{I}_s - B_m^{11}. \end{cases} \tag{7}$$

Proof. With the above notation, we have $P = (V \quad W)$ with $W \in \mathbb{M}_{n \times (n-s)}(K)$. Thus, the relation $Y = P\,Z$ can be written $Y = (V \quad W)\,(Z_1^T \quad Z_2^T)^T = V\,Z_1 + W\,Z_2$, with $Z_1 \in K^s$, $Z_2 \in K^{n-s}$, so that $Y = V\,\Gamma$ leads to $Z_1 = \Gamma$ and $Z_2 = 0$. Now Y is a solution of $[A_1, \ldots, A_m]$ if and only if Z is a solution of $[B_1, \ldots, B_m]$. Plugging $Z = (\Gamma^T \quad 0^T)^T$ into this system and using Proposition 5, we get the result. $\square$

A consequence of this proposition is that rational solutions of (2) can be found recursively. More precisely, we have reduced the problem of computing rational solutions of the original connection (2) of size n in m variables by that of computing rational solutions of the connection (7) of size $s \leq n$ in $m - 1$ variables.

We now give an efficient method for computing the matrices B_i^{11} needed for applying recursion without computing the whole matrix P. With the above notation, we can write $P\,B_i = A_i\,P - \partial_i(P)$ which leads to $V\,B_i^{11} = A_i\,V - \partial_i(V)$ with $V = (v_1 \;\ldots\; v_s) \in \mathbb{M}_{n\times s}(K)$. Now V belongs to $\mathbb{M}_{n\times s}(K)$ and has rank s (by definition) so, there exists an invertible matrix $T \in \mathbb{M}_n(K)$ such that $T\,V = (H^T \quad 0^T)^T$ with $H \in \mathbb{M}_s(K)$ and $\mathrm{rank}(H) = s$. Let $T = (T_1^T \quad T_2^T)^T$ with $T_1 \in \mathbb{M}_{s\times n}(K)$. We then have:

$$\forall\, i \in \{1,\ldots,m\},\; B_i^{11} = H^{-1}\,T_1\,(A_i\,V - \partial_i(V)). \qquad (8)$$

Note that H and T_1 can be computed by performing Gaussian elimination on the matrix $(V \quad \mathbb{I}_n) \in \mathbb{M}_{n\times(s+n)}(K)$.

4.2 Algorithm and complexity analysis

By iterating the process described in Subsection 4.1, we obtain the following recursive algorithm:

ALGORITHM 1. **Input**: *the square matrices $A_1,\ldots,A_m$ of size n and the variables $x_1,\ldots,x_m$ defining (2).*
Output: *A matrix whose columns form a basis of rational solutions of (2) or $\{\}$ if no non-trivial rational solution exists.*

1. *Compute a basis $(v_1,\ldots,v_s)$ of rational solutions of $\Delta_1\,Y = 0$ considered as an ODS in one variable x_1,* i.e., *$x_2,\ldots,x_m$ are viewed as transcendental constants;*
2. **If** *$s = 0$,* **then Return** *$\{\}$,* **else**
   - *Let V be the matrix having $v_1,\ldots,v_s$ as columns;*
   - **If** *$m = 1$,* **then Return** *V,* **else**
     - *Perform Gaussian elimination on $(V \quad \mathbb{I}_n)$ to compute $T = (T_1^T \quad T_2^T)^T$ with $T_1 \in \mathbb{M}_{s\times n}(K)$ such that $T\,V = (H^T \quad 0^T)^T$ with $H \in \mathbb{M}_s(K)$ and* $\mathrm{rank}(H) = s$;
     - *Compute $B_2^{11},\ldots,B_m^{11}$ using Formula (8);*
     - **Return** *V multiplied by the result of applying the algorithm to $([B_2^{11},\ldots,B_m^{11}],[x_2,\ldots,x_m])$.*

PROPOSITION 7. *Let $\prod_{i=1}^s q_i^{r_i+1}$ denote the factorization of the least common multiple of the denominators of the entries of the A_i's into irreducible factors in $C[x_1,\ldots,x_m]$, $d = \sum_{i=1}^s (r_i+1)\deg(q_i)$ and, for each A_i, $r_{\infty,i}$ be as in (5). Algorithm 1 computes a basis of rational solutions of (2) using at most $\mathcal{O}\left(n^5\,(\max_i(r_i)\,d + \sum_{j=1}^m r_{\infty,j}^2) + n^3\,\sum_{j=1}^m N_j^2\right)$ operations in $k = C(x_1,\ldots,x_m)$, where $N_j \in \mathbb{N}$ is a degree bound (in x_j) for the numerators of rational solutions of $\partial Y/\partial x_j = A_j Y$.*

