Division and Slope Factorization of p-Adic Polynomials

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RIKKYO UNIVERSITY

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My personal (long-term) motivation

Computing (some) moduli spaces of p-adic Galois representations.

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- Precision for **slope factorization** algorithms.

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The p-adic integers form a subring $\mathbb{Z}_p$ of $\mathbb{Q}_p$.

Summary on p-adics

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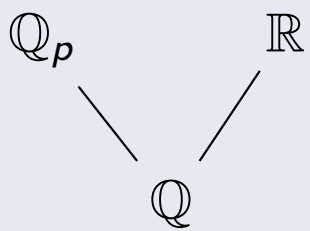

$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Z}/p\mathbb{Z} & \to & \mathbb{Z}/p\mathbb{Z} \end{array}$$

Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=k}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $k \in \mathbb{Z}$ and p a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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Example

The order of $3 * 7^{-1} + 4 * 7^0 + 5 * 7^1 + 6 * 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^k)) + (b + O(p^k)) = a + b + O(p^k).$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1}p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1}p^{-c} + O(p^{d-2c})$$

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Question:

How can we obtain **satisfying** / **optimal** behaviour regarding to precision?

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Geometrical meaning

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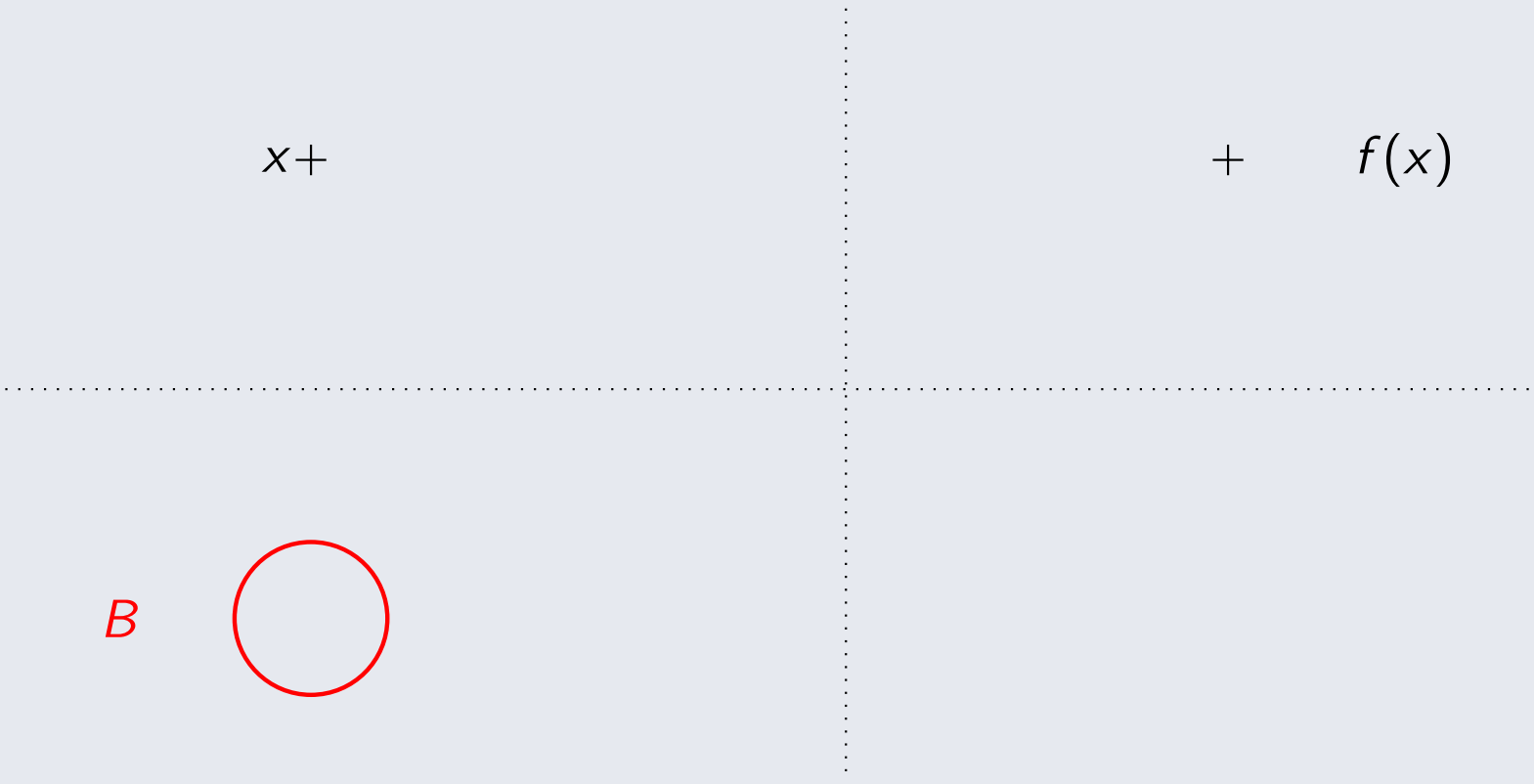

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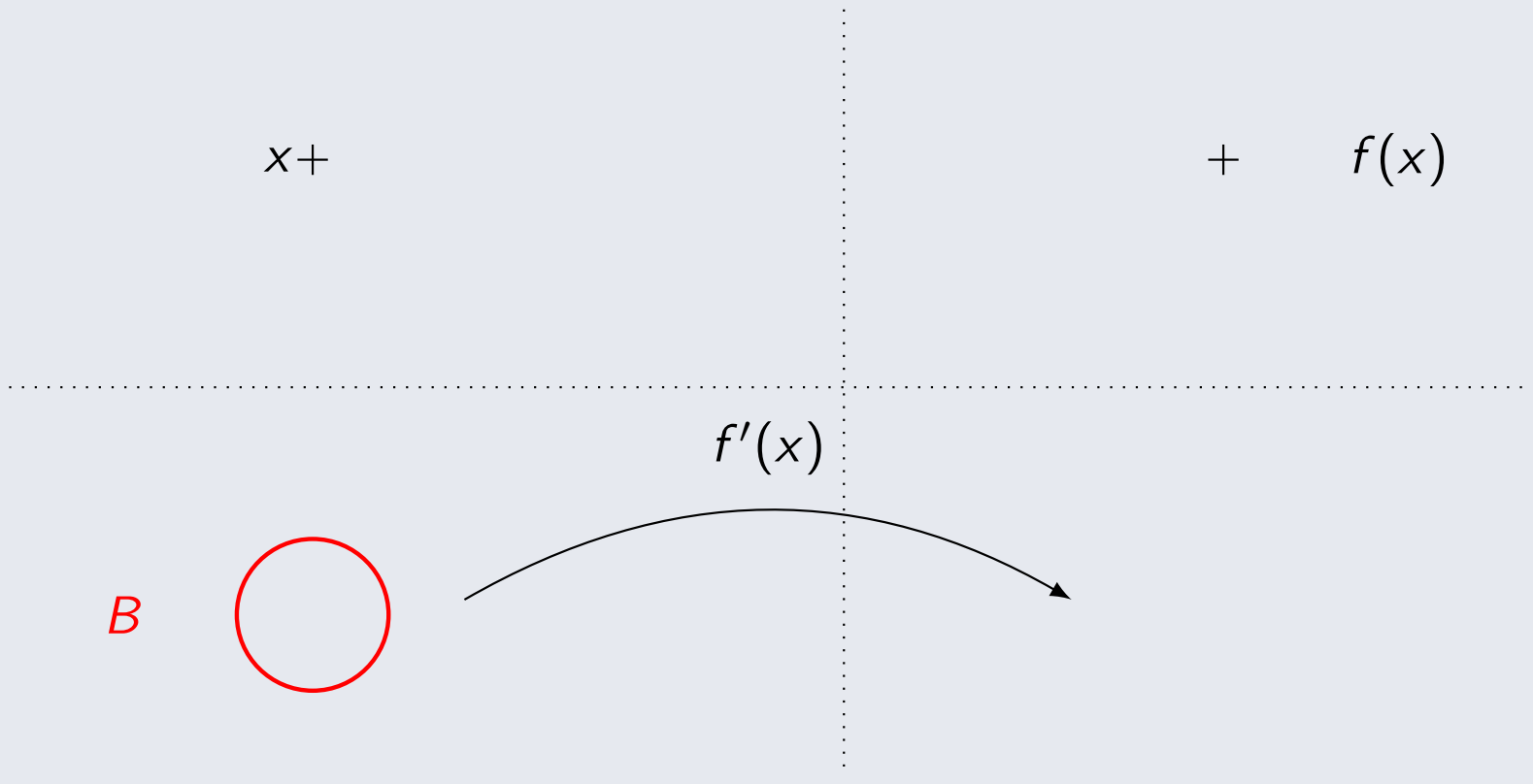

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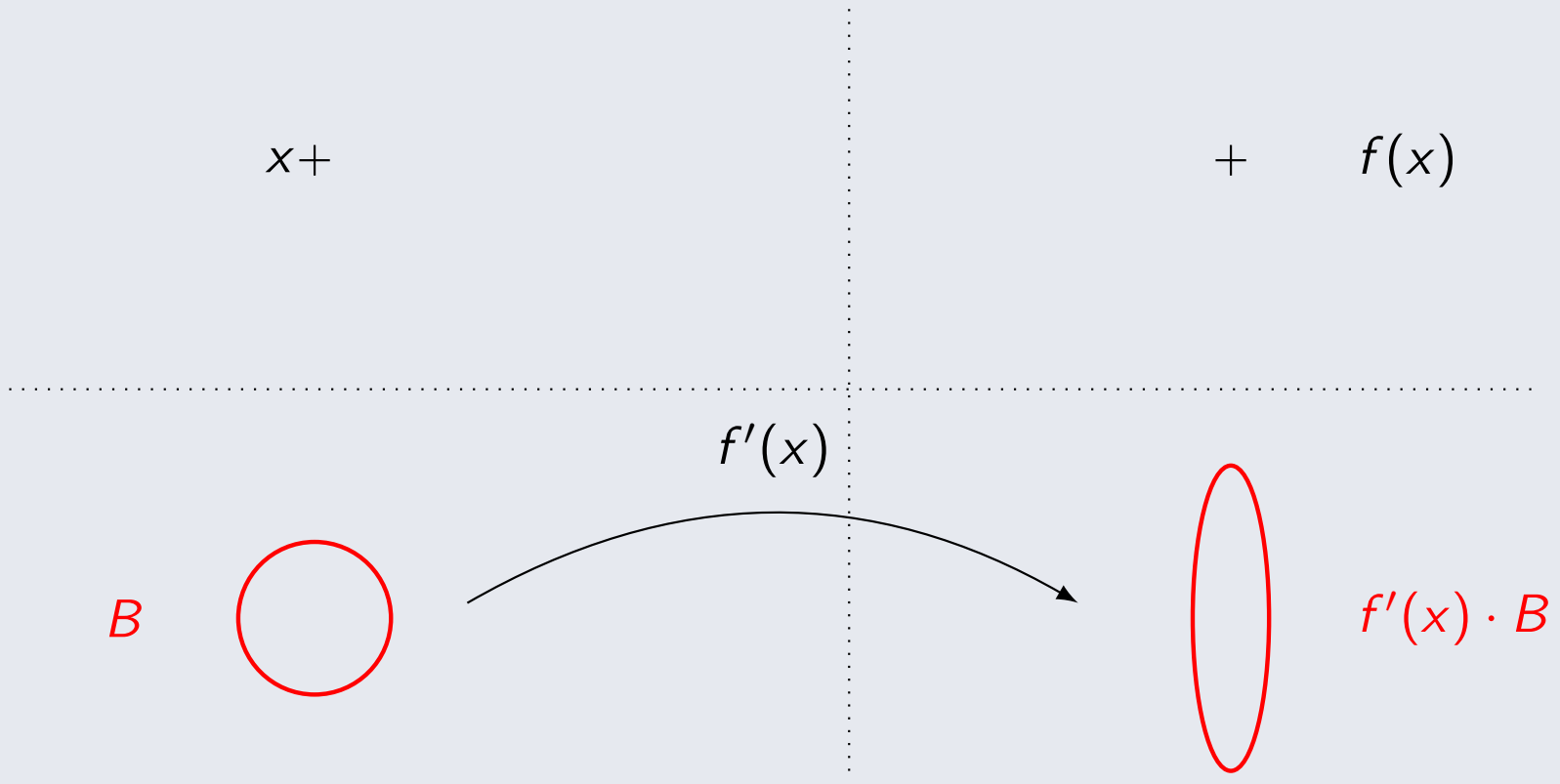

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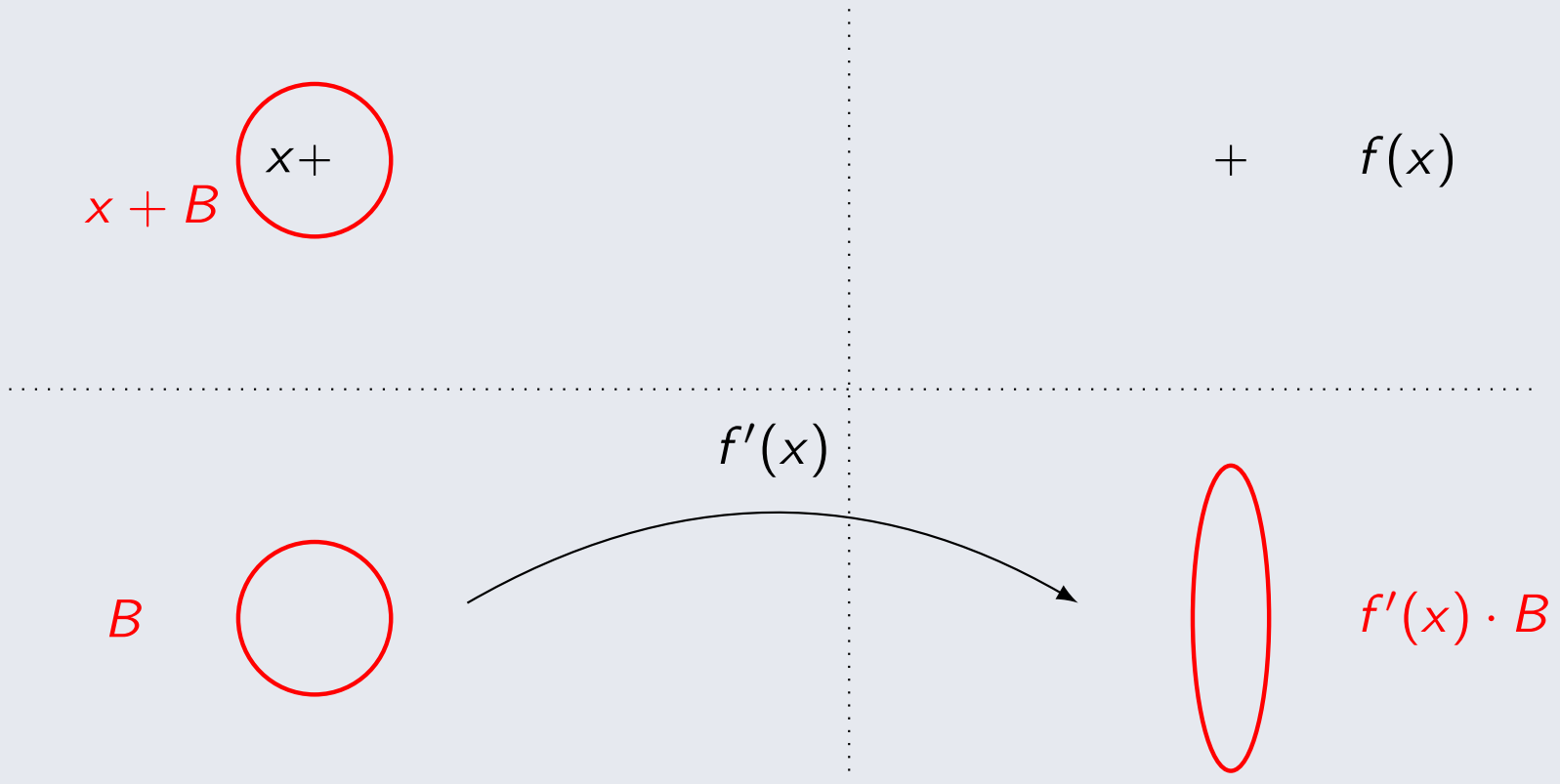

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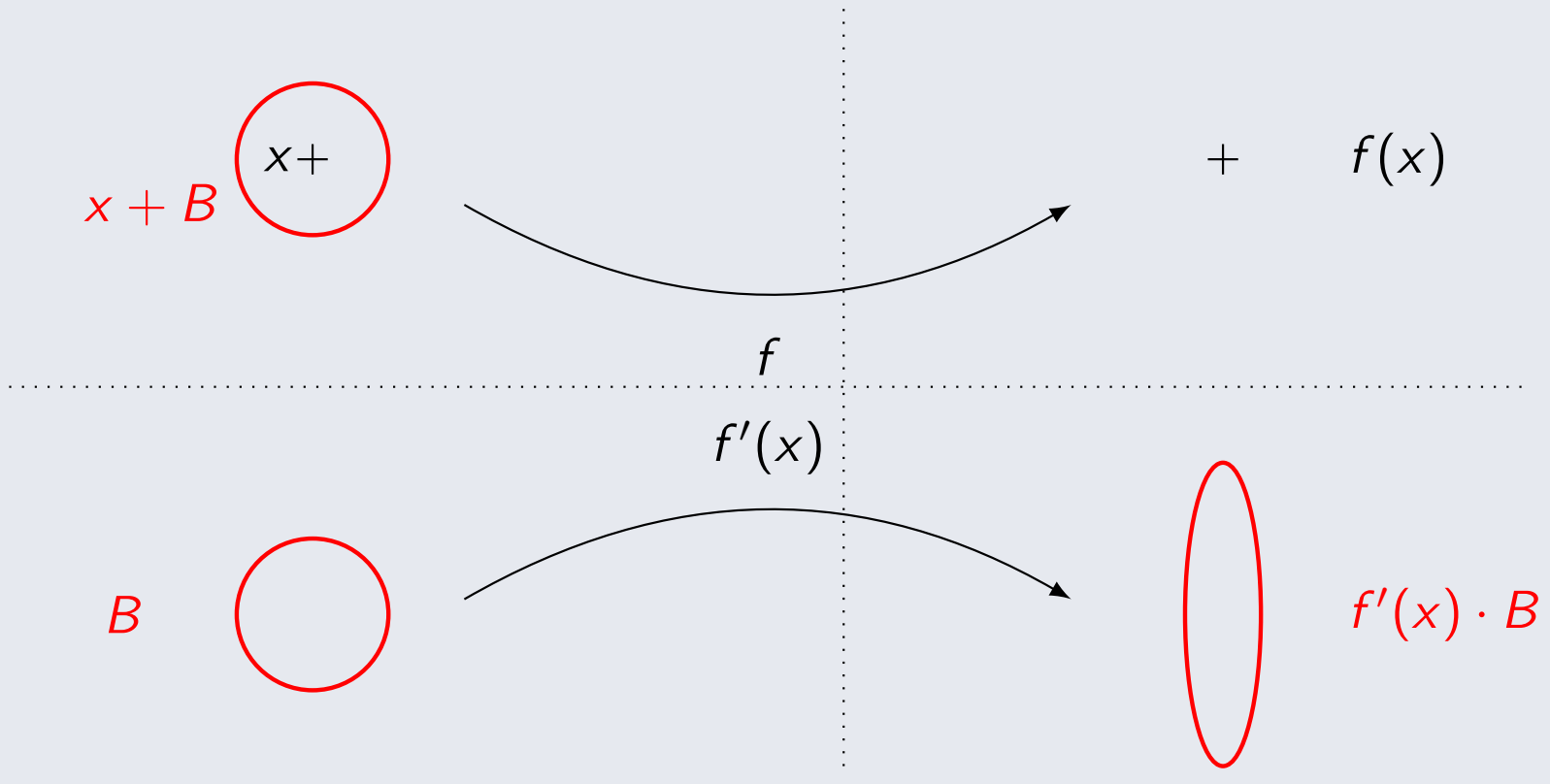

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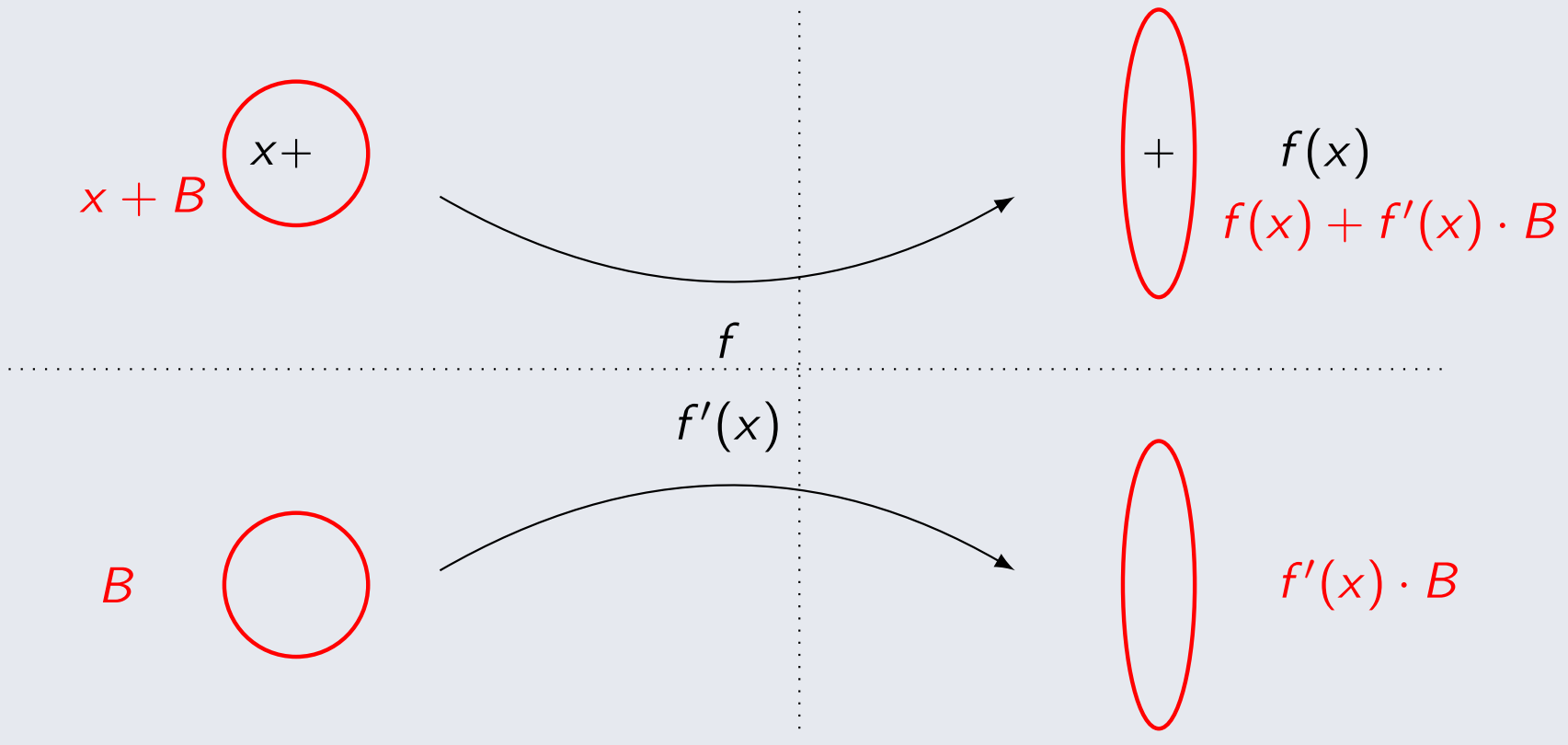

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Our framework can be extended to **(complete) ultrametric K-vector spaces** (*e.g.* being $\mathbb{F}_p((X))^n$, $\mathbb{Q}((X))^m$, $\mathbb{R}((\varepsilon))^s$).

