This is an example of an arbitrary pseudo linear system \( A \delta Y + B \Phi Y = 0 \) where \( \delta \) and \( \Phi \) are defined below.

restart;

### We load our package

```maple
with(PseudoLinearSystems):
```

### We give the following K-automorphism

```maple
PhiAction := proc(M, z) RETURN(subs(z = a*z + b, M)); end proc;
```

### We give the Phi-derivation

```maple
DeltaAction := proc(M, z) RETURN(M - PhiAction(M, z)); end proc;
```

### For a singularity at infinity we assign the local parameter \( t \) to be the following

```maple
t := 1/z;
```

### For a pseudo-linear system \( A \delta Y + B \phi Y = 0 \), the matrices \( A \) and \( B \) are given as

\[
A := \begin{bmatrix}
2z^{-2} + 2z^{-1} + 1 & 2z^{-1} & z^{-3} - z^{-2} - z^{-1} \\
( -2)z^{-3} & -z^{-3} - 2z^{-2} - z^{-1} & z^{-3} + z^{-2} \\
2z^{-3} - z^{-2} + 2z^{-1} & ( -2)z^{-3} + z^{-2} + z^{-1} & ( -2)z^{-3} + 2z^{-2} - z^{-1}
\end{bmatrix}
\]

\[
A := \begin{bmatrix}
\frac{2}{z^2} + \frac{2}{z} + 1 & \frac{2}{z} & \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} \\
\frac{2}{z^3} - \frac{1}{z^2} & \frac{2}{z^3} - \frac{2}{z^2} - \frac{1}{z} & \frac{1}{z^3} + \frac{1}{z^2} \\
\frac{2}{z^3} - \frac{1}{z^2} + \frac{2}{z} & \frac{2}{z^3} + \frac{1}{z^2} + \frac{1}{z} & -\frac{2}{z^3} + \frac{2}{z^2} - \frac{1}{z}
\end{bmatrix}
\]

\[
B := \begin{bmatrix}
-z^{-3} + z^{-2} + z^{-1} & -z^{-3} - z^{-2} + z^{-1} & ( -2)z^{-3} - 2z^{-2} - 2z^{-1} + 1 \\
-z^{-3} + z^{-1} & -z^{-3} + 2z^{-2} & z^{-3} + 2z^{-2} + z^{-1} + 1 \\
z^{-2} + z^{-1} + 1 & z^{-3} + 2z^{-2} - 2z^{-1} & 2z^{-2} - z^{-1} + 1
\end{bmatrix}
\]
We use our procedure to compute the leading matrix pencil of the system

\[
B := \begin{bmatrix}
\frac{-1}{z^3} + \frac{1}{z^2} + \frac{1}{z} & \frac{-1}{z^3} + \frac{1}{z^2} + \frac{1}{z} & \frac{-2}{z^3} - \frac{2}{z^2} - \frac{2}{z} + 1 \\
\frac{-1}{z^3} + \frac{1}{z} & \frac{-1}{z^3} + \frac{2}{z^2} & \frac{1}{z^3} + \frac{2}{z^2} + \frac{1}{z} + 1 \\
\frac{1}{z^2} + \frac{1}{z} + 1 & \frac{1}{z^3} + \frac{2}{z^2} - \frac{2}{z} & \frac{2}{z^2} - \frac{1}{z} + 1
\end{bmatrix}
\]

The system is not simple because the \(\det(L)=0\)

\[
L := \begin{bmatrix}
\lambda & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

We apply our algorithm to compute an equivalent simple system

\[
SF := \text{SimpleForm}(A, B, z, t, \text{DeltaAction}, \text{PhiAction}, \lambda, \text{split system}, \text{Simplify System})
\]
\[- \frac{1}{2z^3(a + b)} \left( 2az^4 + 6az^3 + 2bz^3 + 3z^4 - 5az^2 + 6bz^2 - 6z^3 + 12az \
- 5bz + 6z^2 + 12b - 9z \right),
- \frac{6az^4 + 2az^3 + 6bz^3 + z^4 - az^2 + 2bz^2 - 2z^3 + 4az - bz + 2z^2 + 4b - 3z}{4z^3(a + b)} \right] \]

\[
\left[ \begin{array}{c}
2az^3 + z^4 - az^2 + 2bz^2 - z^3 + 2az - bz + 2z^2 + 2b - 2z \frac{1}{z^4(a + b)} (2az^4 + 4az^3 + 2bz^3 + 3z^4 - 4az^2 + 4bz^2 - 3z^3 + 6az \\
- 4bz + 6z^2 + 6b - 6z) \frac{1}{2z^4(a + b)} (2az^5 - 4az^4 + 2bz^4 + 4az^3 \\
- 4bz^3 - z^4 + 2az^2 + 4bz^2 + z^3 - 2az + 2bz - 2z^2 - 2b + 2z) \end{array} \right] \]

\[
\left[ \begin{array}{ccc}z & -z & 0 \\
0 & -\frac{z}{2} & \frac{z}{2} \\
0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccc}\frac{1}{z} & -\frac{3}{z} & -\frac{1}{2z} \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccc}\lambda + \frac{a - 1}{a} & -\frac{2a - 3}{a} & -\frac{6a - 1}{2a} \\
\frac{1}{2a} & \lambda - \frac{2a + 3}{2a} & -\frac{6a + 1}{4a} \\
0 & 0 & 1 \end{array} \right] \]

\[
\frac{2a^2 - 2a - 5\lambda + 1}{2a} \]

> \textbf{The output is a "simple" pseudo-linear system } A1(\delta Y + B1)\phi Y = 0 \text{ where}

> A1 := SF[1];
> B1 := SF[2];

\[
A1 := \left[ \begin{array}{ccc}
z^3 + 2z^2 + 2z + 2 & -\frac{4z^2 + 5z + 6}{z^3} & -\frac{4z^2 + z + 2}{2z^3} \\
2z^2 - z + 4 & 2z^3 - 3z^2 + 2z - 12 & 3z^2 - 6z - 4 \\
2z^2 - z + 2 & z^3 - 5z^2 + z - 6 & -\frac{z^3 - 3z^2 + 5z + 2}{2z^4} \end{array} \right] \]

\[
B1 := \left[ \begin{array}{ccc} \end{array} \right] \]

(9)
The transformations are given by

\[ S := SF[3]; \]
\[ T := SF[4]; \]

\[ S := \begin{bmatrix} z & -z & 0 \\ 0 & -\frac{z}{2} & \frac{z}{2} \\ 0 & 0 & 1 \end{bmatrix} \]
The Determinant of the new leading matrix pencil is given by

\[
\text{newDET} := SF[6];
\]

\[
\text{newDET} := \frac{2 a \lambda^2 - 2 a - 5 \lambda + 1}{2 a}
\]

We check if the new system is equivalent to the original one

\[
\text{check1} := \text{simplify}(A1 - S \cdot A \cdot T);
\]
\[
\text{check2} := \text{simplify}(B1 - S \cdot (B \cdot \text{PhiAction}(T, z)) + A \cdot \text{DeltaAction}(T, z));
\]