PROOF. The complexity is dominated by the computation of rational solutions of each system. With the above notation, from Proposition 1, a universal denominator of rational solutions of (2) can be computed using $\mathcal{O}(n^5\,\max_i(r_i)\,d)$ operations in k. Now, from Proposition 2, a degree bound N_j w.r.t. the variable x_j can be computed using $\mathcal{O}(n^5\,r_{\infty,j}^2)$ and polynomial solutions of the jth system can be computed in at most $\mathcal{O}(n^3\,N_j^2)$ operations in k. $\square$

Note that the complexity estimate given in Proposition 7 is a worst case estimate since in general the dimension n of the connection decreases as we apply recursion and the computations are performed in fields smaller than k.

REMARK 2. *As the algorithm developed in [3] handles the case of ODS with right-hand sides, Algorithm 1 can be slightly modified in order to be used for computing rational solutions of integrable connections having non-zero right-hand sides $b_i \in k^n$ satisfying the conditions $\Delta_i(b_j) = \Delta_j(b_i)$. Note that this case is handled by our implementation: see* `http://www.ensil.unilim.fr/~cluzeau/PDS.html` *and more precisely the Gaussian cases in* BrycLetacAllCases.

4.3 Denominators of rational solutions

For ODS, we know that the irreducible factors of the denominators of rational solutions are among the irreducible factors of the denominators of the system matrix (see Subsection 3.1). For integrable connections, we have the following more precise result:

PROPOSITION 8. *Let $q \in \overline{C}[x_1,\ldots,x_m]$ be an irreducible factor of the denominator of a rational solution of (2) and assume that q involves some variable x_{i_0},* i.e., *$\partial_{i_0}(q) \neq 0$. Then, q divides the denominator of some entry of A_{i_0}.*

PROOF. Let $Y \in K^n$ be a non-zero rational solution of (2), that is, for all i in $\{1,\ldots,m\}$, $\partial_i(Y) = A_i\,Y$. Let $Y = P/Q$ with $P = (P_1 \ldots P_n)^T \in \overline{C}[x_1,\ldots,x_m]^n$, $Q \in \overline{C}[x_1,\ldots,x_m]$ such that there exists $j_0 \in \{1,\ldots,n\}$ with $\gcd(Q,P_{j_0}) = 1$. We have

$$\forall\, i \in \{1,\ldots,m\},\; \partial_i(Y) = -\frac{\partial_i(Q)\,P}{Q^2} + \frac{\partial_i(P)}{Q} = \frac{A_i\,P}{Q},$$

so that $\forall\, i \in \{1,\ldots,m\}$, $Q\,(\partial_i(P) - A_i\,P) = \partial_i(Q)\,P$. Let A_i^{jk} denote the (j,k)th entry of the matrix A_i ($i \in \{1,\ldots,m\}$, $j,\,k \in \{1,\ldots,n\}$). The latter relation can thus be written:

$$Q\left(\partial_i(P_j) - \sum_{k=1}^n A_i^{jk}\,P_k\right) = \partial_i(Q)\,P_j,\; 1 \le i \le m,\; 1 \le j \le n.$$

Let $q \in \overline{C}[x_1,\ldots,x_m]$ be an irreducible factor of Q satisfying $\partial_{i_0}(q) \neq 0$. We have thus $Q = q^\alpha\,r$ with $\alpha \in \mathbb{N}^*$ and $\gcd(q,r) = 1$ so that $\partial_{i_0}(Q) = \alpha\,q^{\alpha-1}\,\partial_{i_0}(q)\,r + q^\alpha\,\partial_{i_0}(r)$. Then, $\forall\, j \in \{1,\ldots,n\}$,

$$q\,r\left(\partial_{i_0}(P_j) - \sum_{k=1}^n A_{i_0}^{jk}\,P_k\right) = (\alpha\,\partial_{i_0}(q)\,r + q\,\partial_{i_0}(r))\;P_j.$$