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This can be determined with **Newton-polygon** techniques.

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Lattice precision and modular multiplication

Composed derivatives

It is easy to handle in an optimal way the modular multiplication by applying:

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Toward numerical understanding

Displaying precision

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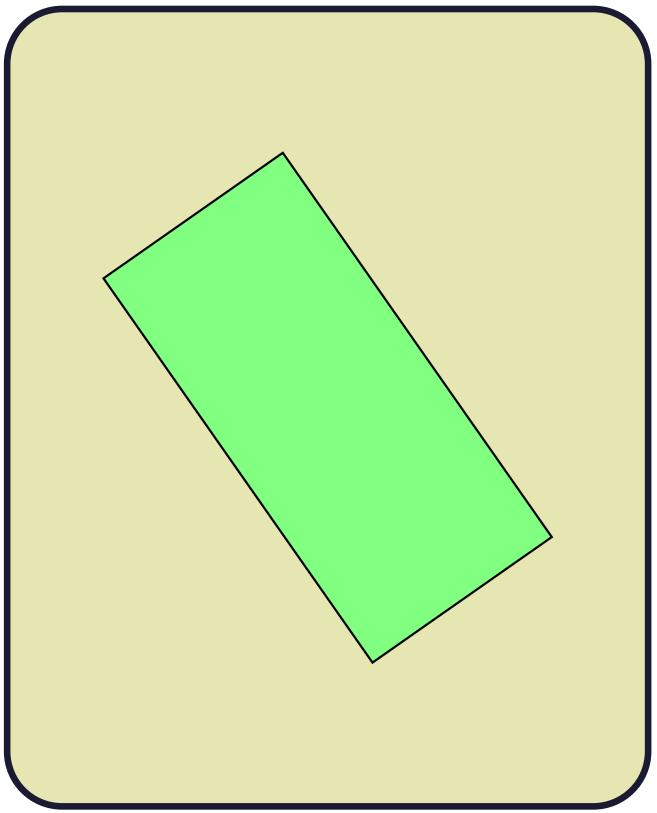
Let $H \subset \mathbb{Q}_p^n$ be a lattice.



Toward numerical understanding

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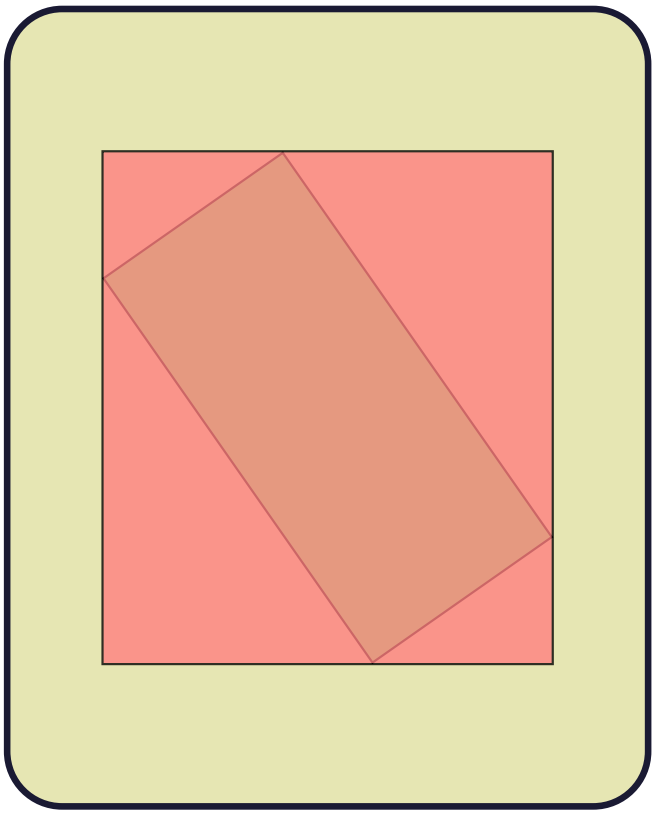
Let $H \subset \mathbb{Q}_p^n$ be a lattice.

$$H_0 = \pi_1(H) \oplus \cdots \oplus \pi_n(H)$$

provides the **best precision on coordinates**.
It is the **smallest diagonal lattice** containing H.



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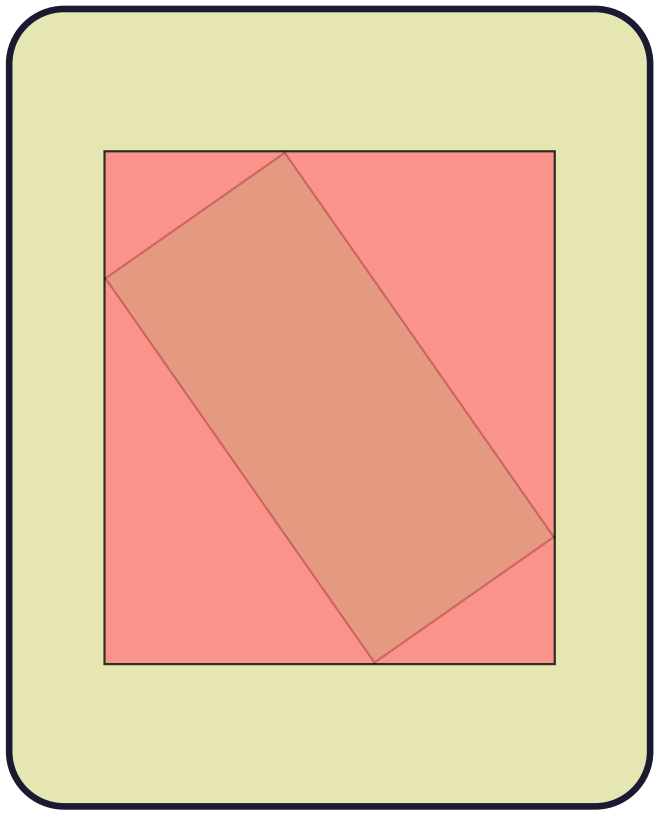
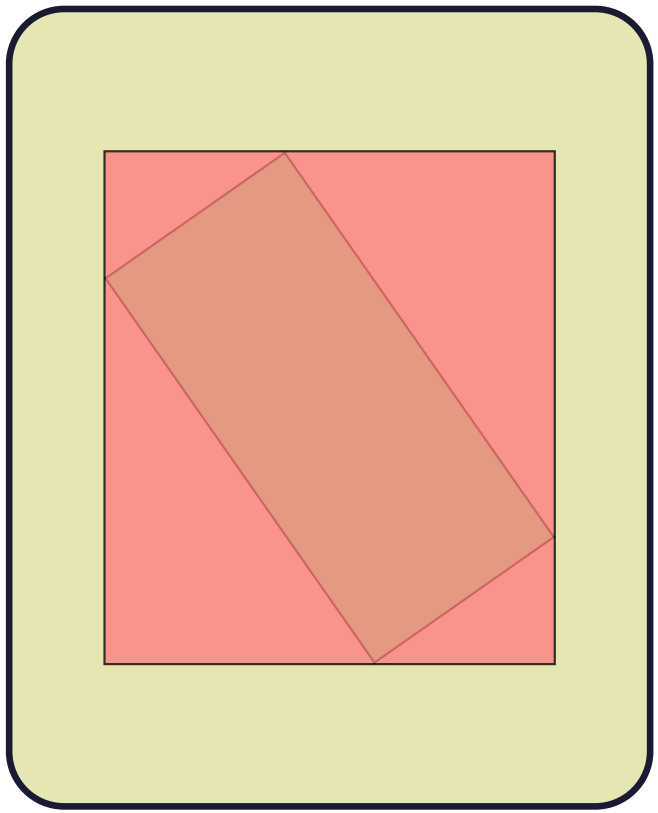


Diffused digits

The number of **diffused digits of precision** of H is the length of H_0/H.

More on diffused digits



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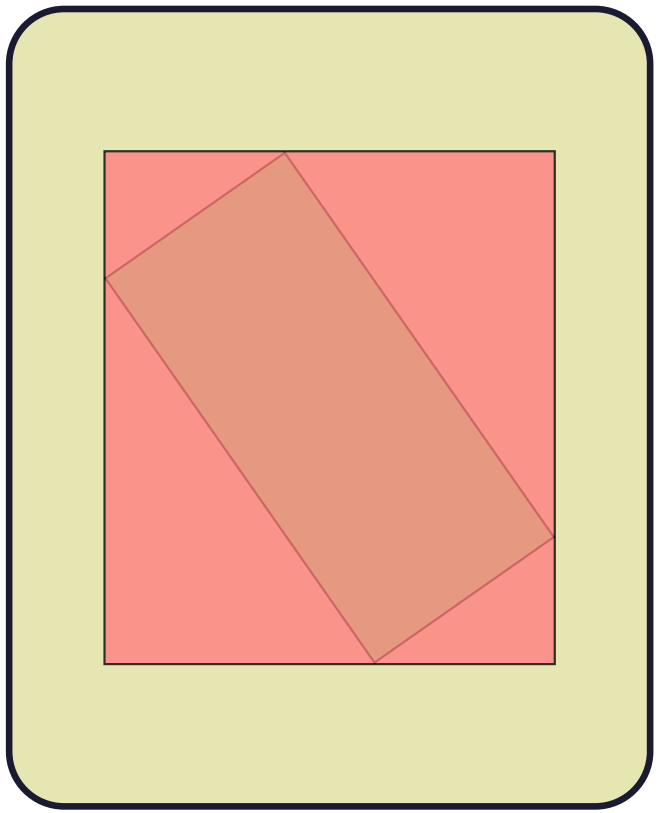

More on diffused digits

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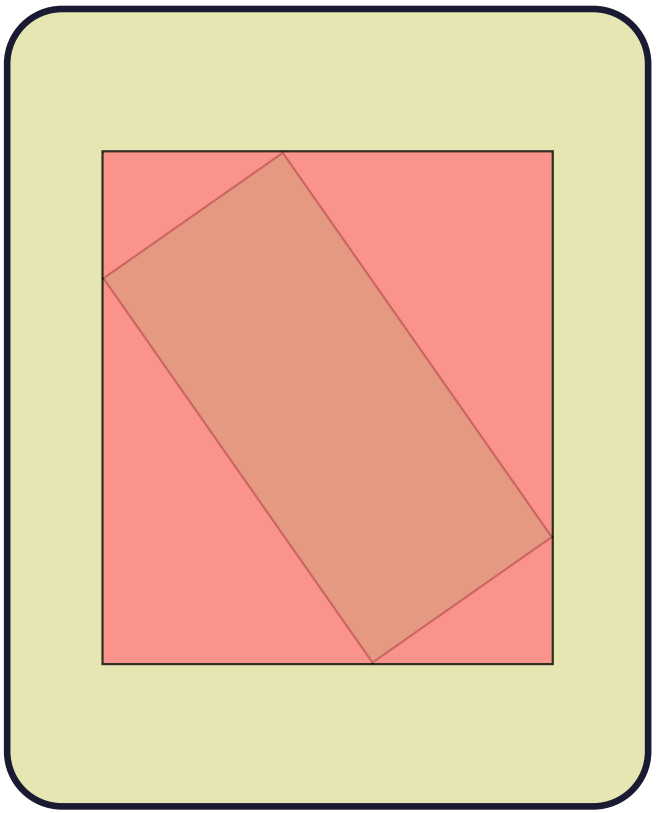
Another definition

Here, the number of diffused digits is:

$$-\log_p\left(|H_0/H|\right).$$



Diffused digits: example

Diffused digits: the lattice

Let

$$H = \left\langle \begin{pmatrix} 1 \\ p \\ p^2 \end{pmatrix}, \begin{pmatrix} p \\ p^2 \\ 2p^4 \end{pmatrix}, \begin{pmatrix} 2p \\ p^2 \\ 2p^3 \end{pmatrix} \right\rangle.$$

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The SNF of H_0/H is: $\begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix}$.

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Number of diffused digits

The SNF of H_0/H is: $\begin{pmatrix} 1 & & \\ & p & \\ & & p \end{pmatrix}$. Hence **2 diffused digits**.

Comparison: $\prod_{i=1}^{n} A_i \mod M$

Modulus M	n	Gain of precision	
		Jagged	Lattice (not dif. + dif.)
$X^5 + X^2 + 1$ (Irred. mod 2)	10	0.2	0.2 + 0.0
	50	4.2	4.2 + 0.0
	100	11.2	11.2 + 0.0
$X^5 + 1$ (Sep. mod 2)	10	0.4	0.9 + 6.0
	50	5.6	11.1 + 42.0
	100	13.6	27.0 + 87.0
$X^5 + 2$ (Eisenstein)	10	6.2	6.2 + 0.0
	50	44.0	44.0 + 0.0
	100	92.5	92.5 + 0.0
$(X+1)^5 + 2$ (Shift Eisenstein)	10	0.6	4.7 + 1.4
	50	7.1	42.6 + 1.4
	100	15.1	91.8 + 1.4
$X^5 + X + 2$ (Two slopes)	10	1.7	7.9 + 9.8
	50	8.1	70.7 + 59.8
	100	16.1	152.6 + 125.9

Figure: Precision for modular multiplication

Qualitative understanding: two possibilities

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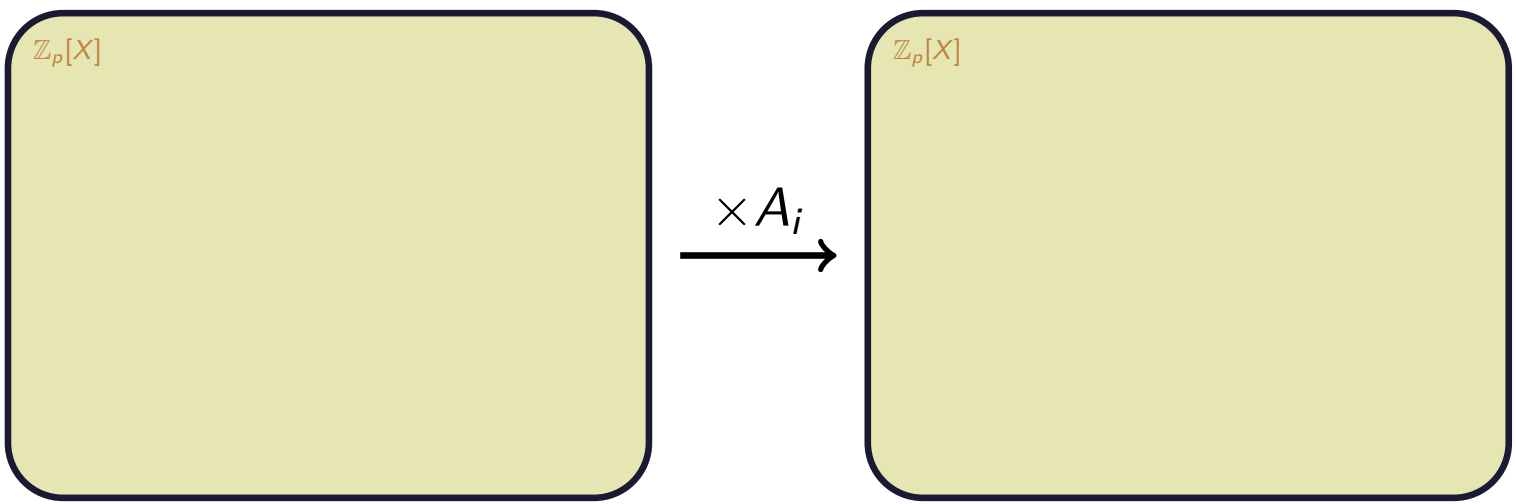

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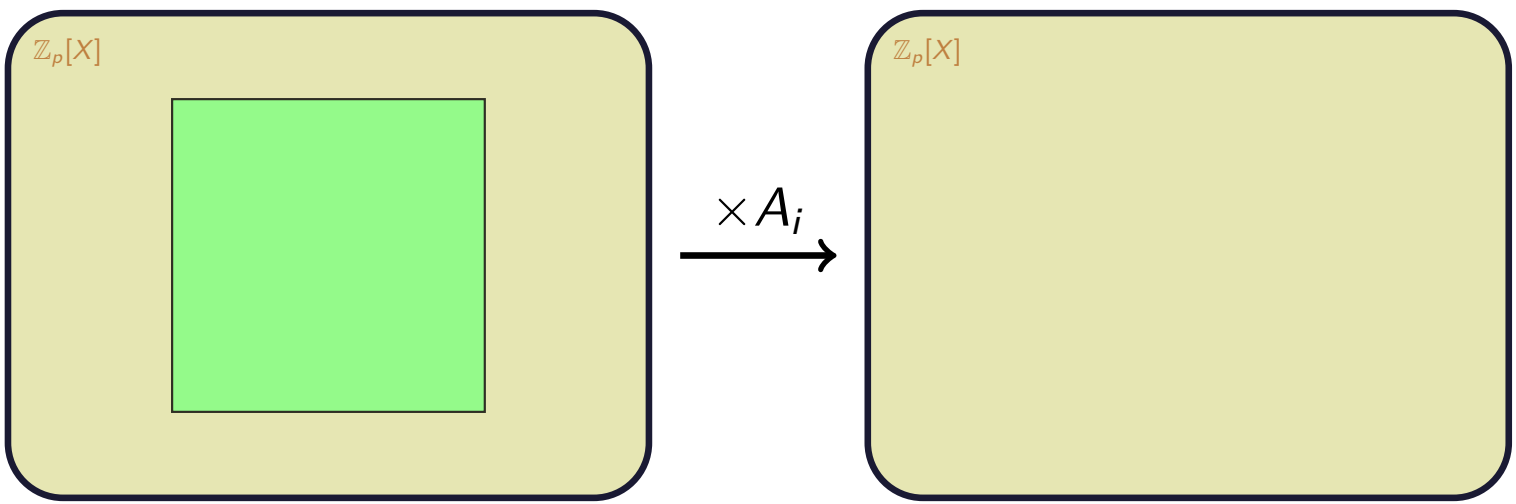

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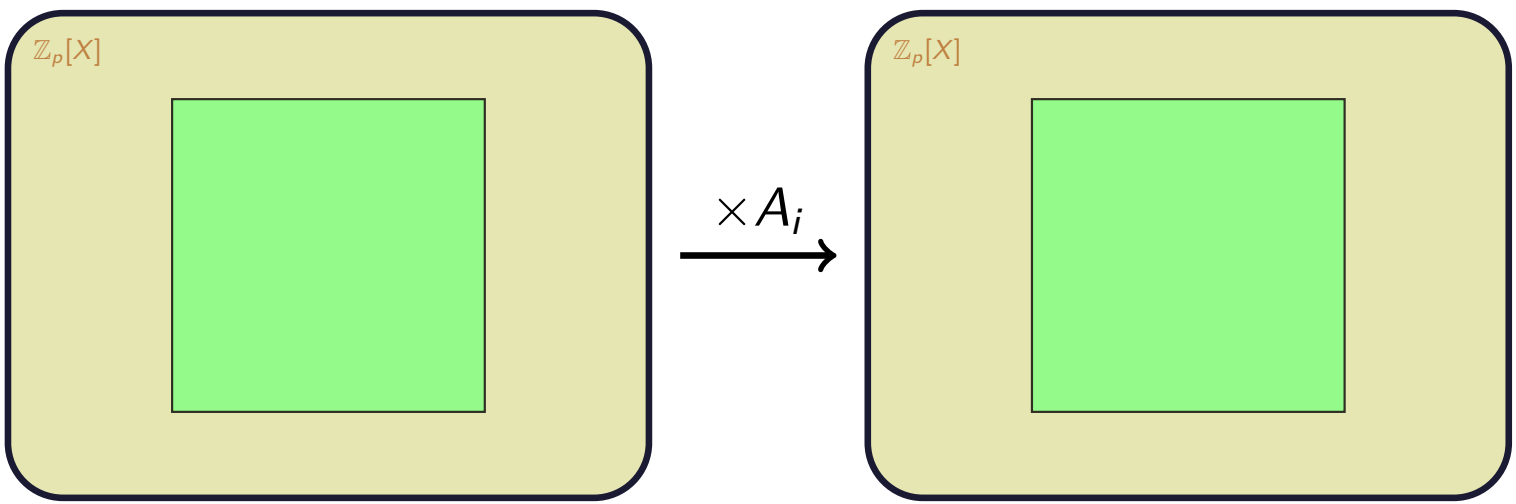

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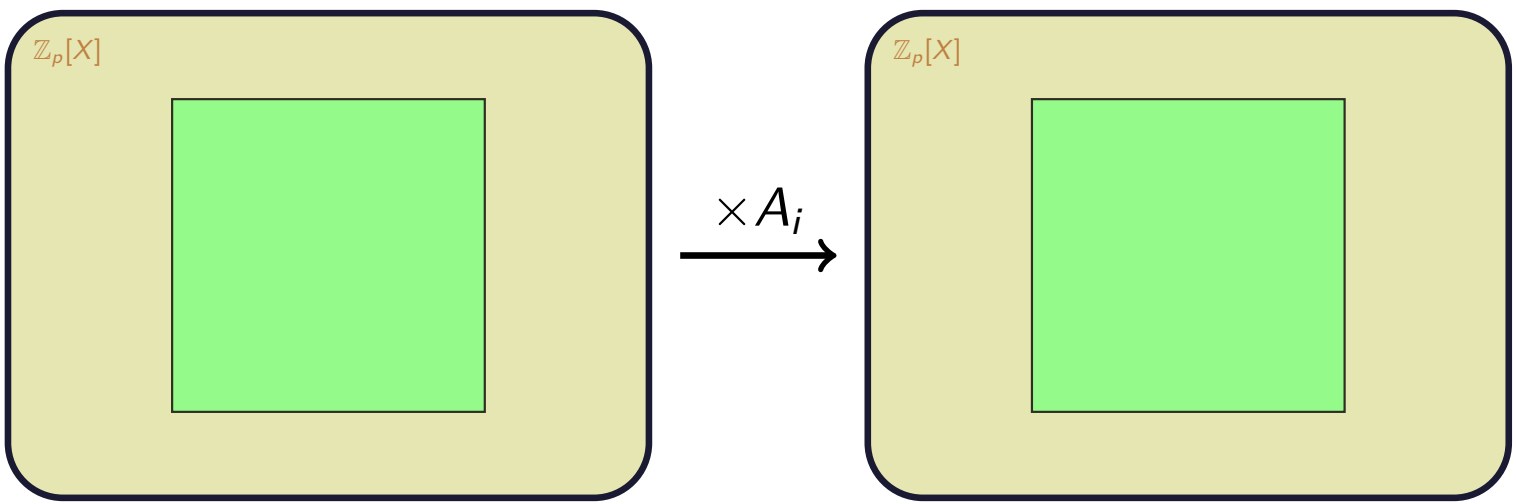

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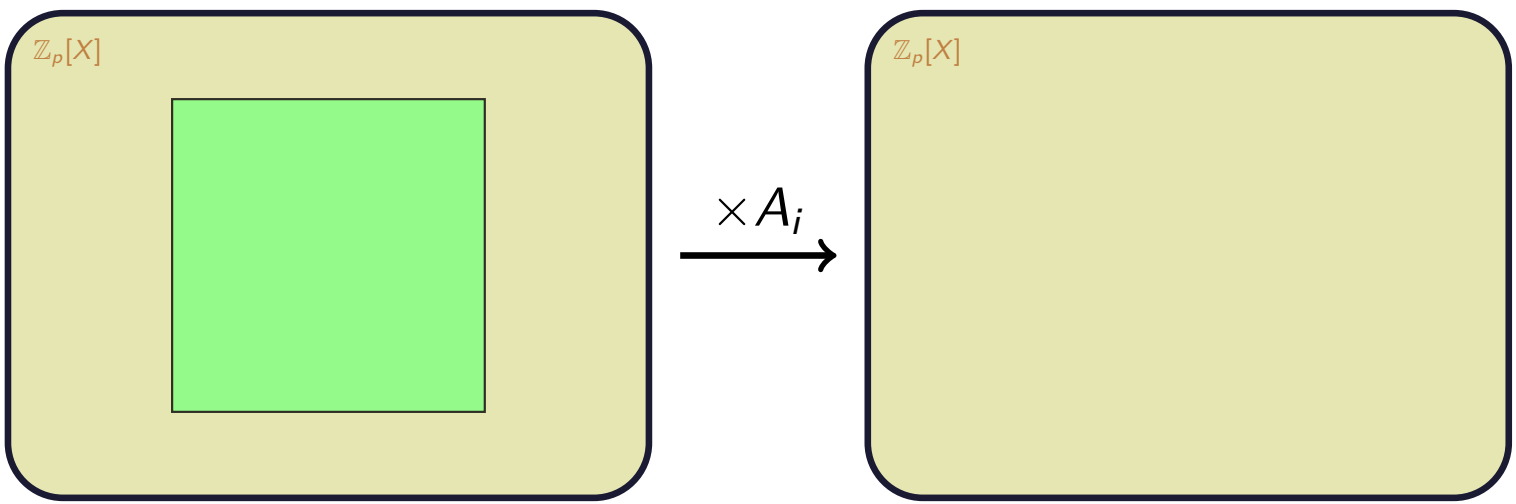

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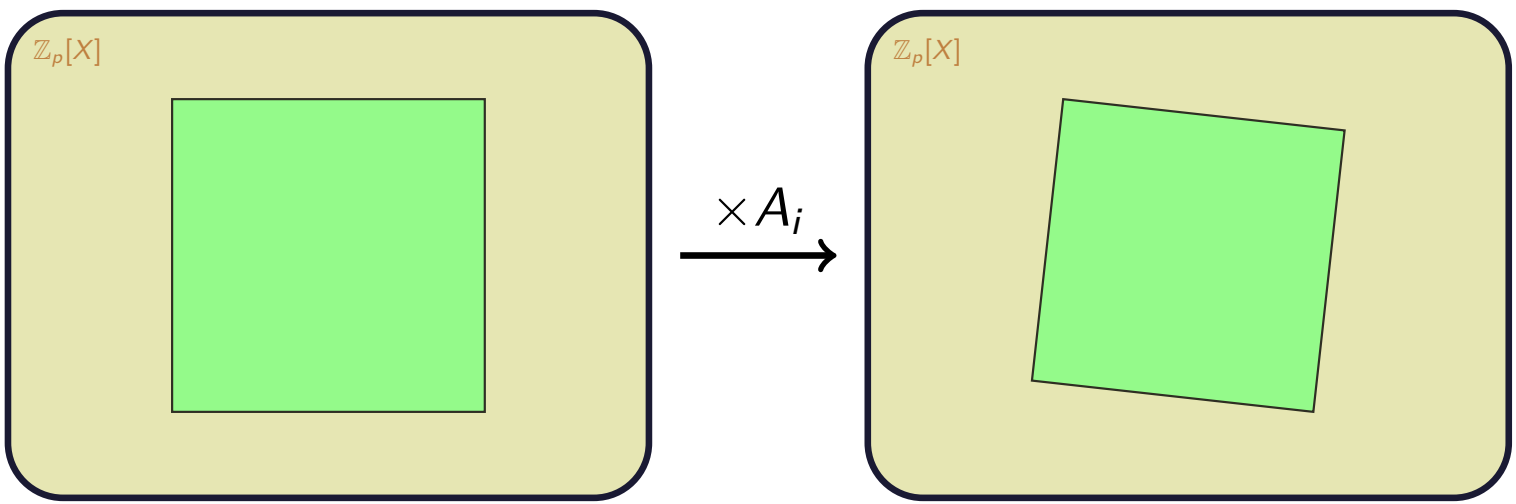

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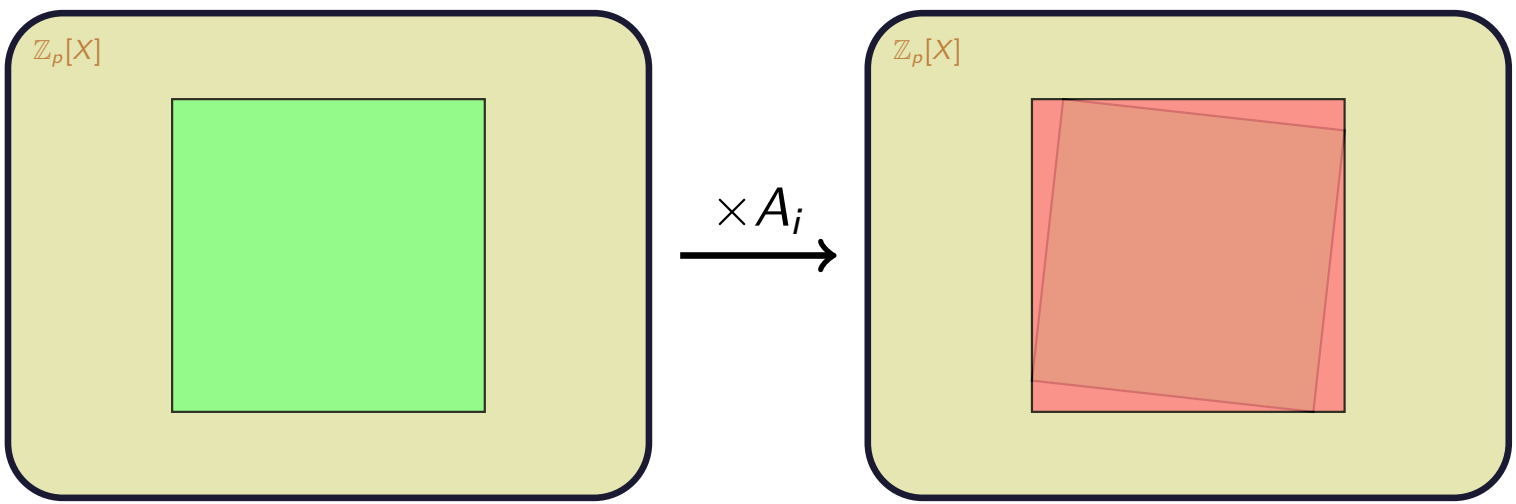

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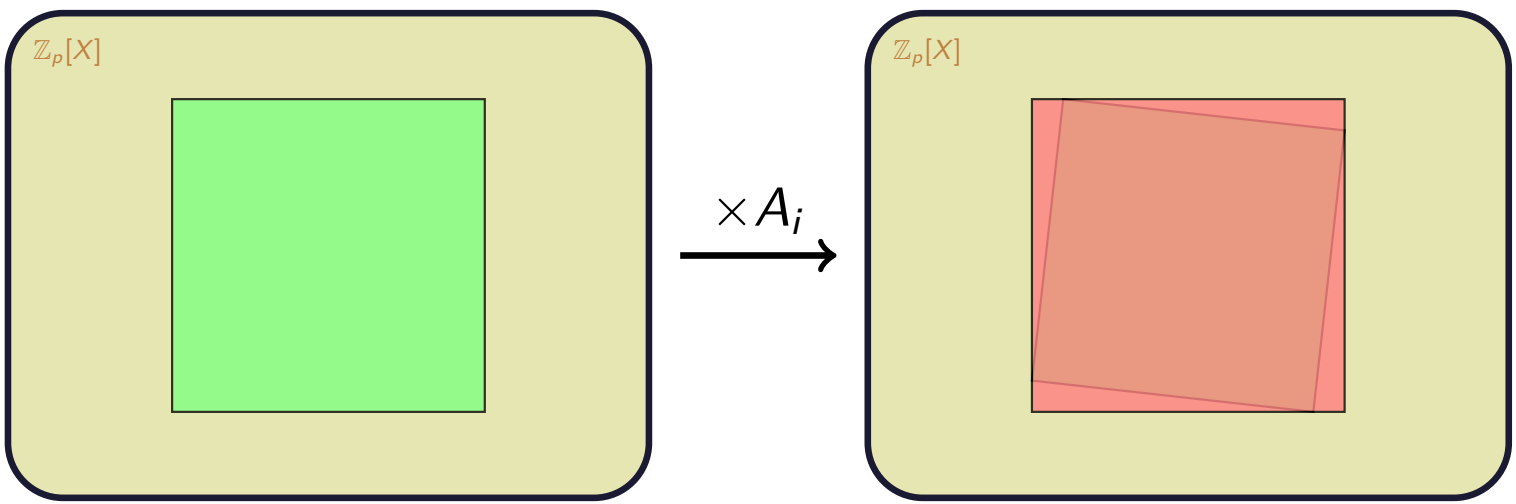

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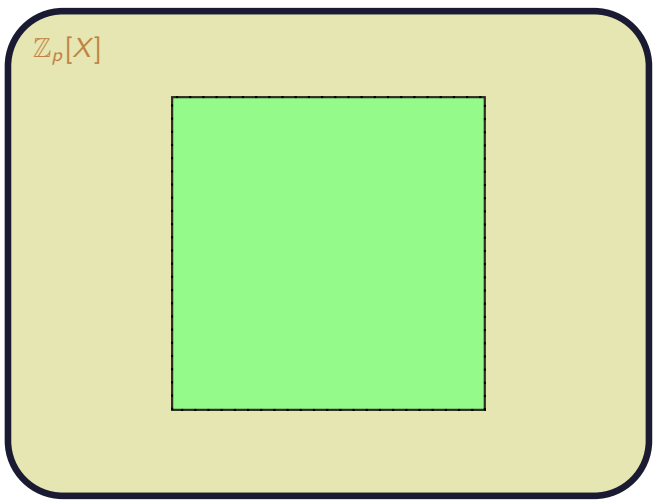

without lattice

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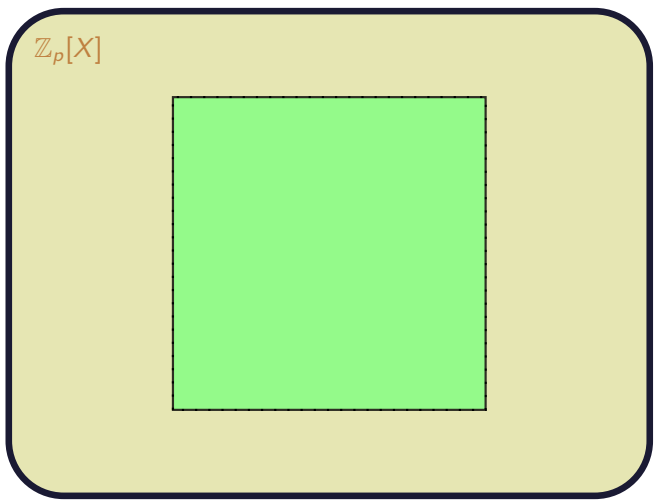

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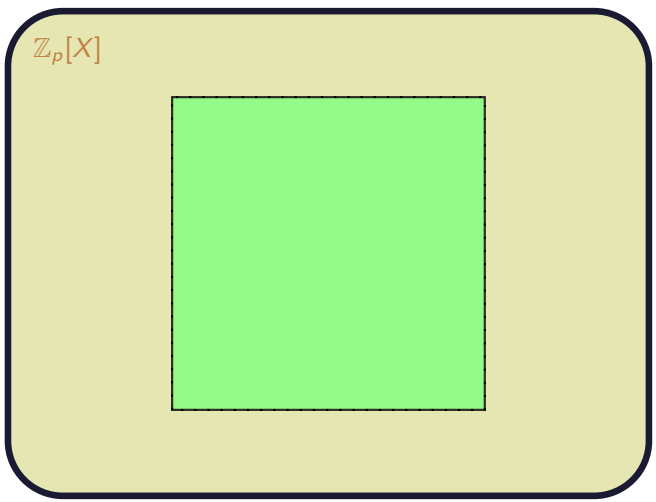

without lattice

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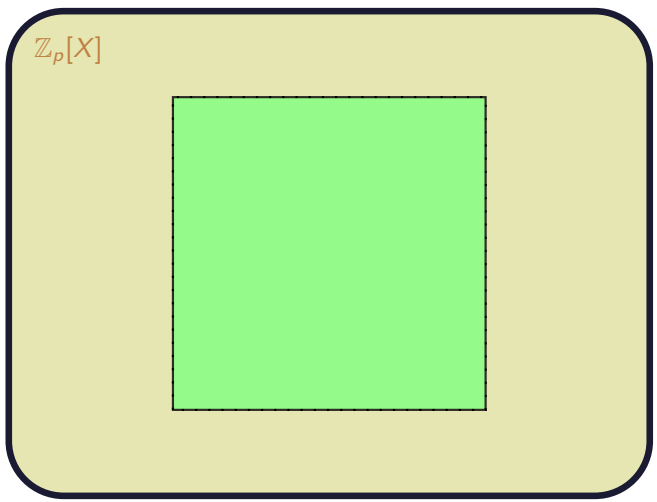

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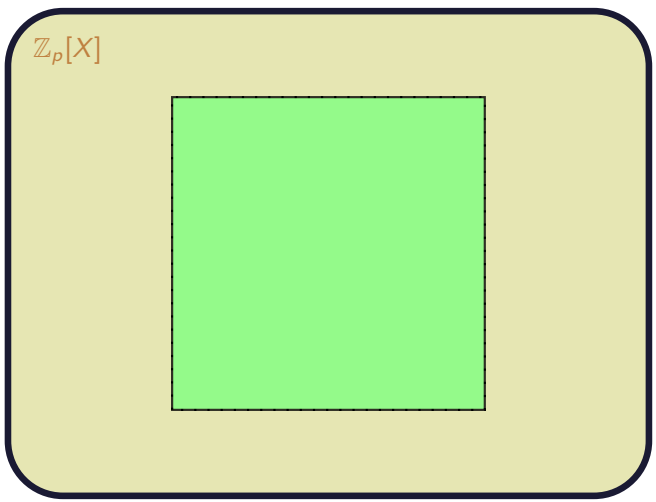
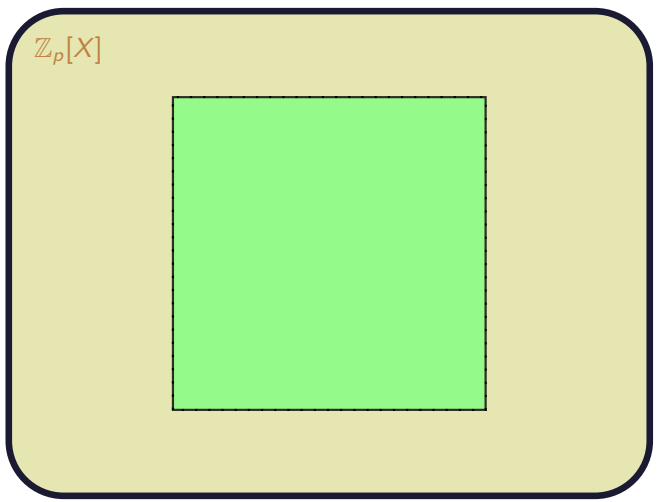


without lattice

$i = 2$



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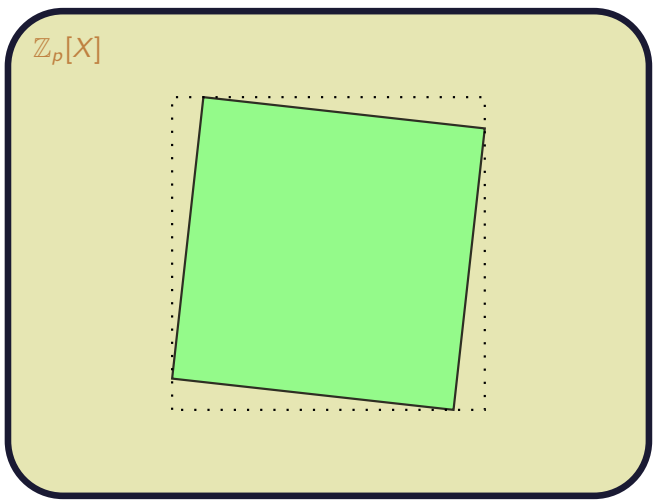
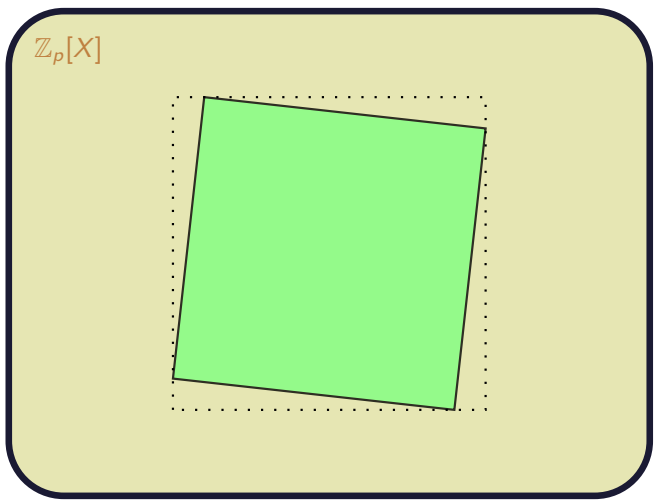


without lattice

$i = 3$



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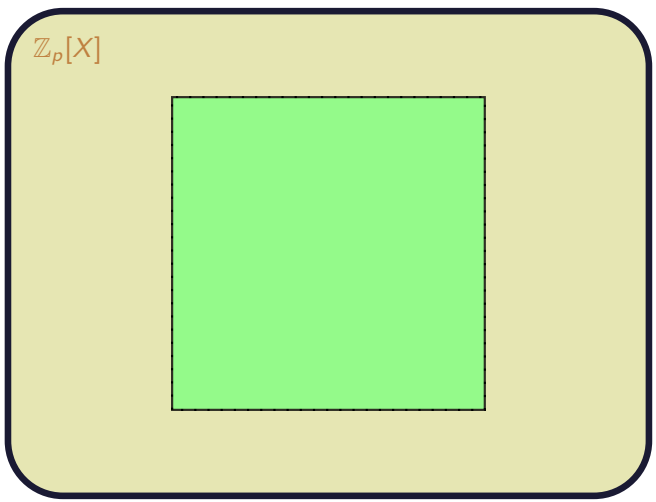

without lattice

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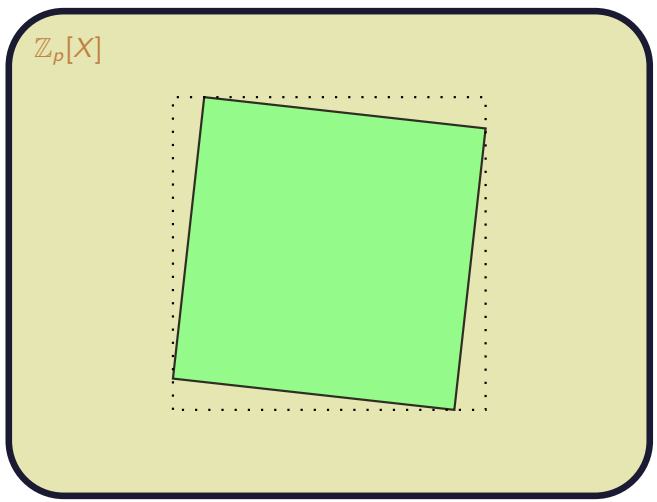