For j in $\{1,\ldots,n\}$, let $D_{i_0}^j$ denote the denominator of the jth row of A_{i_0} (that is the least common multiple of the denominators of the elements of the jth row of A_{i_0}). We thus obtain: $\forall\, j \in \{1,\ldots,n\}$,

$$q\,r\left(D_{i_0}^j\,\partial_{i_0}(P_j) - \sum_{k=1}^n N_{i_0}^{jk}\,P_k\right) = (\alpha\,\partial_{i_0}(q)\,r + q\,\partial_{i_0}(r))P_j\,D_{i_0}^j,$$

for some polynomials $N_{i_0}^{jk}$ ($k \in \{1,\ldots,n\}$) obtained from the numerators of the elements of the jth row of A_{i_0}. Now, we know by hypothesis that there exists $j_0 \in \{1,\ldots,n\}$ such that $\gcd(Q,P_{j_0}) = 1$, so, necessarily, q divides $D_{i_0}^{j_0}$ since $\partial_{i_0}(q) \neq 0$ implies $\gcd(q,\alpha\,\partial_{i_0}(q)\,r + q\,\partial_{i_0}(r)) = 1$. $\square$

This result has useful consequences on our recursive algorithm. The first one is that irreducible factors of the denominators of rational solutions of (2) that depend on all the variables $x_1, \ldots, x_m$ necessarily appear in the denominator of the first matrix A_1. They are thus removed when we apply the first recursion since the variable x_1 then disappears. Moreover, with the above notation, assume that a new irreducible factor of $\overline{C}[x_2, \ldots, x_m]$ is introduced in the denominator of one of the B_i^{11}'s for $i = 2, \ldots, m$ but was not contained in the denominator of one of the A_i's for $i = 2, \ldots, m$ (from Formula (8), such a factor can come from the term H^{-1}). Then it does not need to be taken into account since from Proposition 8, if this factor appears in the denominator of a rational solution of System (2), then it already appears in one of the denominators of the matrices A_i's for $i = 2, \ldots, m$ of the original system.

Remark 3. *Another useful way of using Proposition 8 is the following: by performing a randomized linear change of independent variables,* i.e., $x_j = \sum_{i=1}^{m} c_{j,i}\, t_i$, $1 \leq j \leq m$, *where the constants $c_{j,i}$'s are chosen randomly, we can suppose that, generically, all irreducible factors of the denominators of the matrices of the integrable connection depend on all the new independent variables $t_1, \ldots, t_m$. Therefore, from Proposition 8, once we have computed the rational solutions of the first system, then, in the following steps of the recursion, we only have to search for polynomial solutions.*

4.4 Implementation and applications

Algorithm 1 has been implemented in Maple and is available at `http://www.ensil.unilim.fr/~cluzeau/PDS.html` with some examples of calculations. For the computation of rational solutions of an ODS, we have updated the code implementing the algorithm developed in [3] and contained in the Maple package Isolde ([7]).

Example 2. *Consider again the D-finite PDS studied in [9, Sections 3 and 4]. The authors are, in particular, interested in closed form solutions of these systems.*
In two variables, running our implementation on the integrable connection given in Example 1, we get the matrix

$$\begin{bmatrix} \frac{1}{4}\, {x_1}^2\, \beta + x_2 & x_1 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2}\, x_1\, \beta & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose columns form a basis of the rational solutions space of the integrable connection. This means that a basis of rational solutions of the original system (4) is given by 1, x_1 *and* $\frac{1}{4}\, x_1^2\, \beta + x_2$.
Now, in three independent variables x_1, x_2, x_3, the system is given by:

$$\left\{ \begin{array}{rcl} -\frac{\beta}{2}\, \partial_3 y + 2\, \partial_1 \partial_2 y + x_1\, \partial_2^2 y - x_3\, \partial_3^2 y & = & 0, \\ -\beta\, \partial_2 y + \partial_1^2 y - x_2\, \partial_2^2 y - 2\, x_3\, \partial_2 \partial_3 y & = & 0, \\ \partial_2^2 y + 2\, \partial_1 \partial_3 y + 2\, x_1\, \partial_2 \partial_3 y + x_2\, \partial_3^2 y & = & 0. \end{array} \right. \quad (9)$$

See [9, Theorem 3.2, (i)] in the case $a^2 - 4\, b = 0$. The associated integrable connection is of size $n = 8$ and applying our algorithm, we get that a basis of the rational solutions space of System (9) is given by 1, x_1, $x_2 + \frac{1}{2}\, \beta\, x_1^2$ *and* $x_3 + \frac{1}{24}\, \beta^2 {x_1}^3 + \frac{1}{4}\, \beta\, x_2\, x_1$.
See `http://www.ensil.unilim.fr/~cluzeau/PDS.html` *and more precisely* BrycLetacRationalSolutions.