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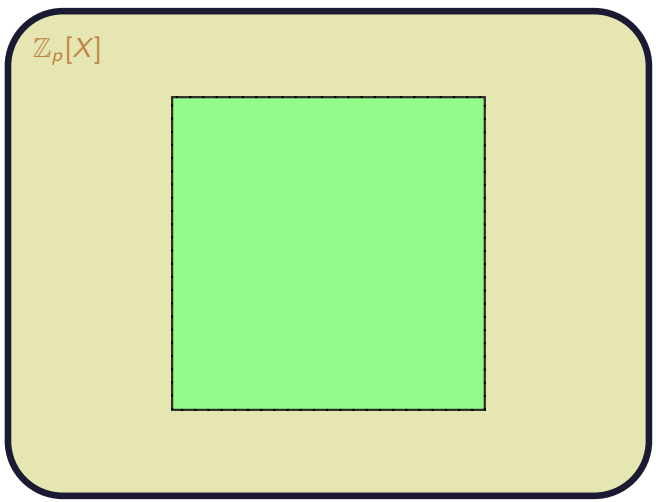
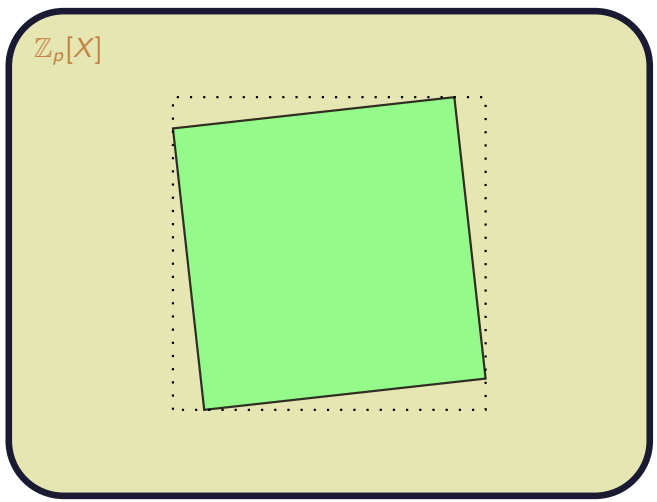


without lattice

$i = 4$



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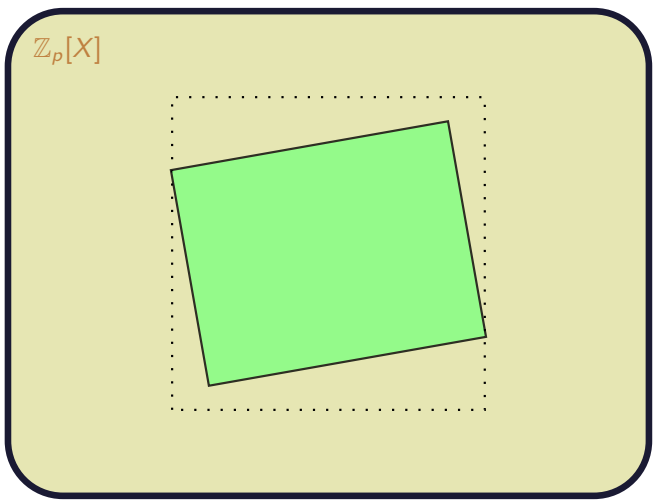
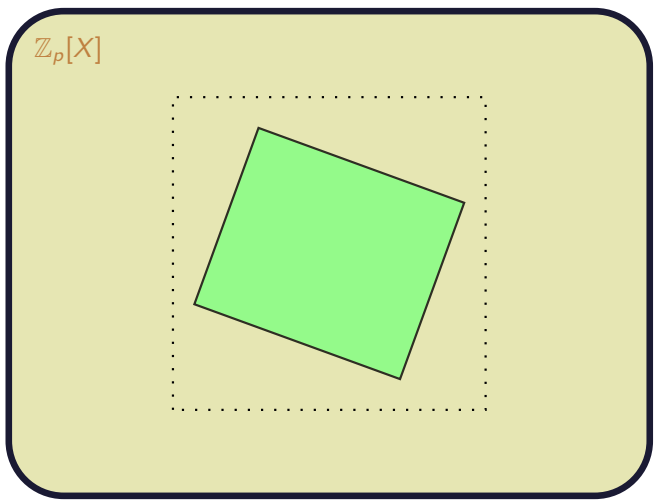


without lattice

$i = 5$



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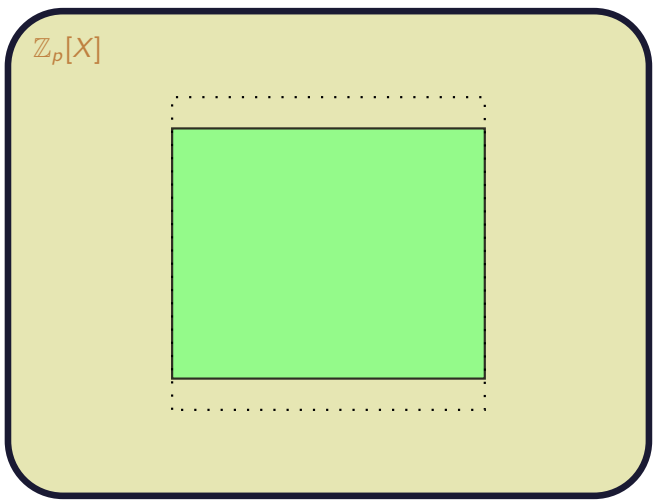

without lattice

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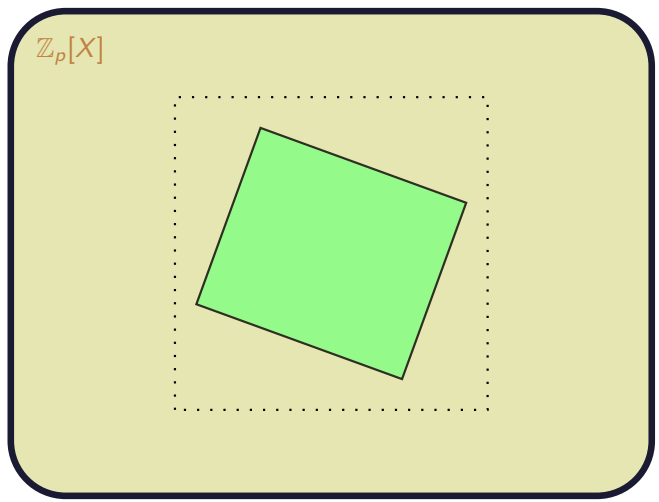

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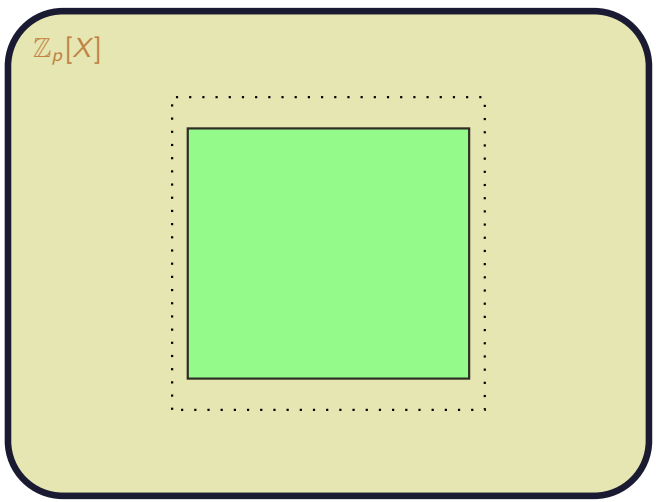
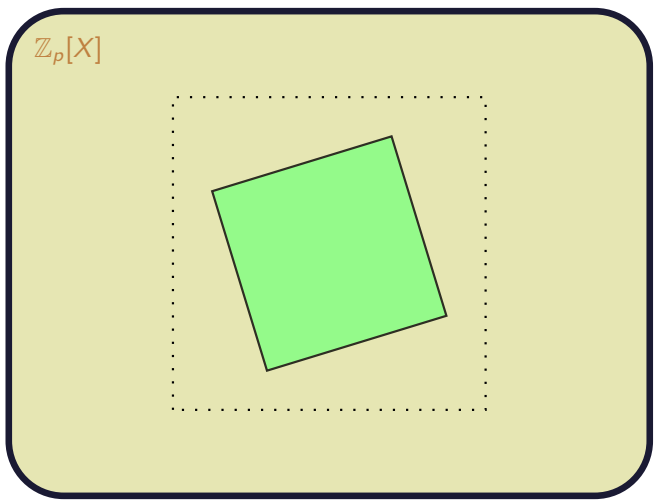


without lattice

$i = 10$

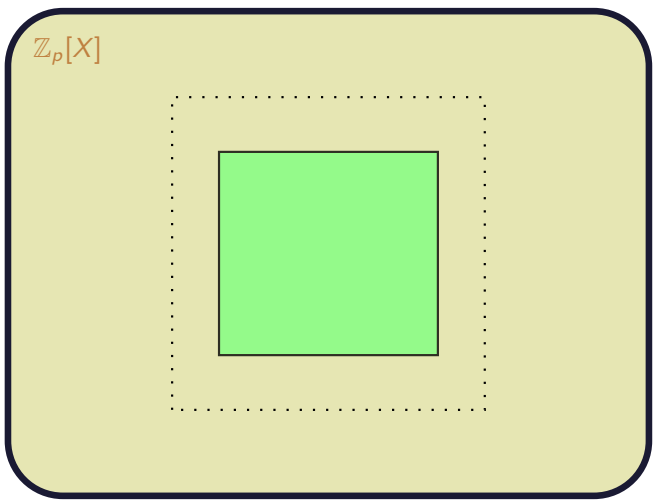


with lattice

Qualitative understanding: long-term

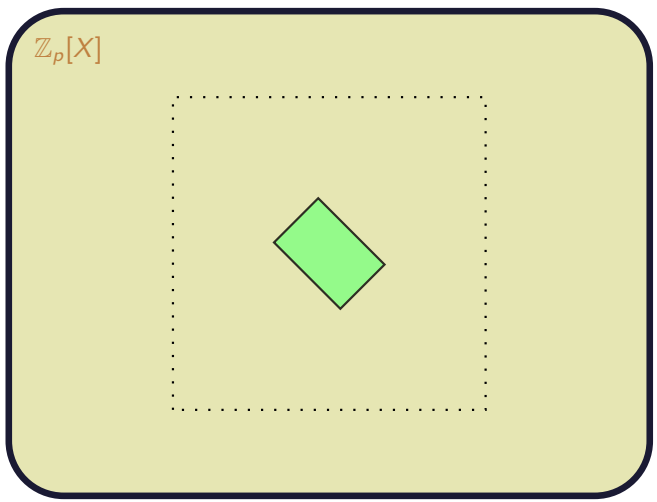
Input: $P, A_1, \ldots, A_n \in \mathbb{Z}_p[X]_d$ known at precision $O(p^N)$
Output: the product $A_1 A_2 \cdots A_n \mod P$

1. $A = 1 + O(p^N)$
2. for i in $1, 2, \ldots n$:
3. $\quad A = A \cdot A_i$
4. return A



without lattice

$i = 100$



with lattice

Table of contents

Newton polygon of a polynomial

Definition

Let $f(X) = a_0 + \cdots + a_n X^n \in \mathbb{Q}_p[x]$ (remark: $\mathbb{Q}_p[\![x]\!]$ would also be fine).

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By lower convex hull, we mean the points of the convex hull of U below the straight line from $(0, (a_0))$ to $(n, v(a_n))$.

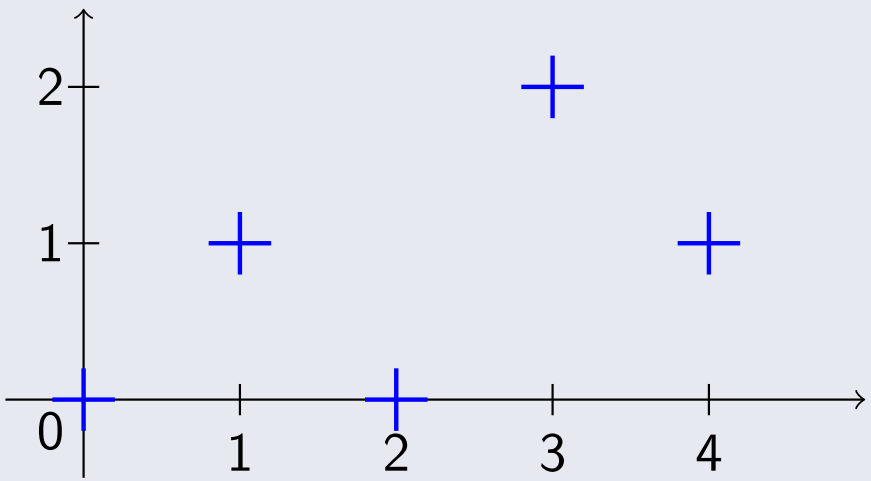
Proposition

Let $Newt_f$ be the (point-wise) biggest convex mapping below U. Then the graph of $Newt_f$ is (the lower frontier of) $NP(f)$.

例

A Newton polygon

$P = 2 + 3X + 7X^2 + 9X^3 + 3X^4$, over $\mathbb{Q}_3$



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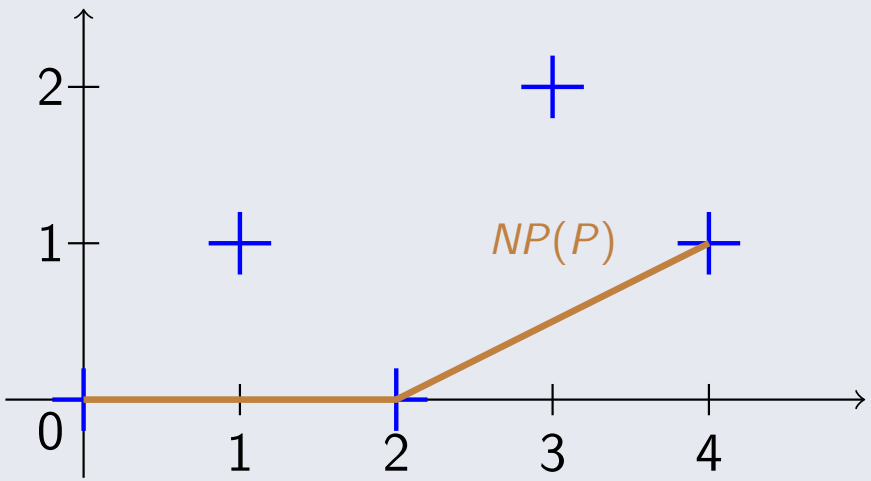

Vocabulary

Definition

A *slope of the Newton polygon* of f is an element of $Newt'_f([0, n])$.

Definition

If λ is a slope of $Newt_f$, we call *segment of slope* λ of $Newt_f$ the set $\{(x, Newt_f(x)) / Newt'_f(x) = \lambda\}$.

Definition

The *length* of this slope is the length of its projection on the x-axis.

Fundamental theorem of Newton polygons

Theorem

*f has a root of valuation λ **iff** $-\lambda$ is a slope of $Newt_f$.*

Fundamental theorem of Newton polygons

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*f has a root of valuation λ **iff** $-\lambda$ is a slope of $Newt_f$.*
Moreover, the number of roots of f (with multiplicity) of valuation λ, is the length of the segment of slope $-\lambda$ of $Newt_f$.

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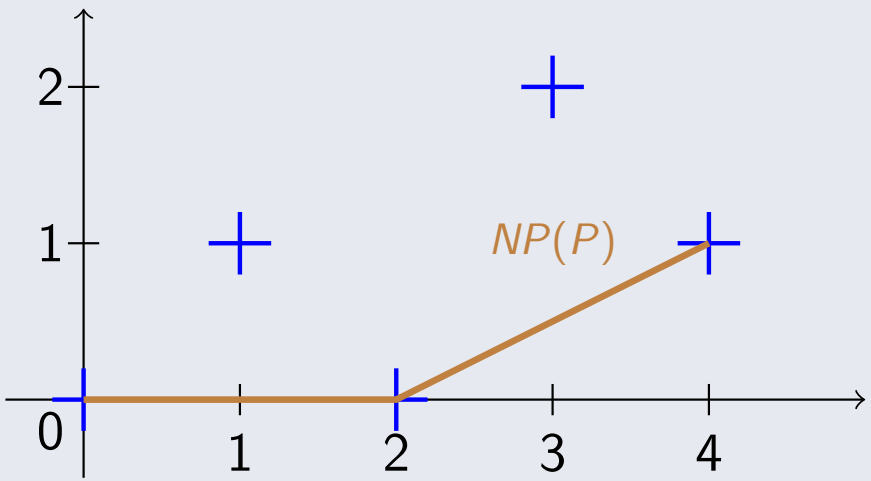

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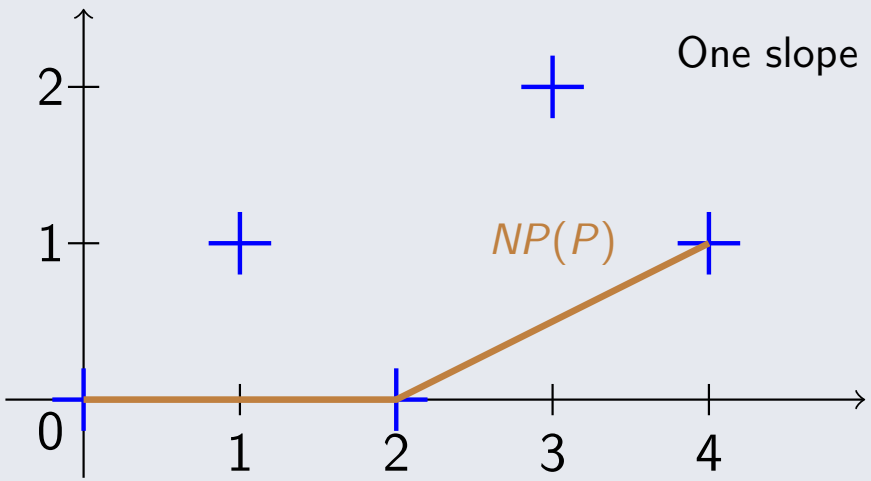

One slope for 0 of length 2, one slope 1/2 of length 2.

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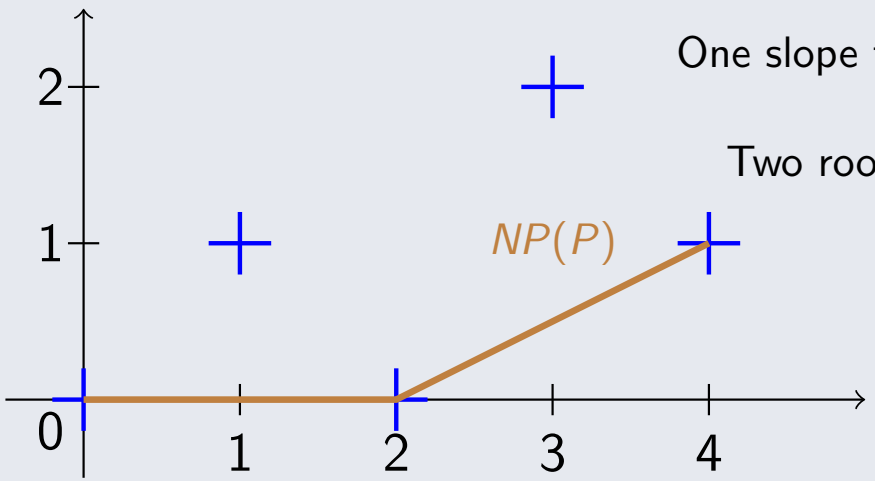

One slope for 0 of length 2, one slope $1/2$ of length 2.

Two roots of valuation 0, two of valuation $-1/2$.

Basic operations

Proposition (Addition)

If f and g are two polynomials, then the Newton polygon of $f+g$ can be ***lower-bounded*** *by taking the lower convex hull for the vertices of $Newt_f$ and $Newt_g$.*

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If f and g are two polynomials, then the Newton polygon of $f+g$ can be ***lower-bounded*** *by taking the lower convex hull for the vertices of Newt_f and Newt_g.*

Proposition (Multiplicativity)

If f and g are two polynomials, then the Newton polygon of fg has for slopes that of f and g, with length the sum of that of f and g.

Some remarks

Proposition

If $P \in \mathbb{Q}_p[X]$ is irreducible, then all its roots have the same valuation. Hence, $Newt_P$ has only one slope.

Remark

The converse is false. For instance, $(X-1)(X-2)$ over $\mathbb{Q}_5$.

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If $Newt_P$ has more than one slope, P is not irreducible.

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Corollary

If $Newt_P$ has more than one slope, P is not irreducible.

Remark

There are good irreducibility criterion based on testing whether one slope can be obtained by multiplication (namely, Dumas and Eisenstein).

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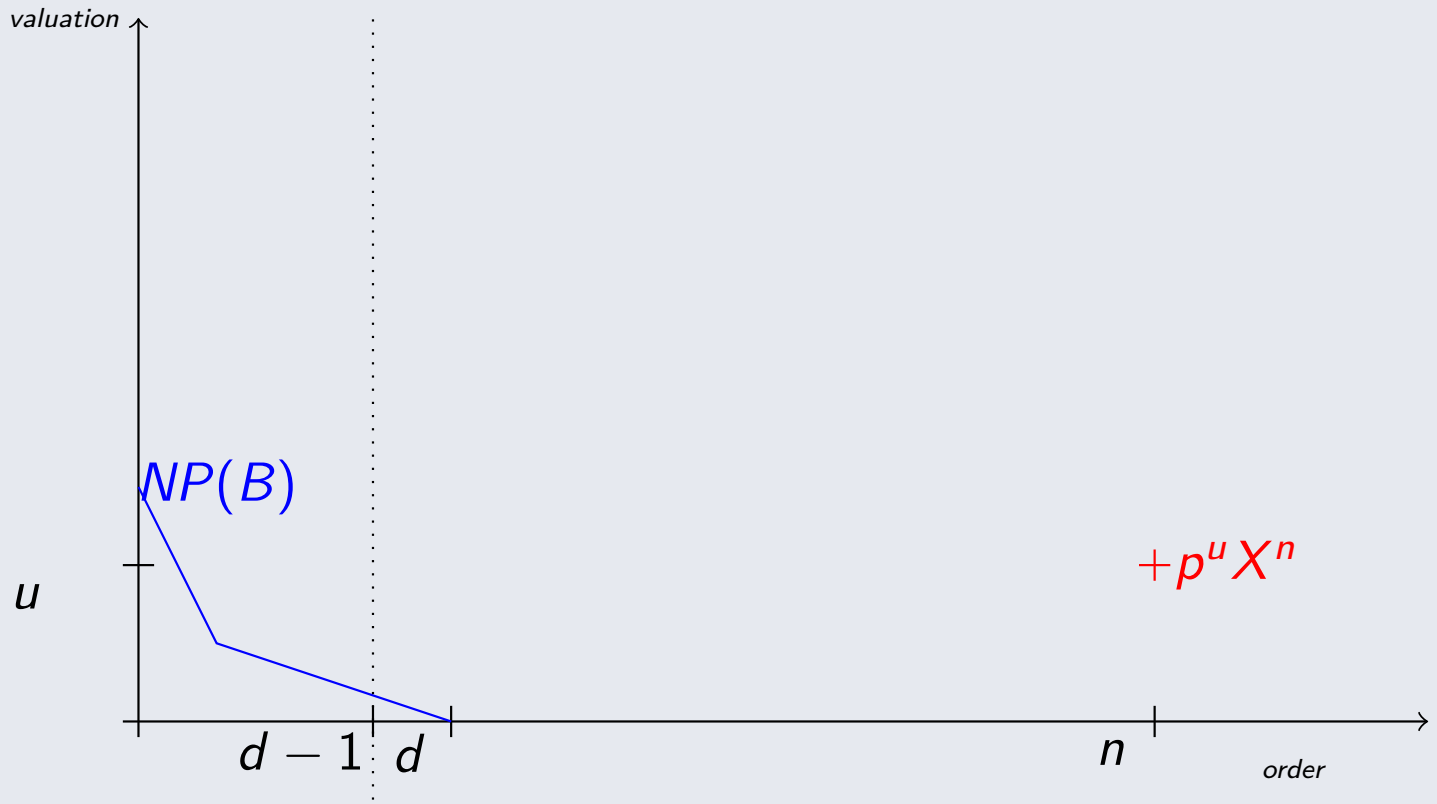
Euclidean division and Newton polygon

Lemma (Division lemma)

- *What is the Newton polygon of the* ***remainder*** *in the division of A by B (in $\mathbb{Q}_p[X]$)?*
- *What is the Newton polygon of the* ***quotient*** *in the division of A by B (in $\mathbb{Q}_p[X]$)?*

Euclidean division and Newton polygon

Lemma (Division lemma)