Computing rational solutions of integrable connections is useful for computing hyperexponential solutions (see Example 4). It can also be used for computing the so-called *eigenring* of an integrable connection of the form (2). The eigenring is defined as the set of matrices $P \in \mathbb{M}_n(K)$ such that $\partial_i(P) = A_i\, P - P\, A_i$, for all $i \in \{1, \ldots, m\}$. It is useful for decomposing (or reducing) systems of the form (2) (see [13, 6, 4, 23] and references therein).

Example 3. *Let us consider the integrable connection of the form (2) with $n = m = 2$, $k = \mathbb{Q}(x_1, x_2)$ and defined by*

$$A_1 = \begin{bmatrix} 0 & \frac{2}{(x_1+x_2)(-x_2+x_1)} \\ \frac{-x_1}{-x_2+x_1} & \frac{-1}{(x_1+x_2)} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \frac{-2}{(x_1+x_2)} & \frac{-2}{(x_1+x_2)(-x_2+x_1)} \\ \frac{{x_2}^2+2\,{x_1}^2-x_2\,x_1}{(x_1+x_2)(-x_2+x_1)} & \frac{1}{(x_1+x_2)} \end{bmatrix}.$$

Applying our algorithm for computing rational solutions of the integrable connection of size $n = 4$ in two variables defined by $A_1 \otimes \mathbb{I}_2 - \mathbb{I}_2 \otimes A_1^T$ and $A_2 \otimes \mathbb{I}_2 - \mathbb{I}_2 \otimes A_2^T$, we can compute the eigenring of this system. We get:

$$[\begin{bmatrix} \frac{-2\,x_1}{x_1+x_2} & \frac{-2}{x_1+x_2} \\ \frac{x_1^2+x_2^2}{x_1+x_2} & \frac{-2\,x_2}{x_1+x_2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}].$$

In particular, the eigenring contains a non-trivial element which admits two distinct eigenvalues, namely $-1 \pm i$, so that the integrable connection can be decomposed over $\mathbb{Q}(i)(x_1, x_2)$.
See `http://www.ensil.unilim.fr/~cluzeau/PDS.html` *and more precisely* EigenringExample.

5. HYPEREXPONENTIAL SOLUTIONS

In this section, we consider the problem of computing hyperexponential solutions of (2). The approach that we follow is also the one used in [19] for handling this problem in the more general context of Laurent-Ore algebras. We give here details on how this can be done efficiently for integrable connections and provide an implementation. Following the terminology used in [18, 19], we first define hyperexponential elements and hyperexponential solutions of integrable connections of the form (2).

Definition 6. *Let $K = \overline{C}(x_1, \ldots, x_m)$ and L a differential extension of K having the same field of constants. A non-zero element $u \in L$ is said to be* hyperexponential over *K if for all $i \in \{1, \ldots, m\}$, $f_i := \partial_i(u)/u \in K$.*
A hyperexponential solution of (2) *is a product $u\, z$ of a hyperexponential element u over K by a vector $z \in K^n$ such that $\Delta_i(u\, z) = 0$ for all $i \in \{1, \ldots, m\}$.*

Lemma 2. *With the above notation, every hyperexponential element u over K satisfies $\partial_j(f_i) = \partial_i(f_j)$, for all i, j in $\{1, \ldots, m\}$. Moreover, if $u\, z$ is a hyperexponential solution of $[A_1, \ldots, A_m]$, then z is a rational solution of $[A_1 - f_1\, \mathbb{I}_n, \ldots, A_m - f_m\, \mathbb{I}_n]$.*

PROOF. Straightforward from Definition 6. □

5.1 Description of the algorithm

The recursive process proposed below follows the lines of the one developed in Subsection 4.1. We start by computing exponential solutions of $\Delta_1 Y = 0$ considered as an ODS in one variable x_1 ($x_2, \ldots, x_m$ viewed as transcendental constants) which can be done by adapting the algorithm of [21]. Now, suppose that we have an exponential solution $u\,z$ of the system $\Delta_1 Y = 0$, where u is hyperexponential over K. Let $f_i := \partial_i(u)/u \in K$ and $\Delta_{i,u} := \partial_i - (A_i - f_i\,\mathbb{I}_n)$, for i in $\{1, \ldots, m\}$. We perform the change of variable $Y = u\,Z$. From the integrability conditions (3) and Lemma 2, it is straightforward that $[A_1 - f_1\,\mathbb{I}_n, \ldots, A_m - f_m\,\mathbb{I}_n]$ is integrable. Moreover $\mathcal{W}_u := \{w \in K^n\,;\, \Delta_{1,u}(w) = 0\}$ is a $\mathbb{K}_1$-vector space which is stable under each $\Delta_{i,u}$ for $i = 2, \ldots, m$ (see Lemma 1). Let $w_1, \ldots, w_s$ be a basis of $\mathcal{W}_u$, complete it into a basis of K^n and denote by P the matrix having the elements of this basis as columns. Performing the change of variable $Z = P\,T$ yields a new connection $[B_1, \ldots, B_m]$ with

$$B_i = P^{-1}\left((A_i - f_i\,\mathbb{I}_n)\,P - \partial_i(P)\right). \qquad (10)$$

The matrices B_i can be decomposed into blocks of appropriate dimensions and have the same form and properties as the ones in Proposition 5. Finally, we obtain the following analog of Proposition 6 (compare also to [19, Proposition 11]) which proves that hyperexponential solutions can be computed recursively.

LEMMA 3. *With the above notation, let W_u be the matrix having $w_1, \ldots, w_s$ as columns. A vector $Y = u\,W_u\,\Gamma_u$ is a hyperexponential solution of $[A_1, \ldots, A_m]$ if and only if Γ_u is a hyperexponential solution of $[B_2^{11}, \ldots, B_m^{11}]$ where the B_i's are defined by (10) and $B_i^{11} \in \mathbb{M}_s(K_1)$ denotes the first $s \times s$ submatrix of B_i. Hyperexponential solutions of (2) can thus be found recursively.*

As for rational solutions, a worst case complexity estimate of the algorithm in operations in K can be obtained by taking the sum of the complexity estimate given by Proposition 4 applied to each ODS.

REMARK 4. *During the calculation of exponential solutions of $\Delta_1 Y = 0$, we must discard the local exponential parts involving non-rational functions of $x_2, \ldots, x_m$. Indeed, from Definition 4, only rational functions of x_1 appear in the non-ramified local exponential parts but some extension of the base field, namely $\overline{C}(x_2, \ldots, x_m)$ here, may be introduced. So, each exponential part which is not rational in $x_2, \ldots, x_m$ should be removed since it can not lead to a hyperexponential solution in the sense of Definition 6.*
Another interesting improvement that can reduce significantly the number of combinations of local exponential parts to be considered (see Subsection 3.2) is to use p-curvature calculations as it is done in [14] for the case of a scalar linear differential equation. The p-curvature can be computed using the algorithm given in [6] and combinations can be removed exactly in the same way as in [14].

In [18], the authors propose another algorithm for computing hyperexponential solutions of D-finite PDS. This algorithm proceeds in a different way. It starts by extracting from the D-finite system a rectangular system that is non-zero univariate polynomials $P_i \in C[x_1, \ldots, x_m][\partial_i]$ for $i = 1, \ldots, m$. Then, it computes hyperexponential solutions of each P_i and try to combine them to construct hyperexponential solutions of the original D-finite system. The main difference with our algorithm (and that of [19]) is that once we have computed the exponential solutions of the first system, we use them to reduce both the size of the connection and the number of variables before considering the second system.

5.2 Implementation and examples

The recursive algorithm summarized in Subsection 5.1 has been implemented in Maple and is available with some examples at `http://www.ensil.unilim.fr/~cluzeau/PDS.html`. For the computation of exponential solutions of ODS, we have made small changes to the code implementing the algorithm developed in [21] that is contained in ISOLDE ([7]). This was necessary in order to allow transcendental constants in the coefficients and also to discard, at each recursive call, exponentials parts that can not lead to hyperexponential solutions in the sense of Definition 6.