Division of $p^u X^n$ *by* B : $p^u X^n = BQ + R$

Euclidean division and Newton polygon

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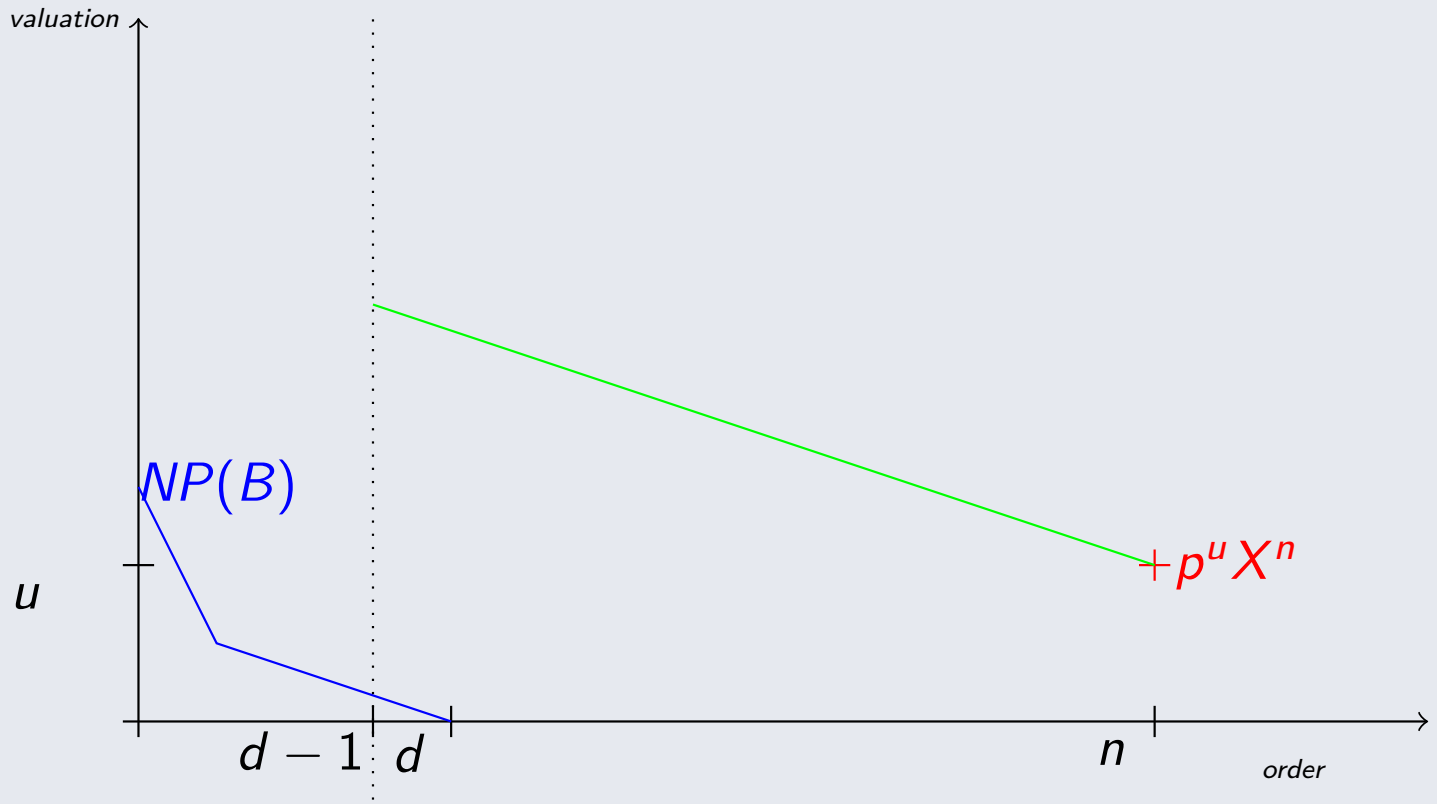

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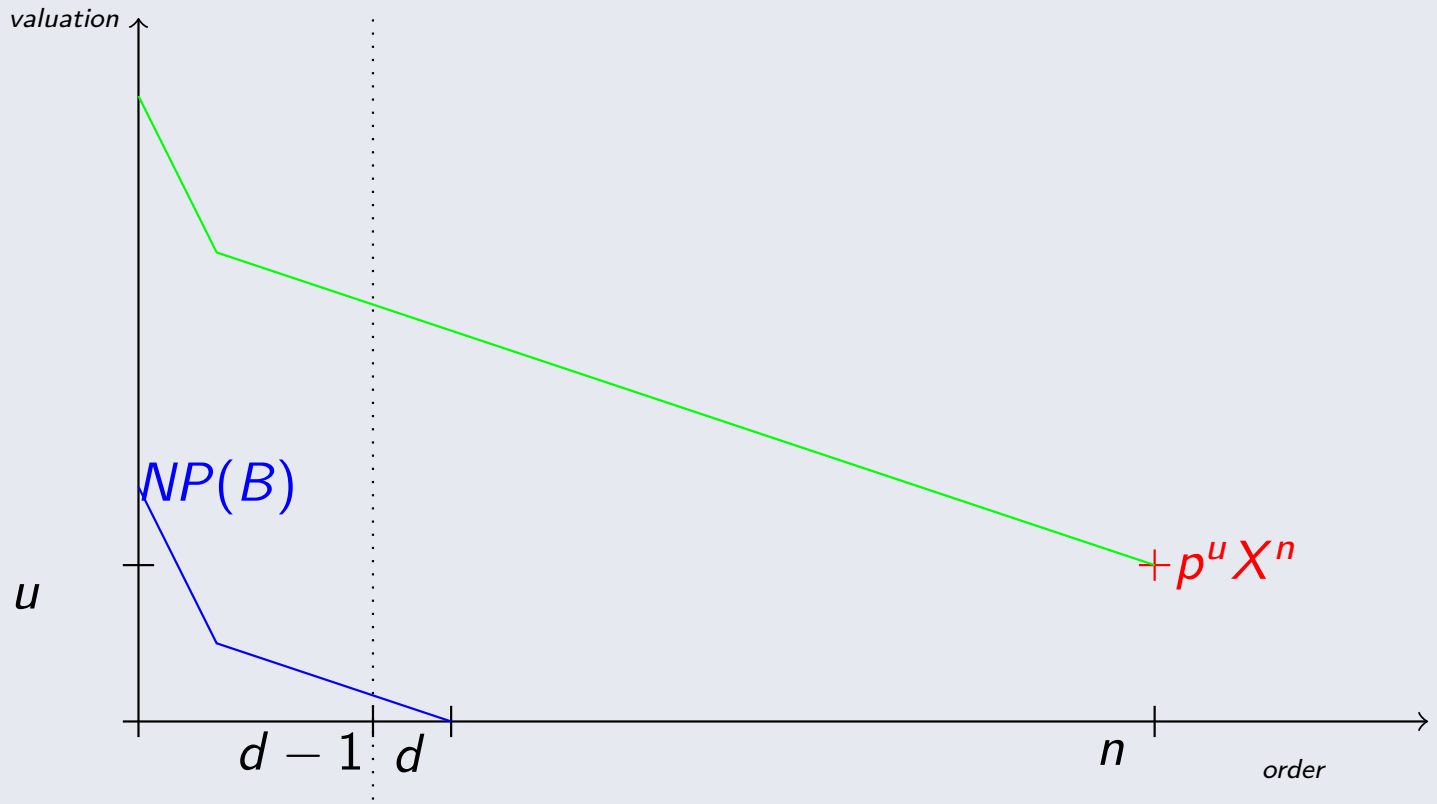

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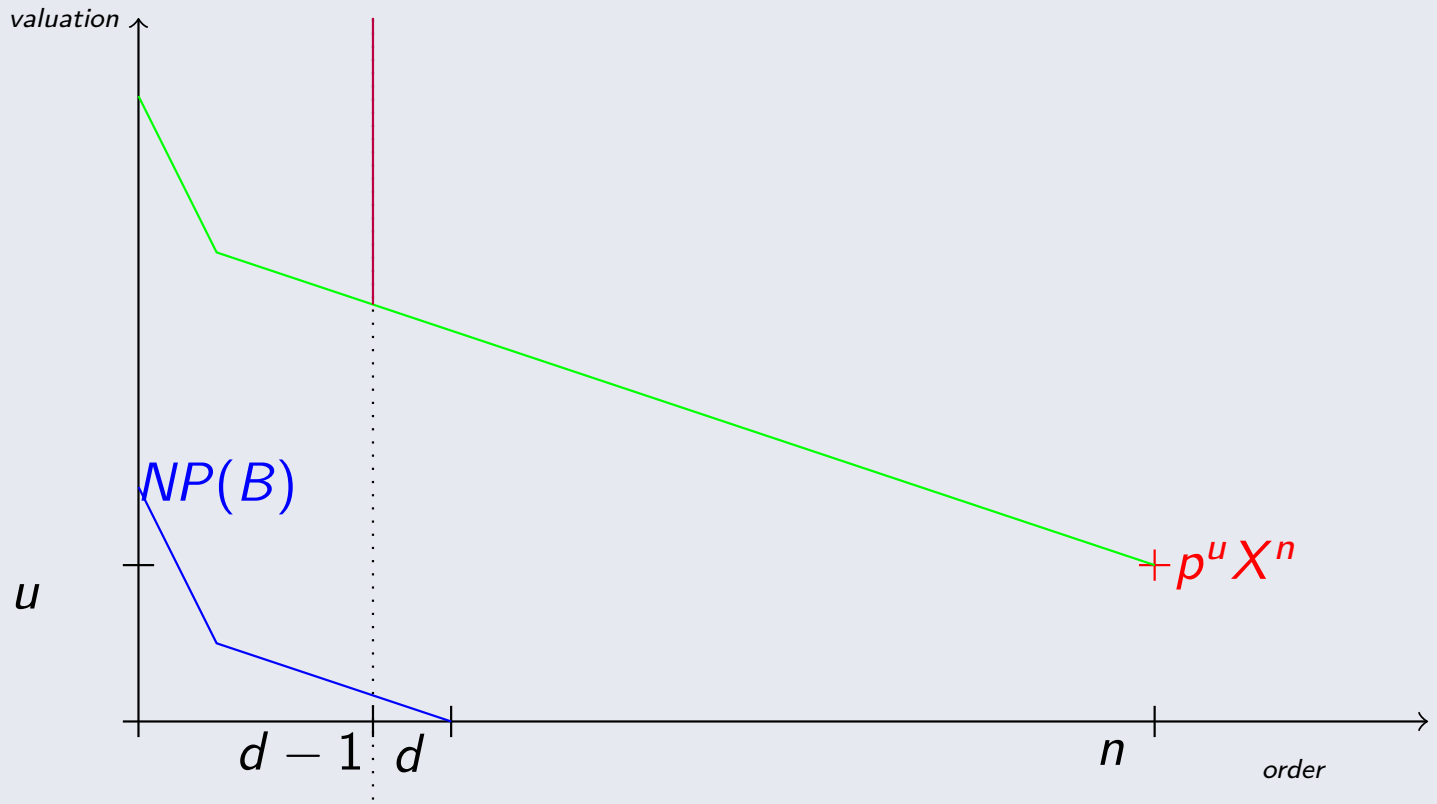

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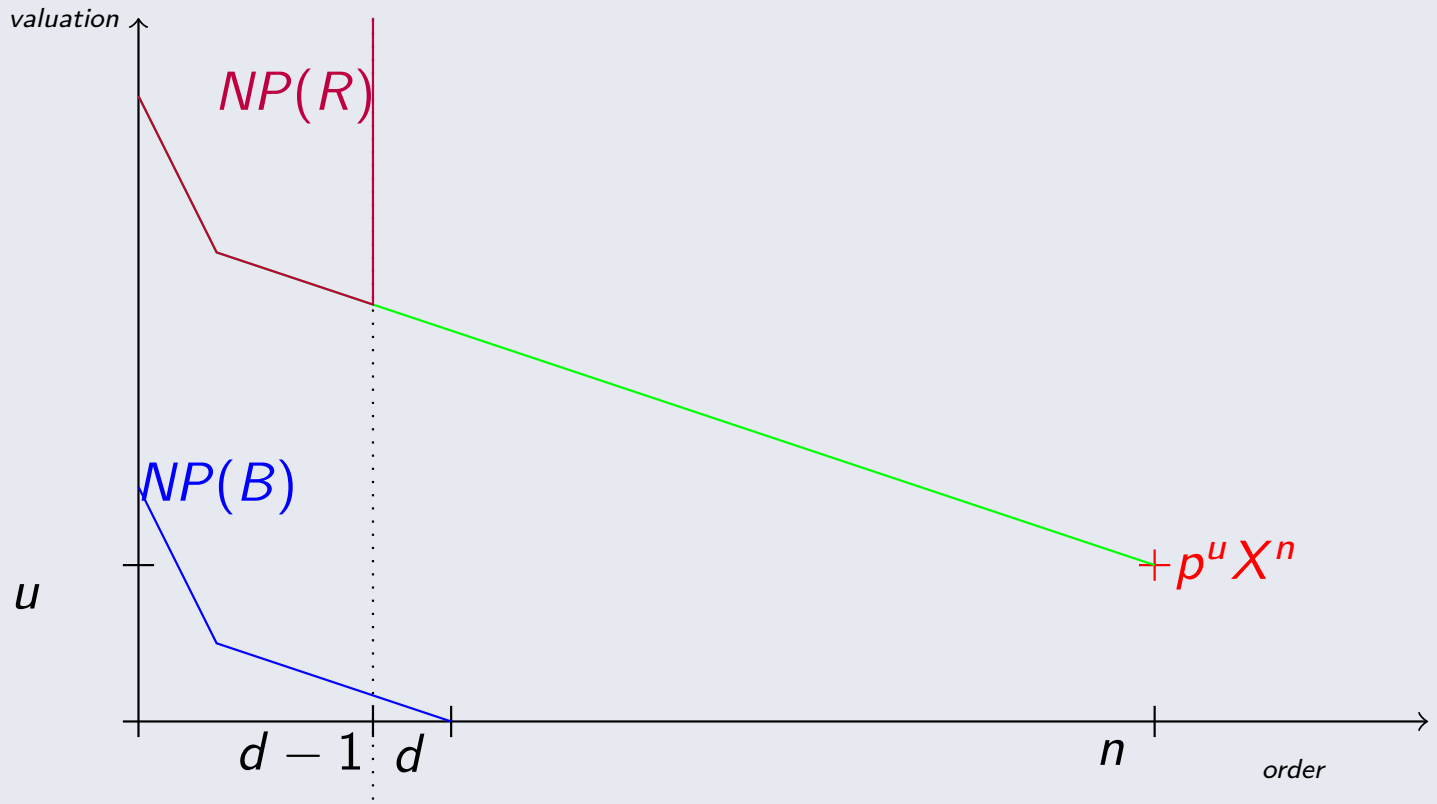

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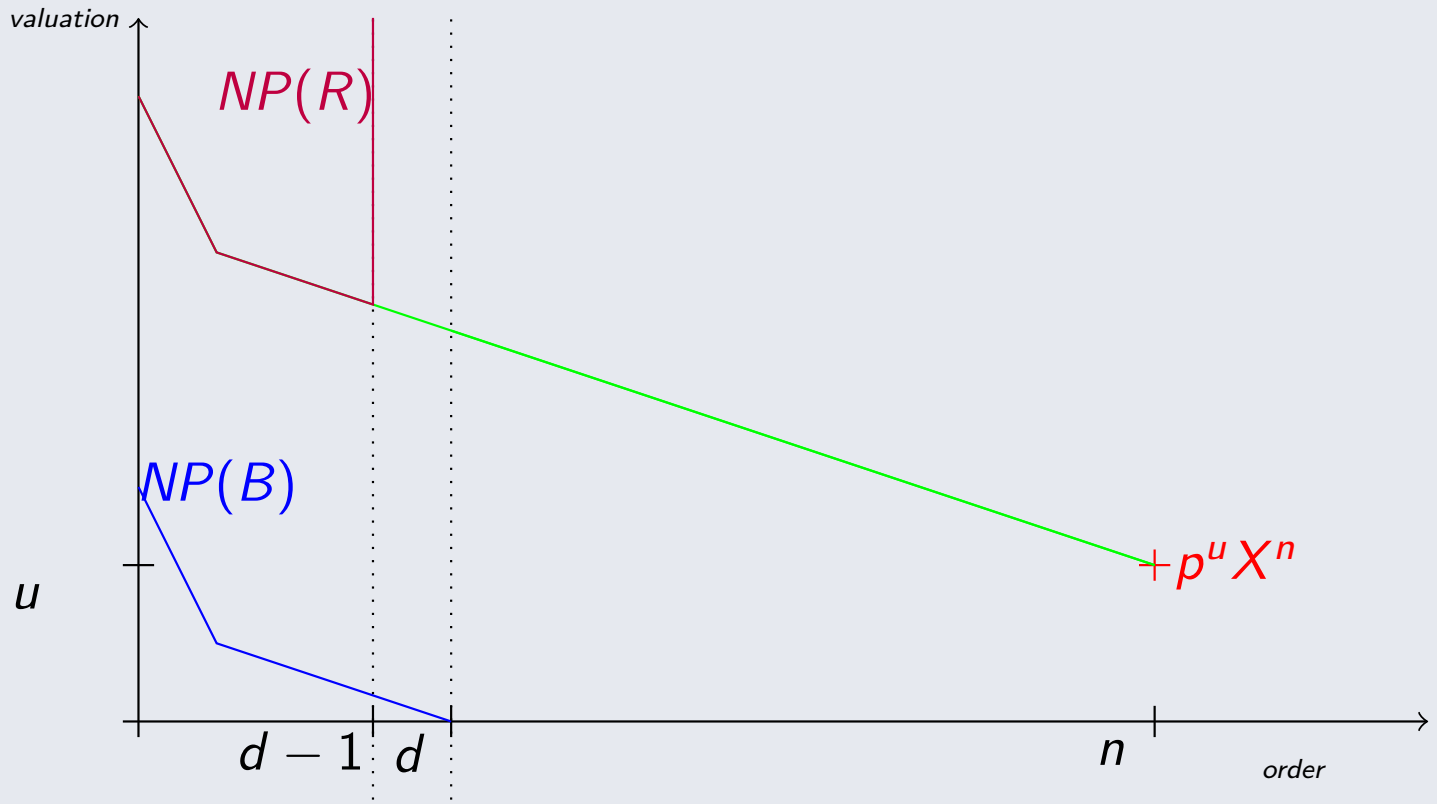

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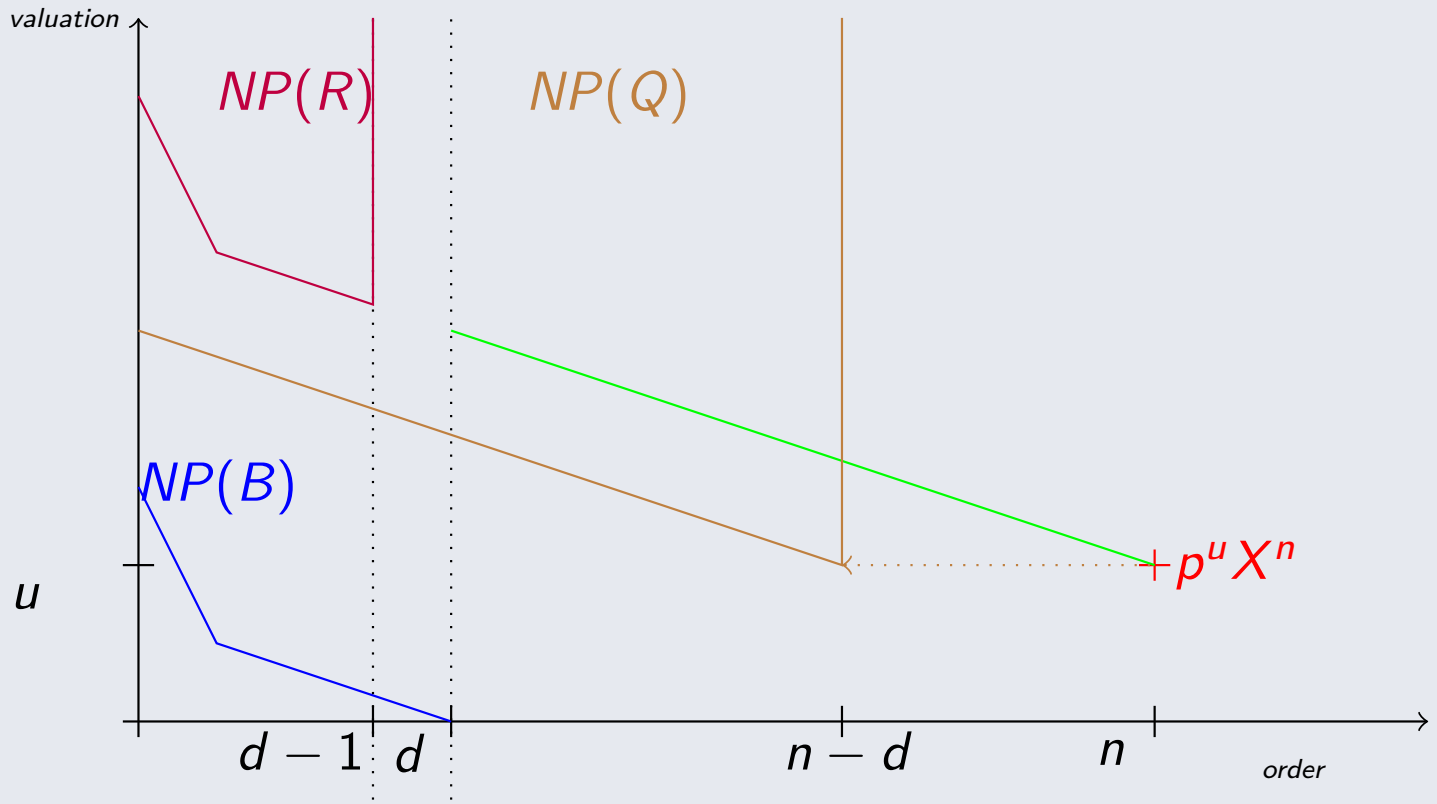

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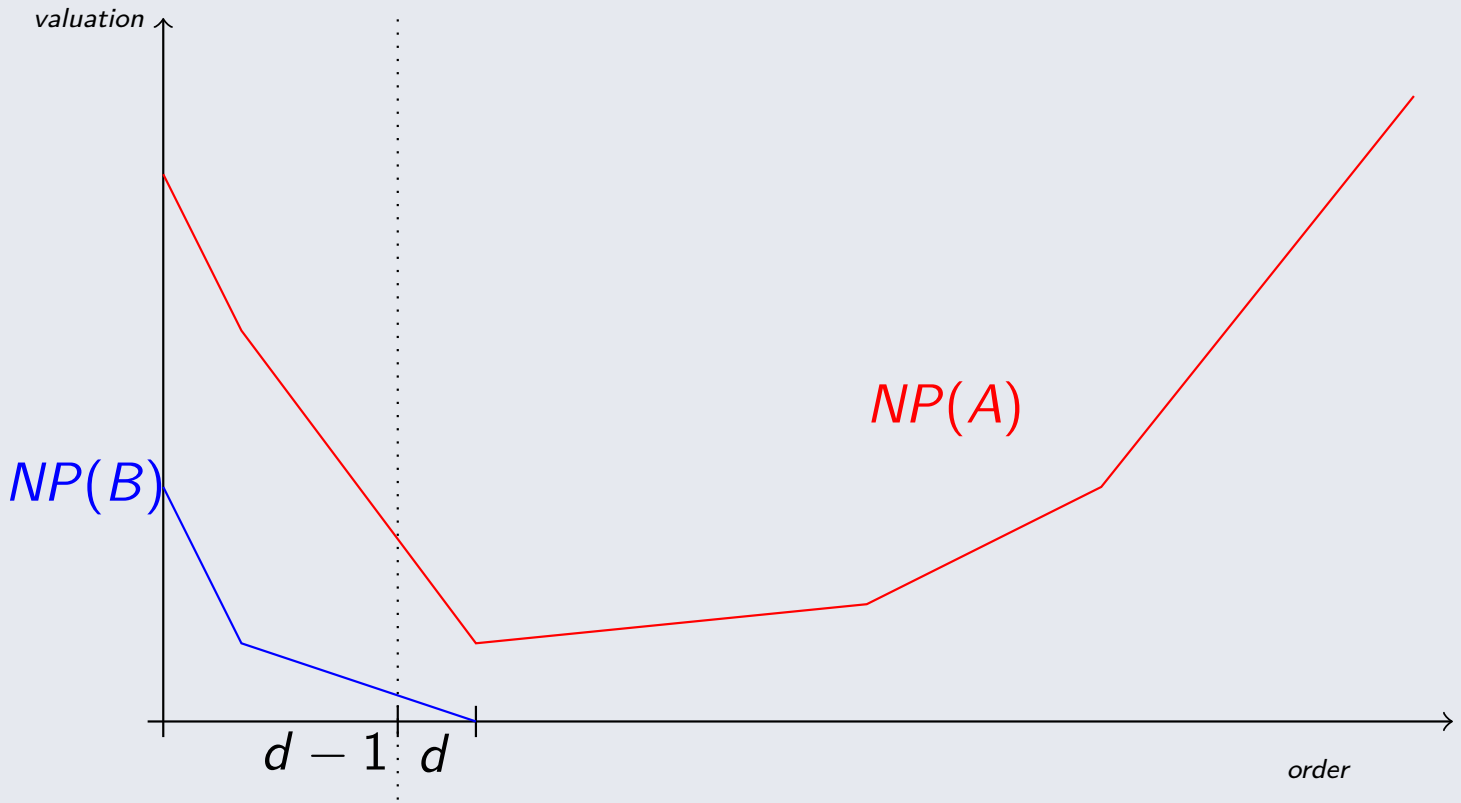

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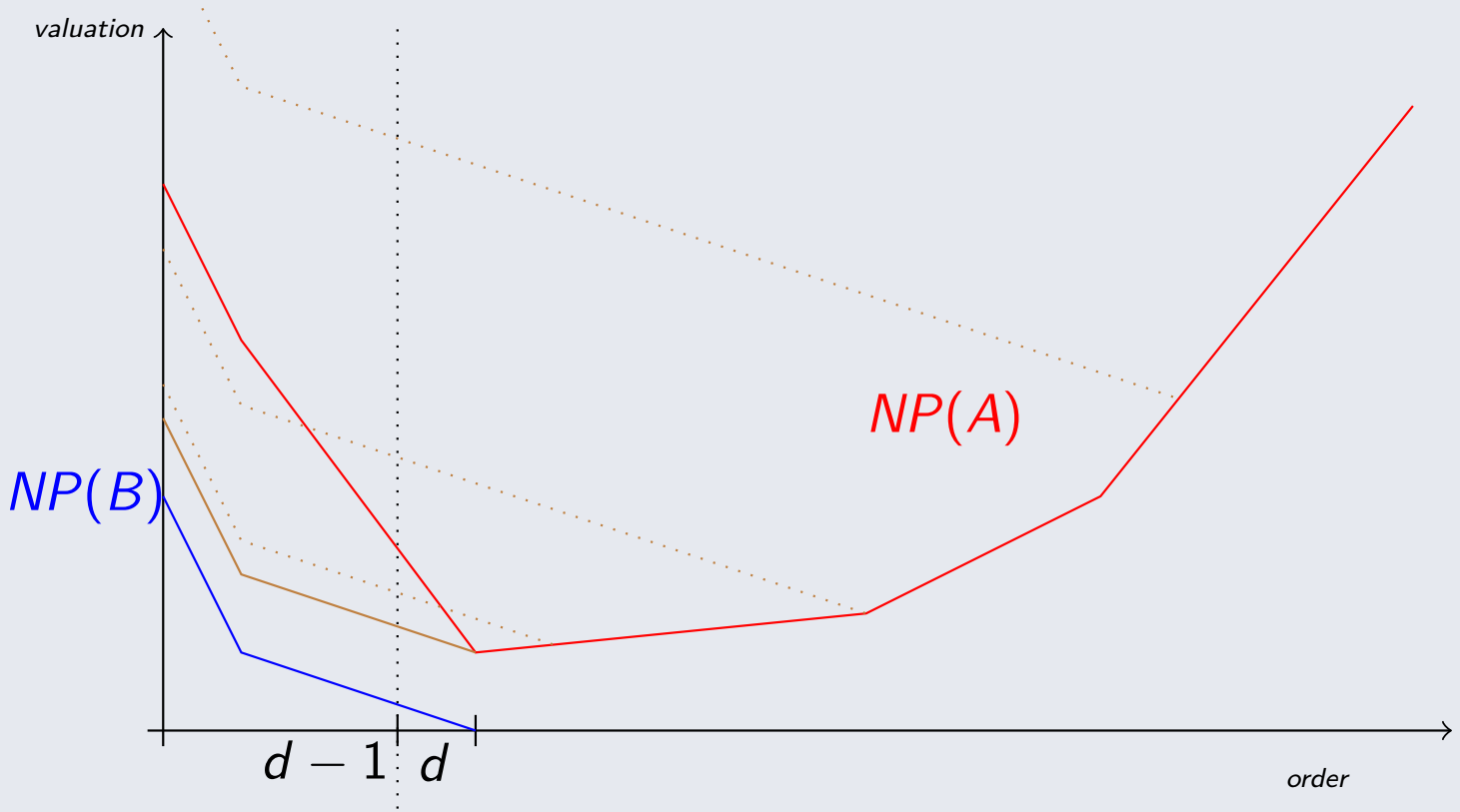

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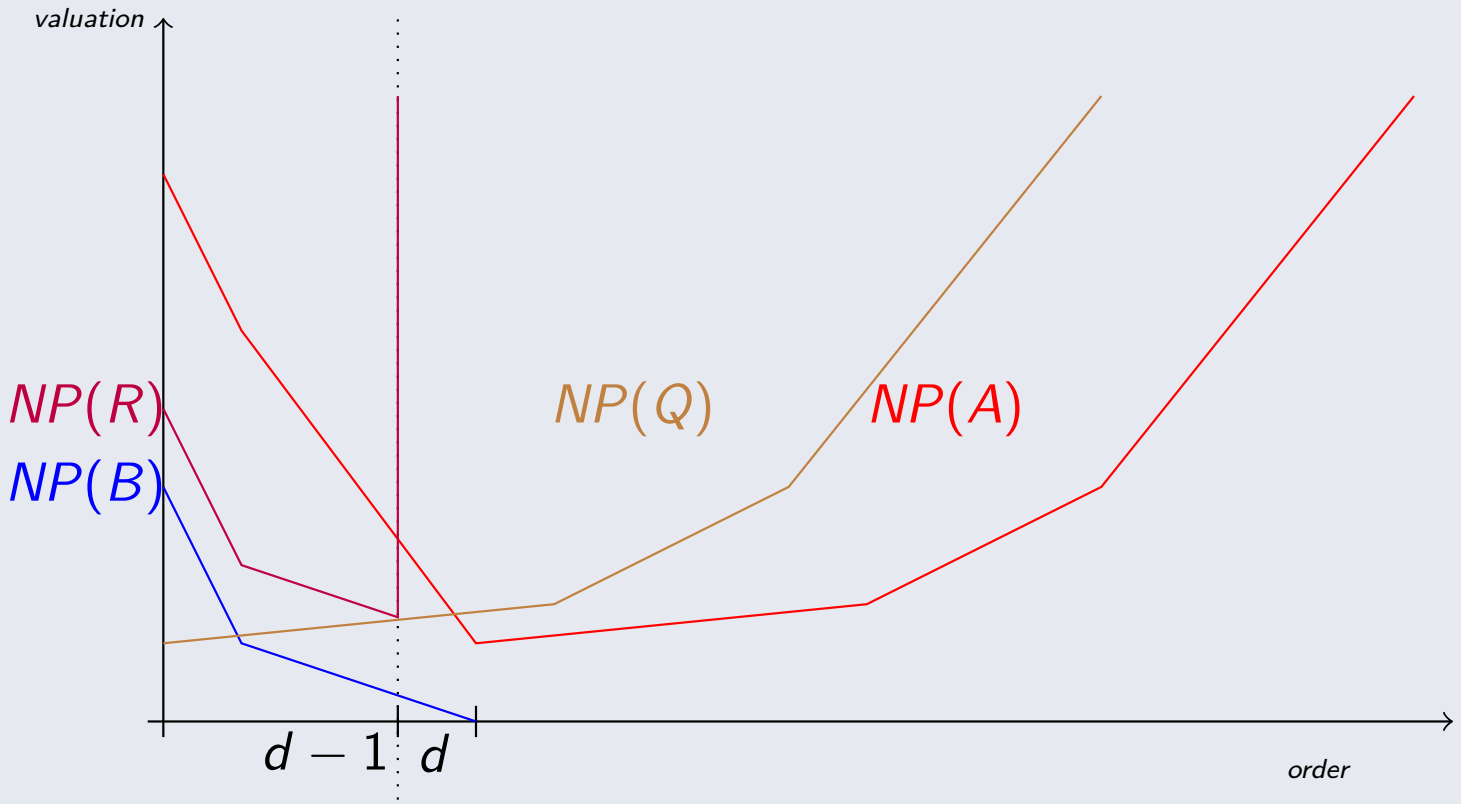

Division of A by B : $A = BQ + R$

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Handling the precision

Lattice precision

A and B are known with precision lattice H_A and H_B. Then (H_Q, H_R) are given by the Euclidean division of $H_A - QH_B$ by B.

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For $A = BQ + R$ with A known with precision-polygon φ, we can apply the previous construction to φ **divided** by B to obtain the precision on Q and R.

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Remark

This proved to be useful to handle precision for the computation of the characteristic polynomial.

Comparison: $\prod_{i=1}^{n} A_i \mod M$

Modulus M	n	Gain of precision		
		Jagged	Newton	Lattice (not dif. + dif.)
$X^5 + X^2 + 1$ (Irred. mod 2)	10	0.2	0.2	0.2 + 0.0
	50	4.2	4.2	4.2 + 0.0
	100	11.2	11.2	11.2 + 0.0
$X^5 + 1$ (Sep. mod 2)	10	0.4	0.4	0.9 + 6.0
	50	5.6	5.6	11.1 + 42.0
	100	13.6	13.6	27.0 + 87.0
$X^5 + 2$ (Eisenstein)	10	6.2	6.2	6.2 + 0.0
	50	44.0	44.0	44.0 + 0.0
	100	92.5	92.5	92.5 + 0.0
$(X+1)^5 + 2$ (Shift Eisenstein)	10	0.6	0.6	4.7 + 1.4
	50	7.1	7.1	42.6 + 1.4
	100	15.1	15.1	91.8 + 1.4
$X^5 + X + 2$ (Two slopes)	10	1.7	1.7	7.9 + 9.8
	50	8.1	8.1	70.7 + 59.8
	100	16.1	16.1	152.6 + 125.9

Figure: Precision for modular multiplication

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Let $f \in \mathbb{Q}_p[X]$. Then,

- *We can write $f = \prod_i f_i$.*
- *The f_i's are all of **one slope**.*
- *They have **respectively different slopes**.*

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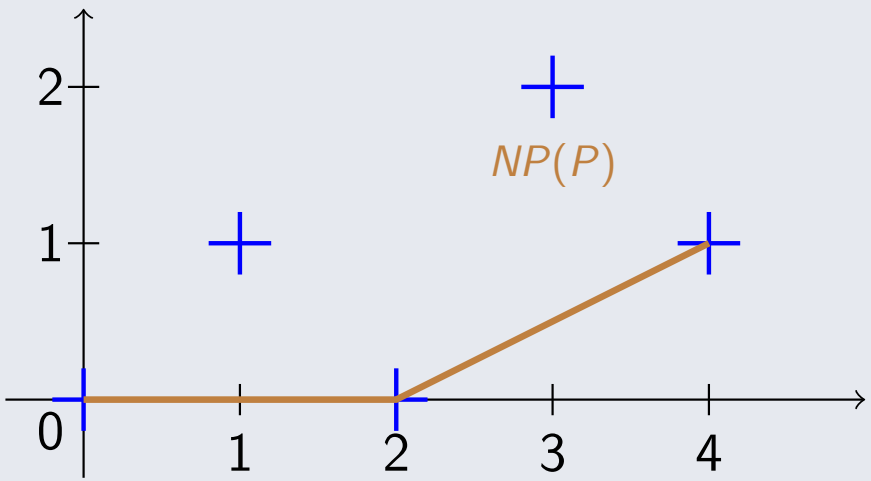

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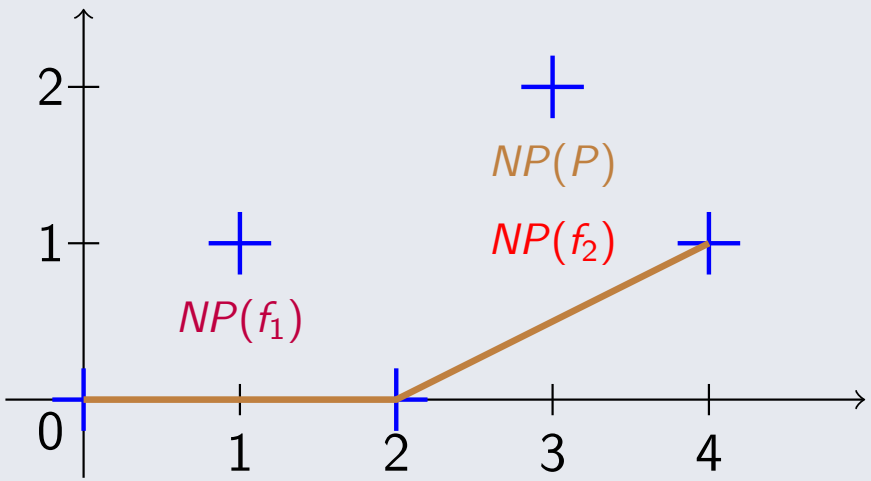

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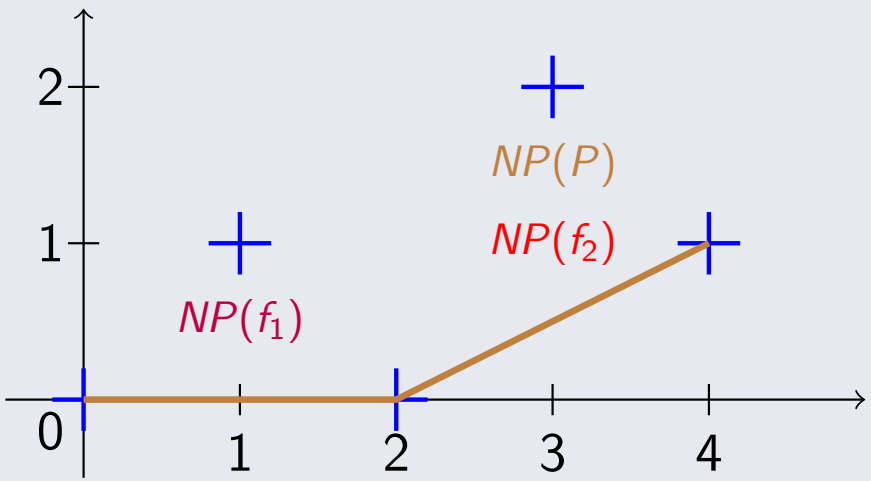

$$P = (2 + 3X + X^2) \times (1 + 3X^2)$$

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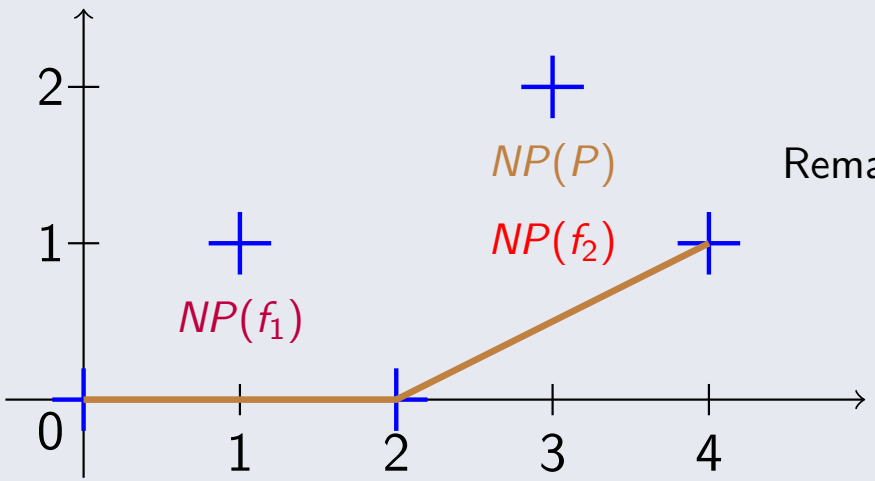


$$P = (2 + 3X + X^2) \times (1 + 3X^2)$$

Remark: $2 + 3X + X^2 = (1 + X)(1 + 2X)$.

A Newton iteration

The iteration

Already found in *Polynomial root finding over local rings and application to error correcting codes* by Berthomieu, Lecerf, Quintin:

$$\begin{aligned} A_{i+1} &:= A_i + (V_i P \mod A_i) \\ B_{i+1} &:= P \ \backslash\text{quo}\ A_{i+1} \\ V_{i+1} &:= (2V_i - V_i^2 B_{i+1}) \mod A_{i+1} \end{aligned}$$

The result

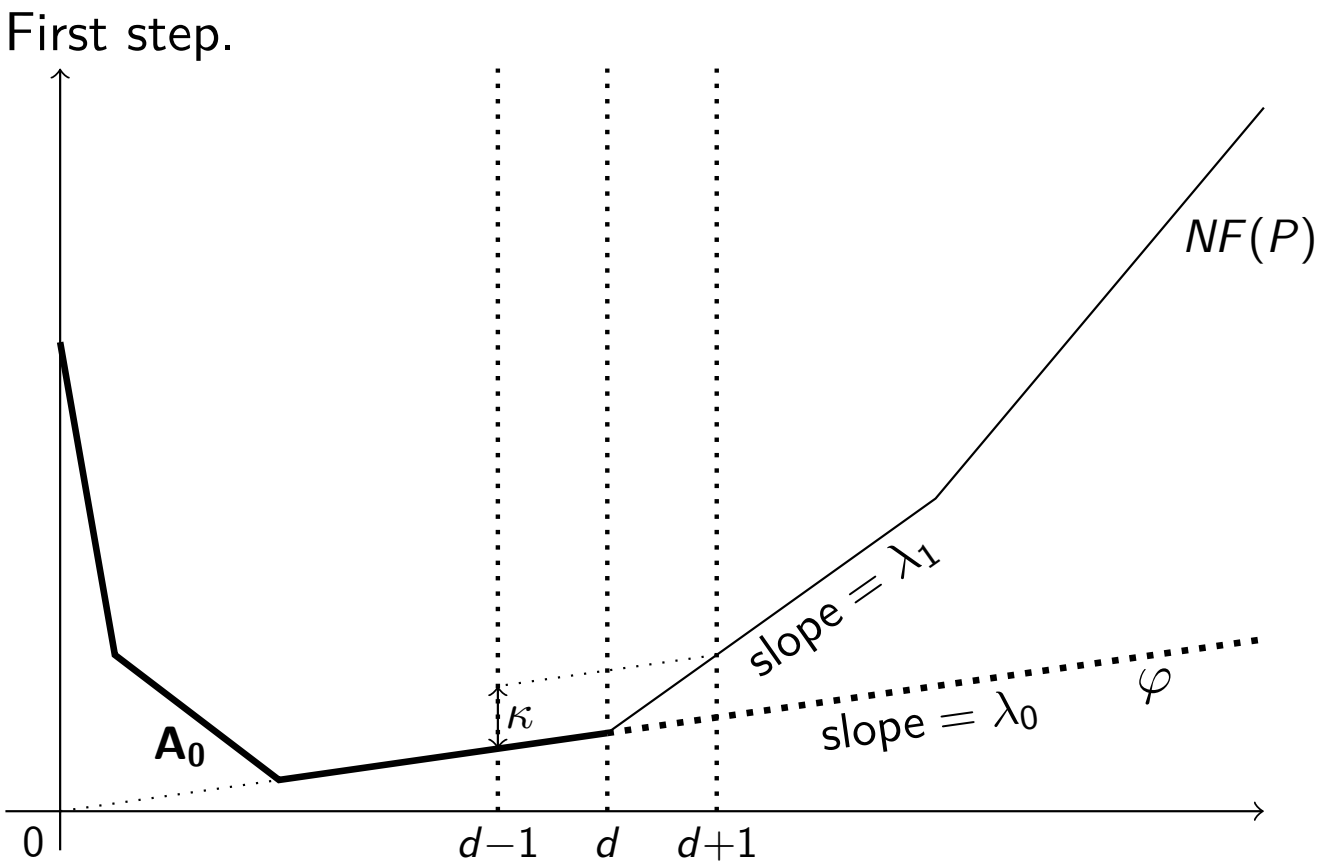
A_i, B_i, V_i converge **quadratically** to A, B, V such that $AB = P$, V is the inverse of B modulo A and A, B have the desired Newton polygons.