EXAMPLE 4. *Let us consider again the PDS studied in [9]. For the two variables case, we have seen in Example 2 that it admits a basis of rational solutions composed of three elements. Now applying directly our algorithm for computing hyperexponential solutions we get the latter three rational solutions and another hyperexponential solution given by $(-4\,x_2 + x_1^2)^{\frac{1-\beta}{2}}$. We thus have found all the closed form solutions of System (4).*
For the three variables case given by (9), we have exhibited, in Example 2, a basis of four linearly independent rational solutions. Now, if we run our algorithm for computing hyperexponential solutions, we get that the system does not admit more hyperexponential solutions. The running time for the latter calculation is 194 seconds[2]*. A faster way of computing a basis of hyperexponential solutions of this system is the following: we start by computing a basis of rational solutions of the integrable connection of size $n = 8$ using Algorithm 1. Then, we construct a basis of K^8 with $K = \mathbb{C}(x_1, x_2, x_3)$ having as first elements the four non-zero rational solutions that we have found. Let $P \in \mathbb{M}_8(K)$ denote the matrix having the vectors of this basis as columns and perform the change of variable $Y = P\,Z$ in the integrable connection. The three matrices of the new integrable connection have zero as first four columns so that we are reduced to computing hyperexponential solutions of an integrable connection of size $n = 4$. Running our implementation for computing hyperexponential solutions on this connection of smaller size, we get that there is no extra-hyperexponential solution so that we can conclude that hyperexponential solutions of System (9) are given by the four non-zero rational solutions given in Example 2. The total running time of performing this alternative method is 22 seconds.*
See `http://www.ensil.unilim.fr/~cluzeau/PDS.html` and more precisely BrycLetacHyperexponentialSolutions.
This suggests the following strategy for computing hyperexponential solutions: first compute rational solutions using Algorithm 1, then reduce the size of the connection as

[2]Computations made on a 2.53 GHz Intel Core 2 Duo

above before computing non-rational hyperexponential solutions. Note that when non-zero hyperexponential solutions of the reduced integrable connection exist, a small "integration step" is needed since, for all $i = 1, \dots, m$,

$$\partial_i \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \Leftrightarrow \begin{cases} \partial_i(Y_1) = A_1 Y_2, \\ \partial_i(Y_2) = A_2 Y_2, \end{cases}$$

so that once Y_2 is known, Y_1 has to be found by integrations. In particular, if we note $Y_j = u\, z_j$, $j = 1, 2$ and $f_i = \partial_i(u)/u$, $i = 1, \dots, m$, then the components of z_1 are rational solutions of the scalar Risch equations given by the rows of $\partial_i(z_1) = f_i\, z_1 + A_1\, z_2$, $i = 1, \dots, m$.

EXAMPLE 5. *Consider the D-finite PDS given in [18, Ex. 3.3] which can be written as an integrable connection of the form (2) of size $n = 2$ in $m = 3$ variables with matrices*

$$A_1 = \begin{bmatrix} \frac{4\,x_3+x_1}{4\,x_1\,x_3+x_1} & \frac{2\,x_1\,x_3-2\,x_3}{4\,x_1\,x_3+x_1} \\ \frac{2\,x_1-2}{4\,x_1\,x_3+x_1} & \frac{4\,x_1\,x_3+1}{4\,x_1\,x_3+x_1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{4\,x_3}{Q_2} & \frac{-2\,x_3}{Q_2} \\ \frac{-2}{Q_2} & \frac{1}{Q_2} \end{bmatrix},$$

with $Q_2 = 4\,x_3\,x_2^2 + x_2^2$, and $A_3 = \begin{bmatrix} 0 & 1 \\ \frac{3+4\,x_3}{4\,{x_3}^2+x_3} & \frac{16\,{x_3}^2-3}{8\,{x_3}^2+2\,x_3} \end{bmatrix}.$

Running our algorithm, we find the matrix

$$[\begin{bmatrix} \frac{-2\,x_1}{\sqrt{x_3}}\, e^{\frac{-1}{x_2}} & \frac{1}{2}\, e^{x_1+2\,x_3} \\ \frac{x_1}{x_3^{3/2}}\, e^{\frac{-1}{x_2}} & e^{x_1+2\,x_3} \end{bmatrix},$$

whose columns contain hyperexponential solutions of the integrable connection. This is consistent with the computations in [18, Ex. 3.3].
See Example LiSchwarzTsarevHyperexponentialSolutions *at* *`http://www.ensil.unilim.fr/~cluzeau/PDS.html`*.

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