Ideas on the proof

About the proof

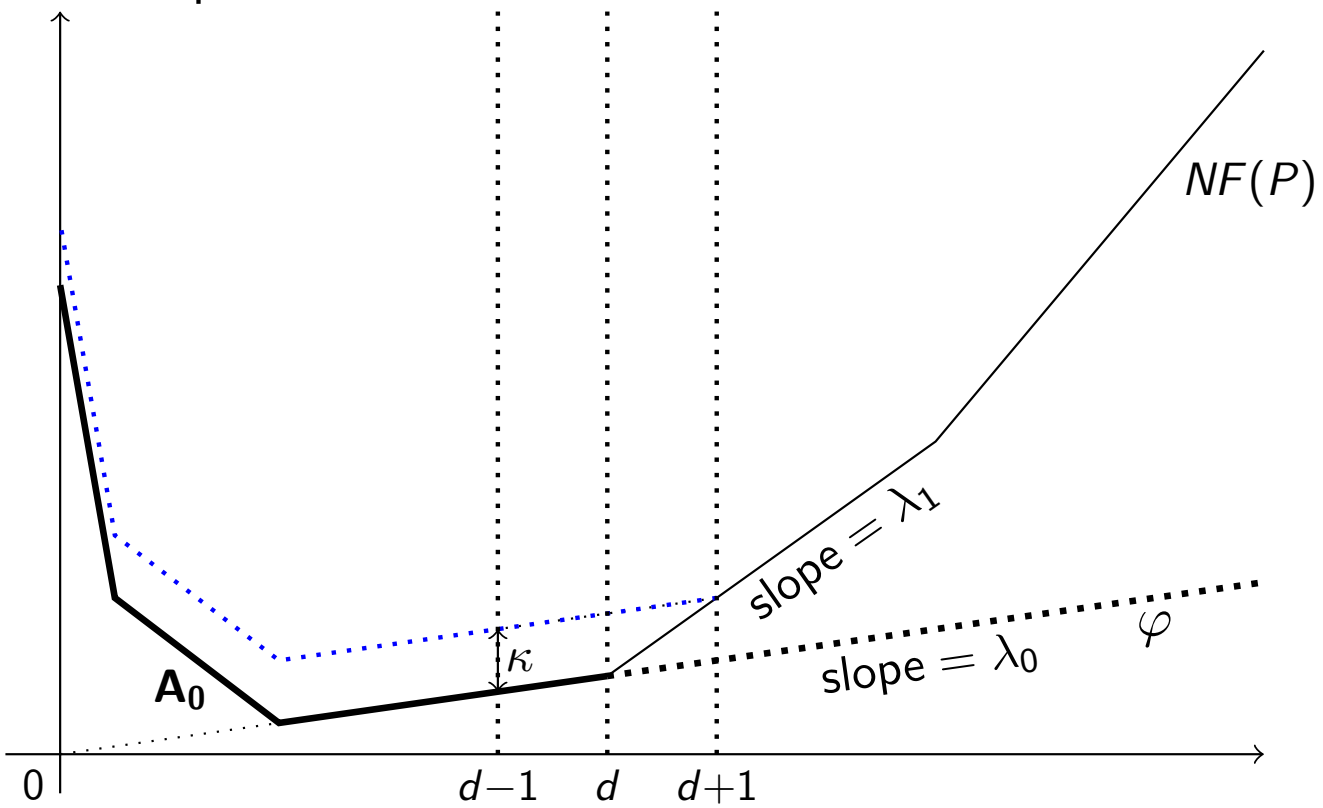
- We monitor $R_i = A_{i+1} - A_i$ and $S_i = P \mod A_i$.
- We prove that both $NP(R_i)$ and $NP(S_i)$ goes to infinity.

Illustration



Illustration

First step. Euclidean division.



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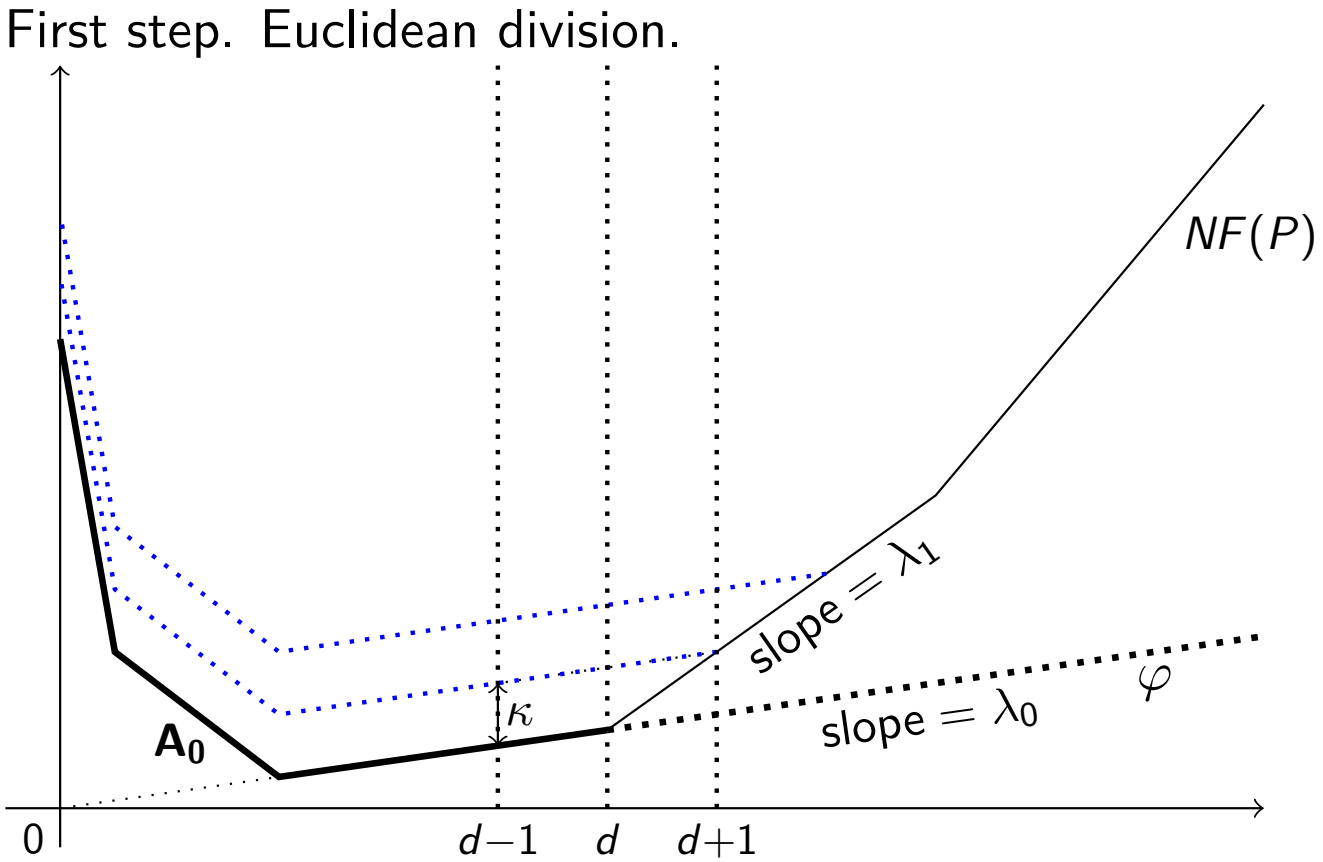

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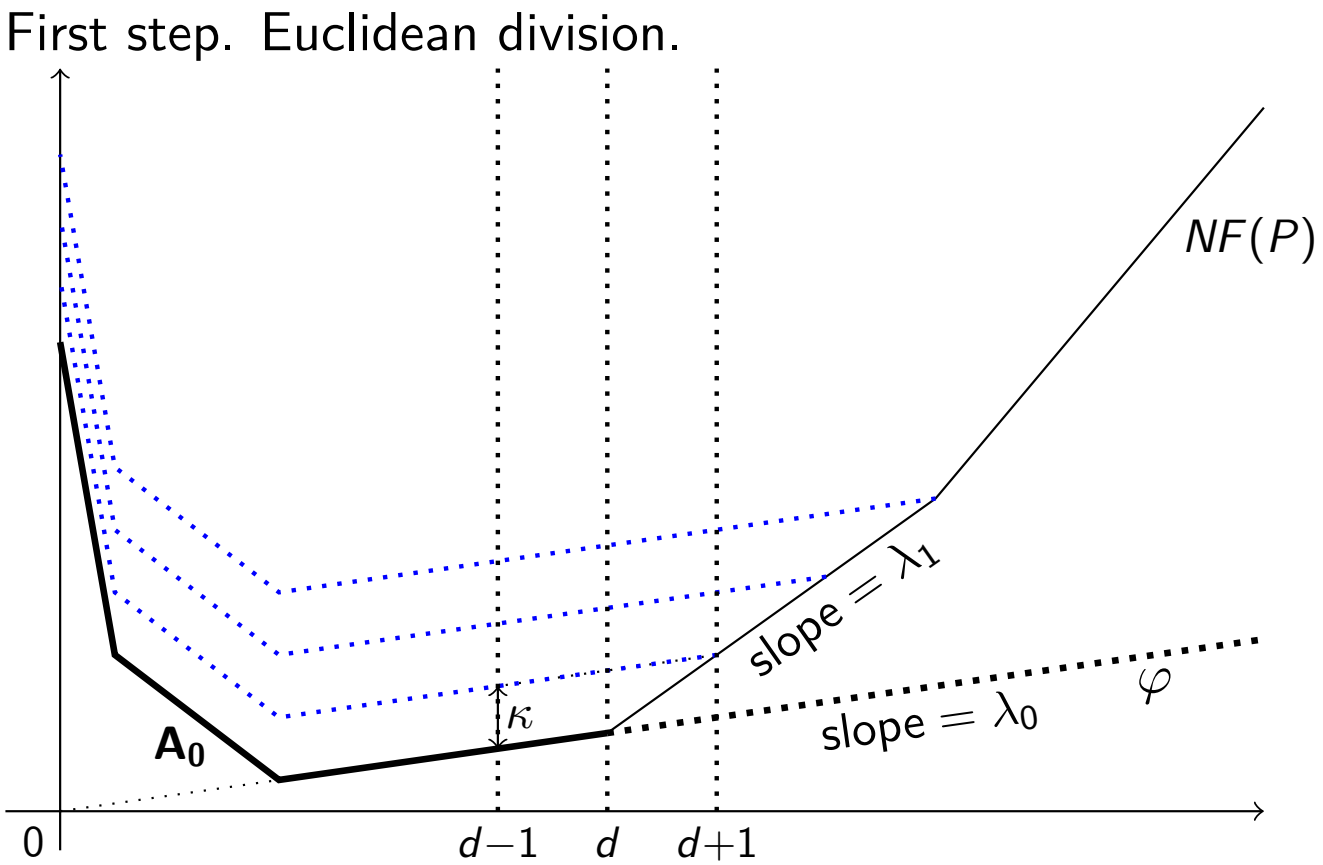

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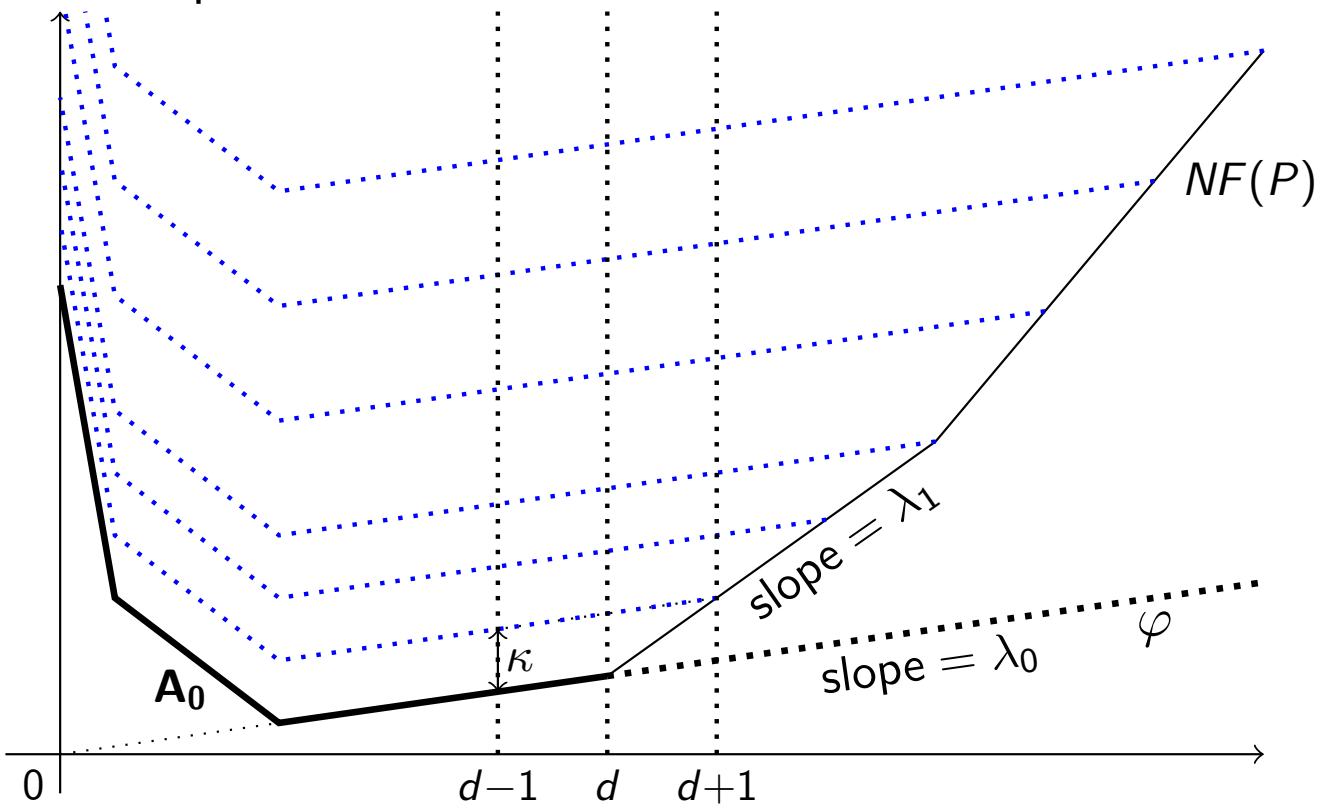

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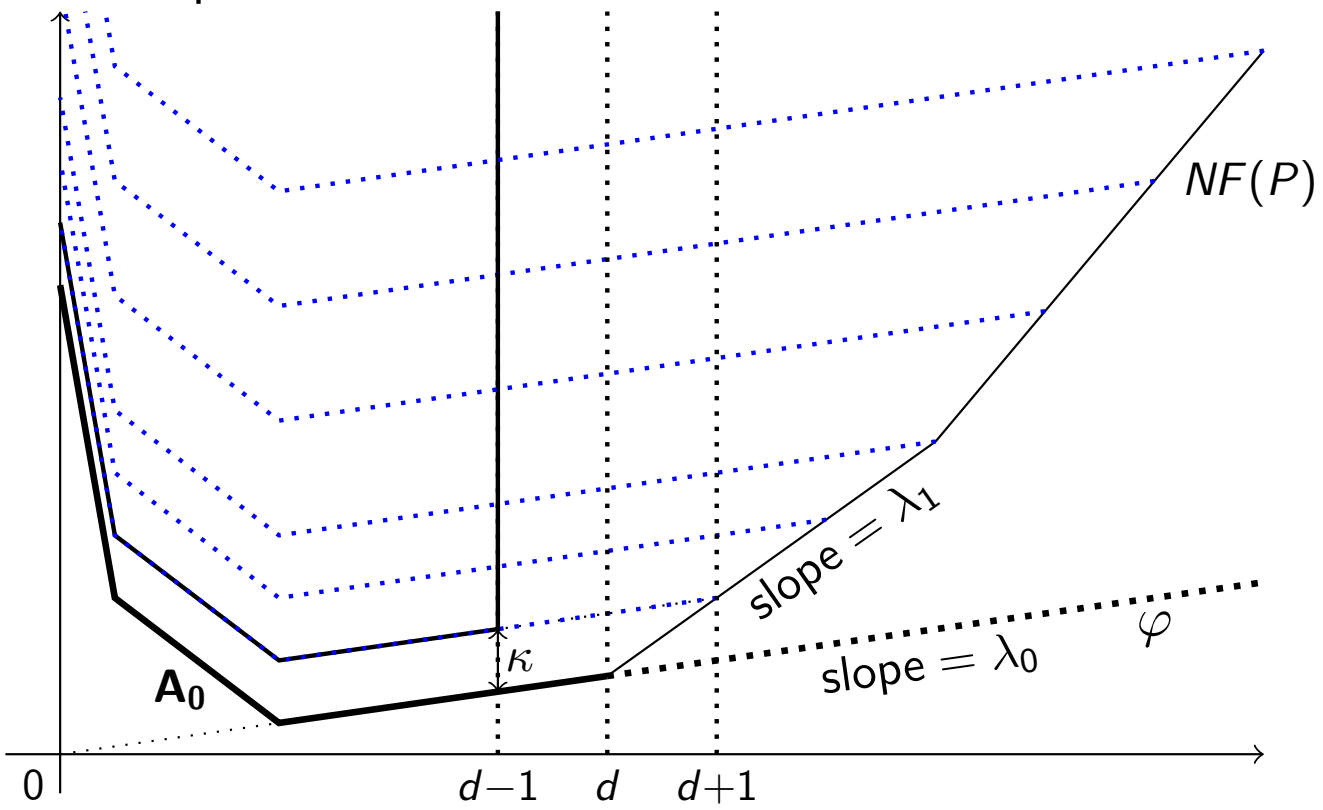

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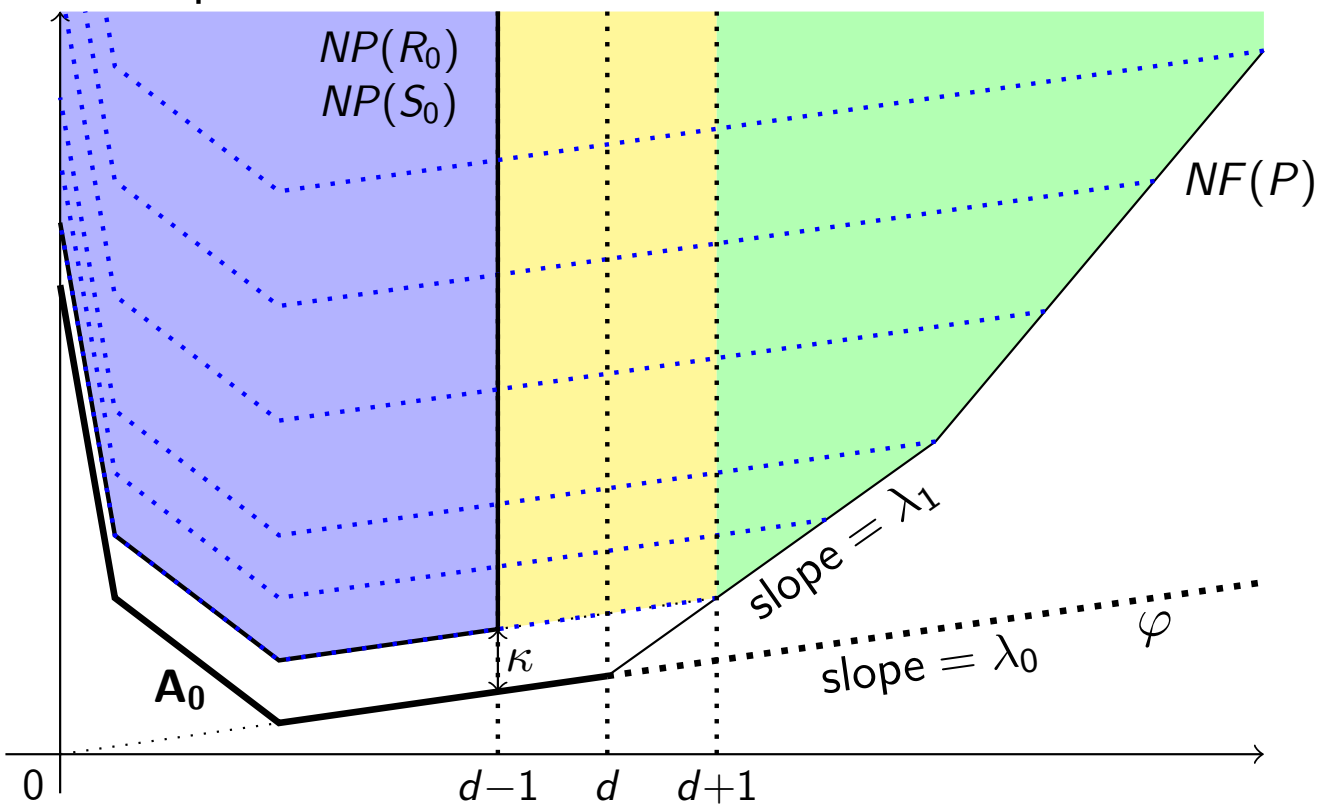

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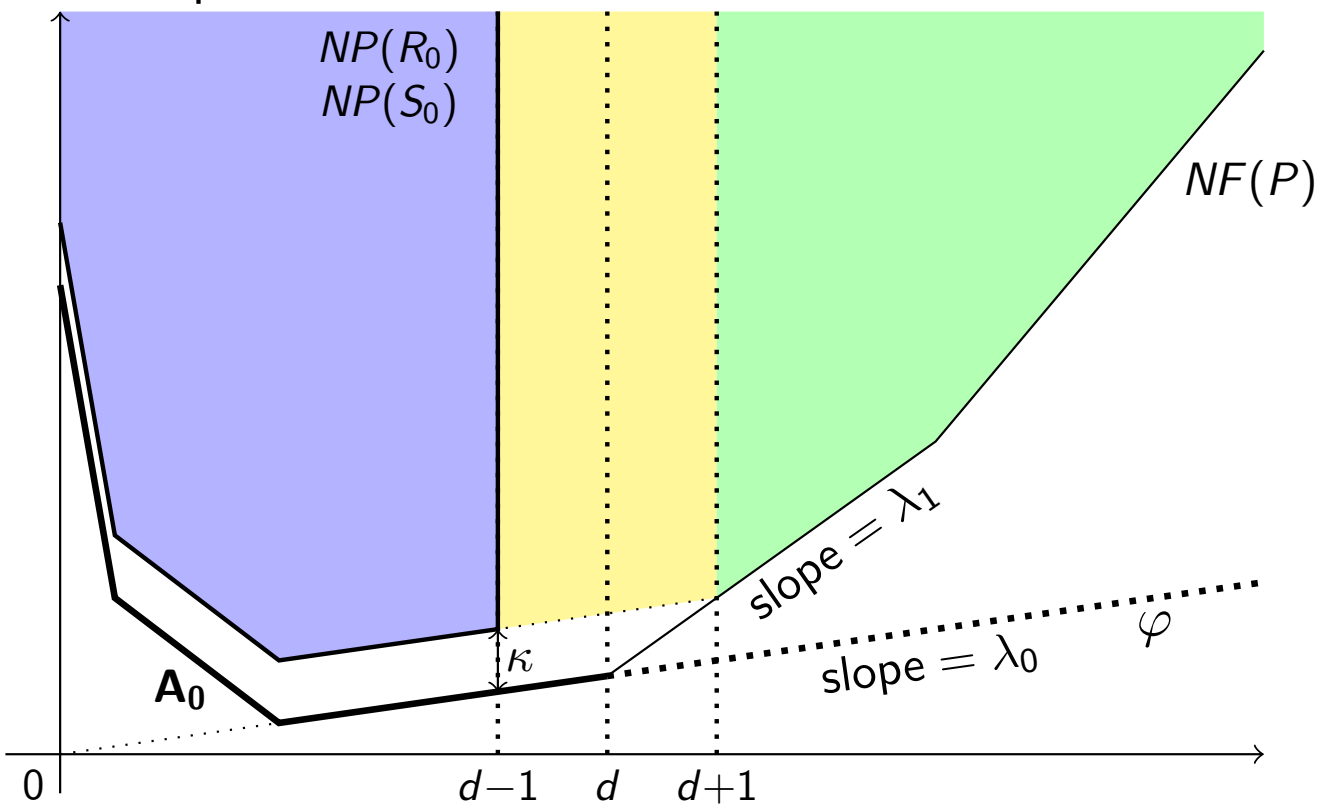

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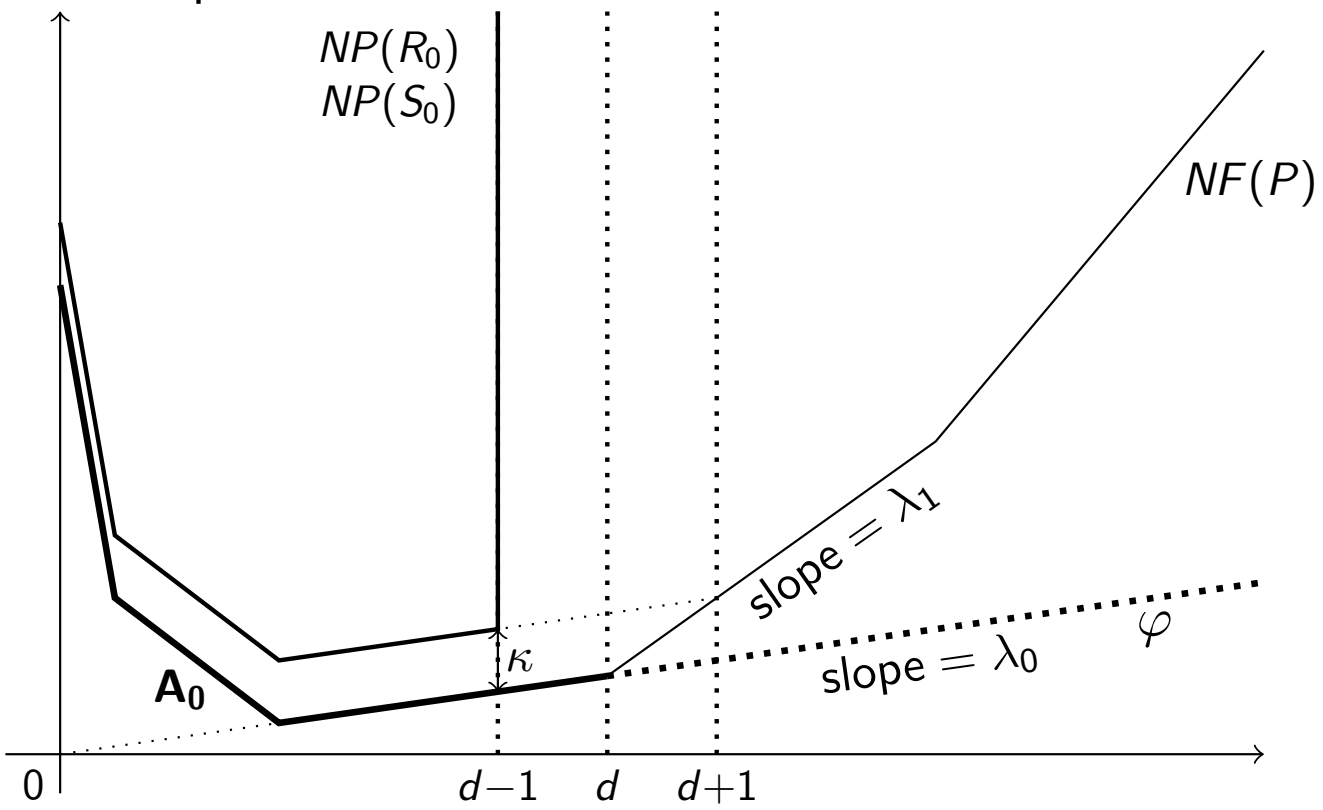

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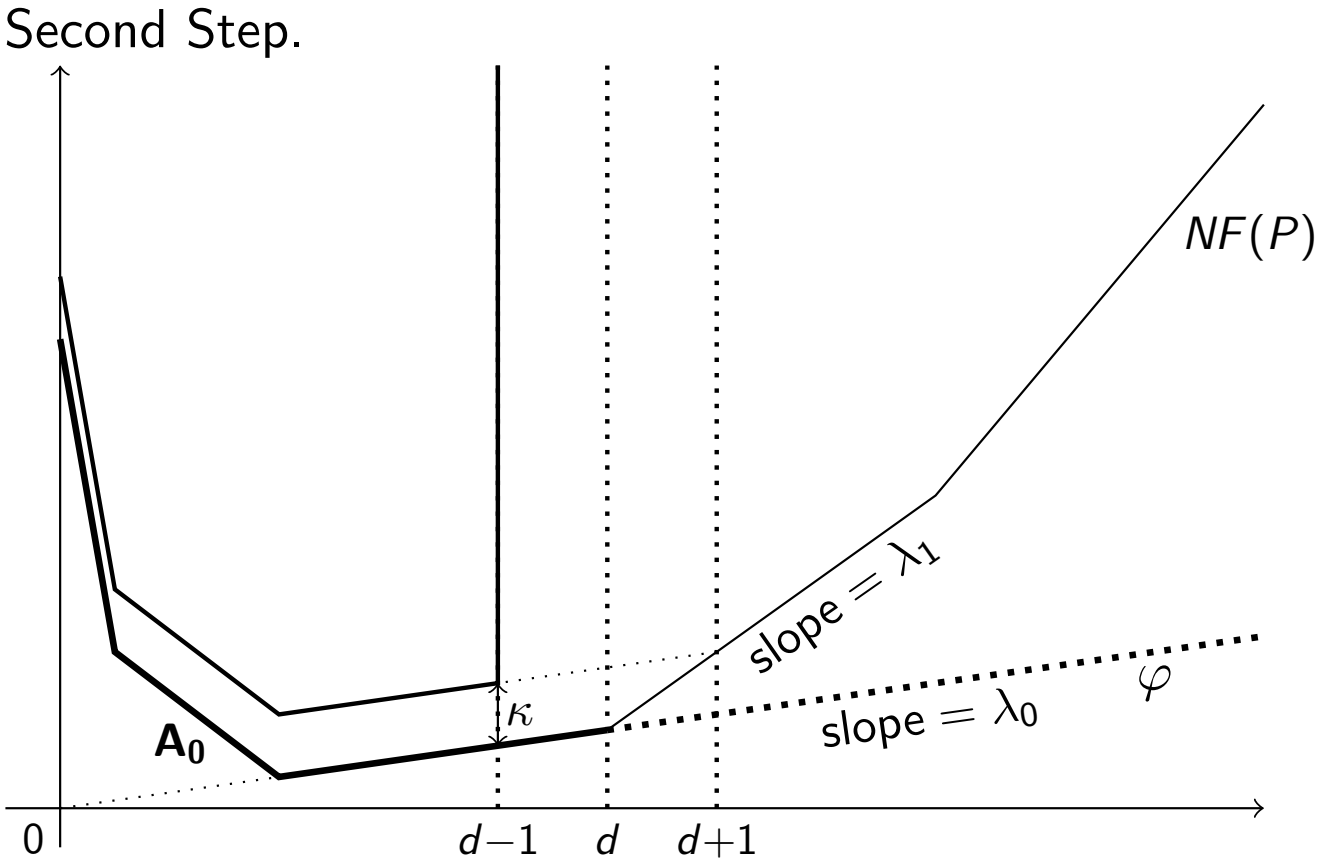
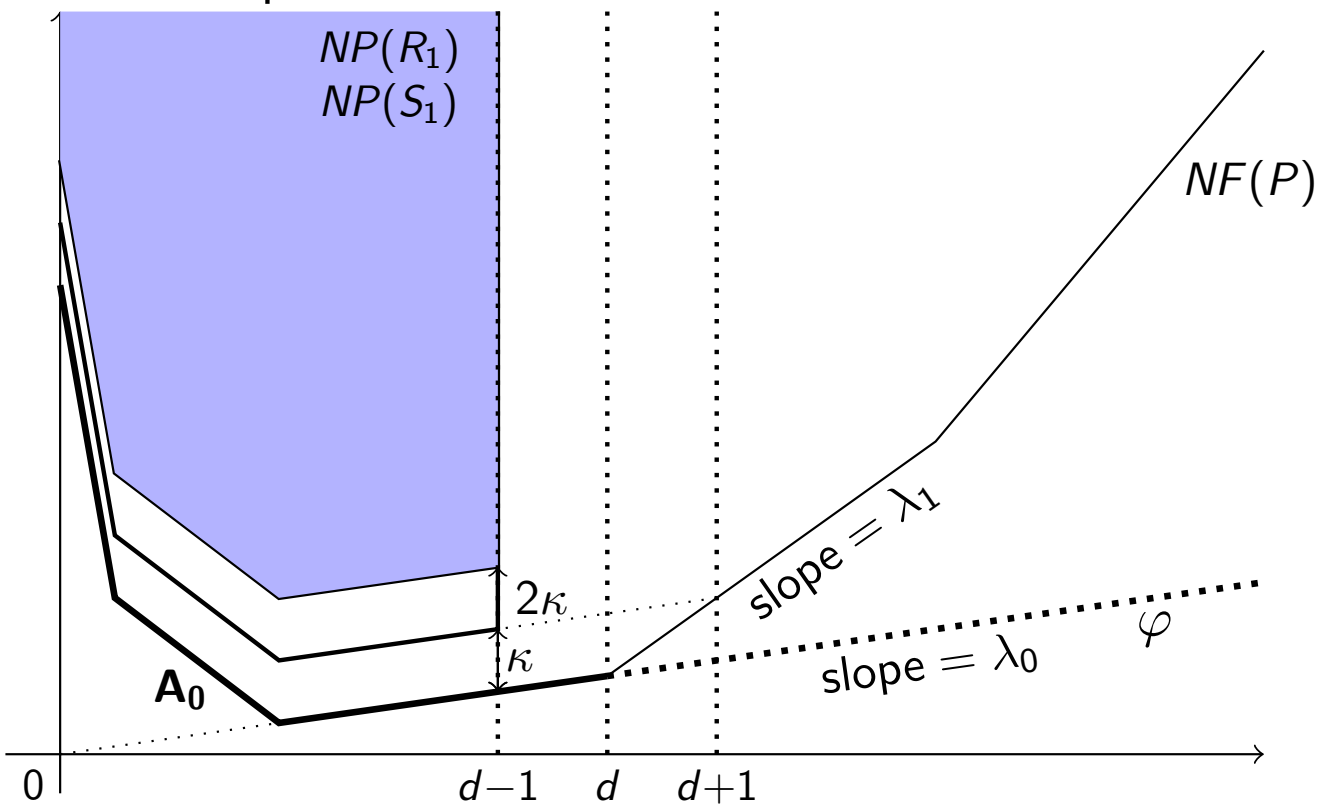
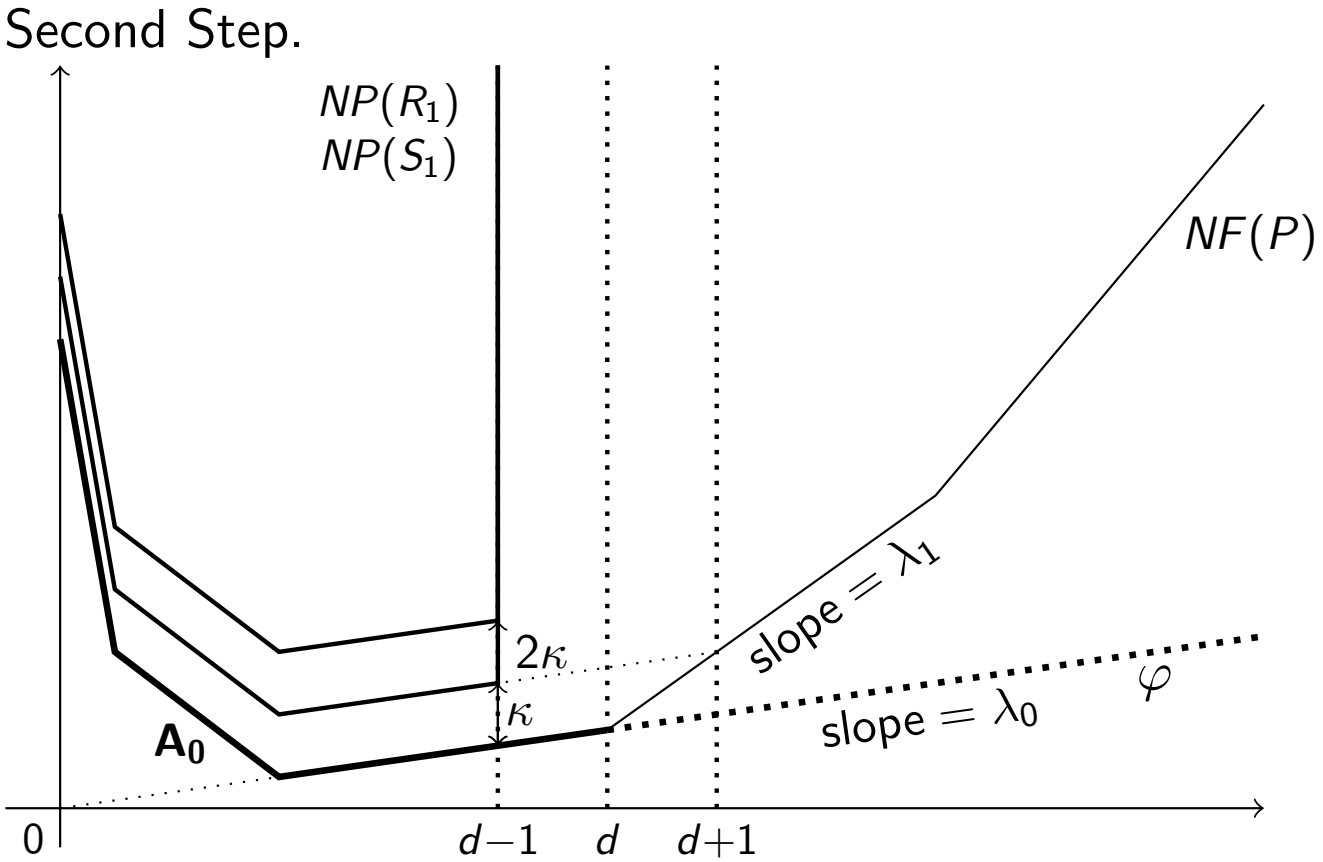


Illustration

Second Step.

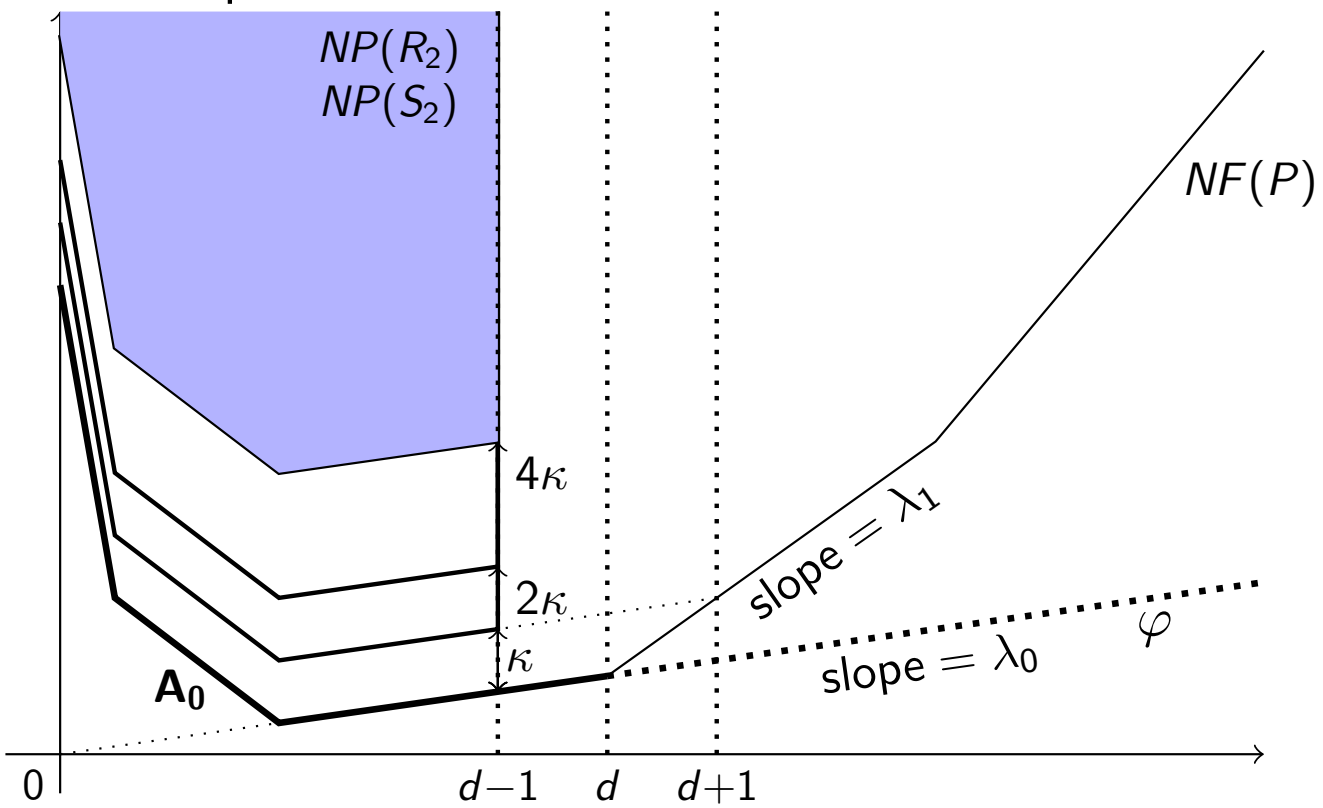


Illustration



Illustration

Third Step.



Illustration

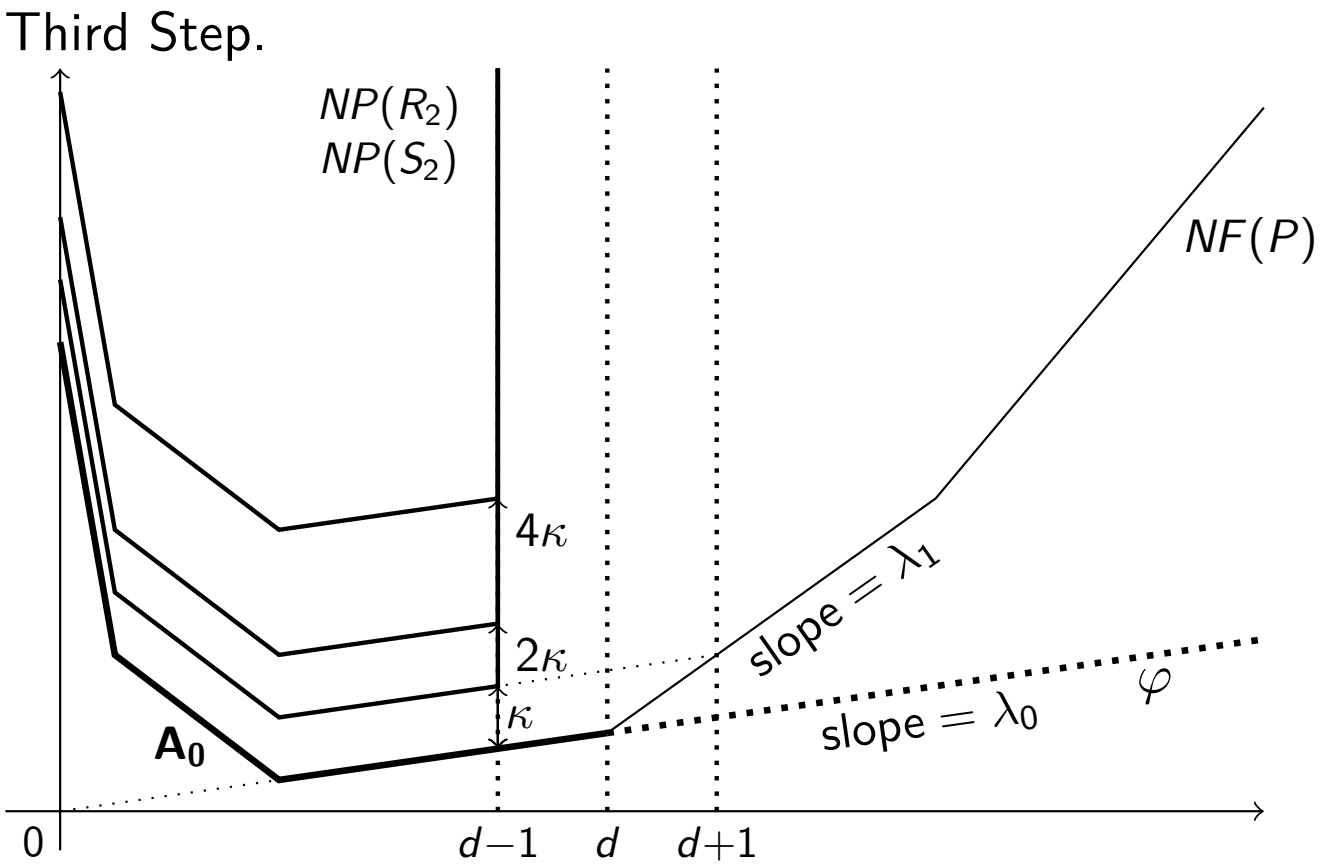


Table of contents

What about precision?

Setting

Let $F_A : P \mapsto A^{(1)}$ be the application such that $P = A^{(1)}B^{(1)}$, with $A^{(1)}, B^{(1)}$ corresponding to the slopes before/after the breakpoint d.

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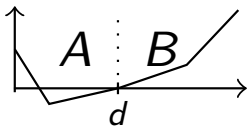
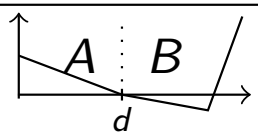
Differential

The application $F_A : P \mapsto A^{(1)}$ is of class C^1. Its differential at some point P is the linear mapping

$$dP \mapsto dA^{(1)} = (V^{(1)}\, dP) \mod A^{(1)}$$

where $A^{(1)}B^{(1)} = P$ and $V^{(1)}$ is the inverse of $B^{(1)}$ modulo $A^{(1)}$.

Some numerical results: $A \mapsto AB \mapsto A$.

Polynomials	Precision	Mean gain of precision		
		Jagged	Newton	Lattice
	absolute	−14.5	−14.7	0.0
	relative	−1.5	−3.6	0.0
	absolute	0.0	0.0	0.0
	relative	−0.3	−1.2	0.0

To sum up

On p-adic precision

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On polynomial computations

- **Diffused digits** for modular multiplication.
- **Newton iteration** for **slope factorisation**.

References

Initial article

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Linear Algebra

- Xavier Caruso, David Roe and Tristan Vaccon p-adic stability in linear algebra, ISSAC 2015.

Polynomial Computations

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Thank you for your attention

Thanks