On some p-adic differential equations

with separation of variables

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Why should one work with p-adic numbers ?

p-adic methods

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My personal (long-term) motivation

Computing (some) moduli spaces of p-adic Galois representations.

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- In particular: key-exchange, proof of identity, public-key encryption... They are candidate for **Post-Quantum Cryptography**.
- Hence, we need deep **understanding of the computation of isogenies**.

How to compute isogenies?

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Isogeny and Differential equations (*cf* Schoof-Elkies-Atkin algorithm, Bostan-Morain-Salvy-Schost 08, Lercier-Sirvent 08, ...)

Let E and $\tilde{E}$ be two elliptic curves over $\mathbb{Z}/p\mathbb{Z}$:

$$E : y^2 = x^3 + Ax + B,$$
$$\tilde{E} : y^2 = x^3 + \tilde{A}x + \tilde{B}.$$

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Let us assume that there exists some normalized isogeny I between E and $\tilde{E}$. Then, for some rational fraction U,

$$I(x, y) = (U(x), yU'(x)),$$

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$$I(x, y) = (U(x), yU'(x)),$$

Writing $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$, we get :

$$(Bx^6 + Ax^4 + 1)S'^2 = 1 + \tilde{A}S^4 + \tilde{B}S^6.$$

Change of variable and the differential equation

The differential equation

Let S be such that $U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}$.

We have the following differential equation for S :

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- **Not easy**:

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Solving a differential equation in $\mathbb{Z}/p\mathbb{Z}$???

- **Not easy**:
$$\int X^{p-1} = \frac{1}{p}X^p ?$$
- We would like to be in **zero characteristic**: let's go **p-adic**!

Previous take on this topic

Linear Differential Equation

Bostan, Gonzalez-Vega, Perdry & Schost in 2005 and Grenet, Lecerf & van der Hoeven in 2016 have tackled this case completely.

Isogeny computation and differential equations

Used inside variants of SEA. Main references are Bostan, Morain, Salvy & Schost in 2008 and Lercier & Sirvent in 2008.

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The differential equation

Let S be such that

$$U = \frac{1}{S(\frac{1}{\sqrt{x}})^2}.$$

Then if $A, B, \tilde{A}, \tilde{B}$ are in $\mathbb{Z}_p$,

$$S \in \mathbb{Z}_p[[t]]$$

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Computing the isogeny

Given E and $\widetilde{E}$, the goal is to compute the isogeny I *via* the differential equation:

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1. Lift (consistently) from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}_p$.
2. Solve the differential equation in $\mathbb{Z}_p$.
3. Reduce mod p to get the solution in $\mathbb{Z}/p\mathbb{Z}$.

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with $0 \leq a_i < p$ for all i.

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The p-adic integers form a subring $\mathbb{Z}_p$ of $\mathbb{Q}_p$.

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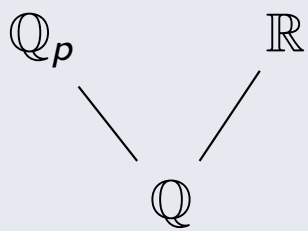
Remark



$$\begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Z}/p\mathbb{Z} & \to & \mathbb{Z}/p\mathbb{Z} \end{array}$$

Definition of the precision

Finite-precision p-adics

Elements of $\mathbb{Q}_p$ can be written $\sum_{i=k}^{+\infty} a_i p^i$, with $a_i \in [\![0, p-1]\!]$, $k \in \mathbb{Z}$ and p a prime number.
While working with a computer, we usually only can consider the beginning of this power serie expansion: we only consider elements of the following form $\sum_{i=l}^{d-1} a_i p^i + O(p^d)$, with $l \in \mathbb{Z}$.

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Example

The order of $\ldots\overline{654,3}^7 = 3 \times 7^{-1} + 4 \times 7^0 + 5 \times 7^1 + 6 \times 7^2 + O(7^3)$ is 3.

p-adic precion *vs* real precision

The quintessential idea of the step-by-step analysis is the following :

Proposition (p-adic errors don't add)

Indeed,

$$(a + O(p^{k})) + (b + O(p^{k})) = a + b + O(p^{k}).$$

That is to say, if a and b are known up to precision $O(p^k)$, then so is $a + b$.

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It is quite the opposite to when dealing with real numbers, because of **Round-off error** :

$$(1 + 5 * 10^{-2}) + (2 + 6 * 10^{-2}) = 3 + 1 * 10^{-1} + 1 * 10^{-2}.$$

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Proposition (addition)

$$(x_0 + O(p^{k_0})) + (x_1 + O(p^{k_1})) = x_0 + x_1 + O(p^{min(k_0,k_1)})$$

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Proposition (division)

$$\frac{xp^a + O(p^b)}{yp^c + O(p^d)} = x * y^{-1}p^{a-c} + O(p^{min(d+a-2c,b-c)})$$

In particular,

$$\frac{1}{p^c y + O(p^d)} = y^{-1}p^{-c} + O(p^{d-2c})$$

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Reduction of the problem

Change of equation

When $p \neq 2$, we can replace $y'^2 \times G = H(y)$ by $y' = g \times h(y)$ with $g, h \in \mathbb{Z}_p[\![x]\!]^\times$.

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We wish to solve the following differential equation, with $g, h \in \mathbb{Z}_p[\![x]\!]^\times$:

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We can write

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for some P_n with integer coefficients.

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Loss in precision

Loss: $val(n!) \geq \frac{n - \log_p n}{p-1}$ at the n-th step.

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$$\int O(p^m) x^k = \frac{O(p^m)}{k+1} x^{k+1}.$$

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$$N_{g,h}(u) \leftarrow u - h(u) \int \left(\frac{u'}{h(u)} - g \right).$$

Remark

$$\int O(p^m) x^k = \frac{O(p^m)}{k+1} x^{k+1}.$$

One loses $O(\log_p N)$ at each step, for N the order of truncation.

Our problem

We wish to solve the following differential equation, with $g, h \in \mathbb{Z}_p[\![x]\!]^\times$:

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Loss in precision **quadratic in the number of steps**.

Summary: precision and p-adic computations

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- If so, can we **achieve this loss**?

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The Main lemma of p-adic differential precision

Lemma (CRV14)

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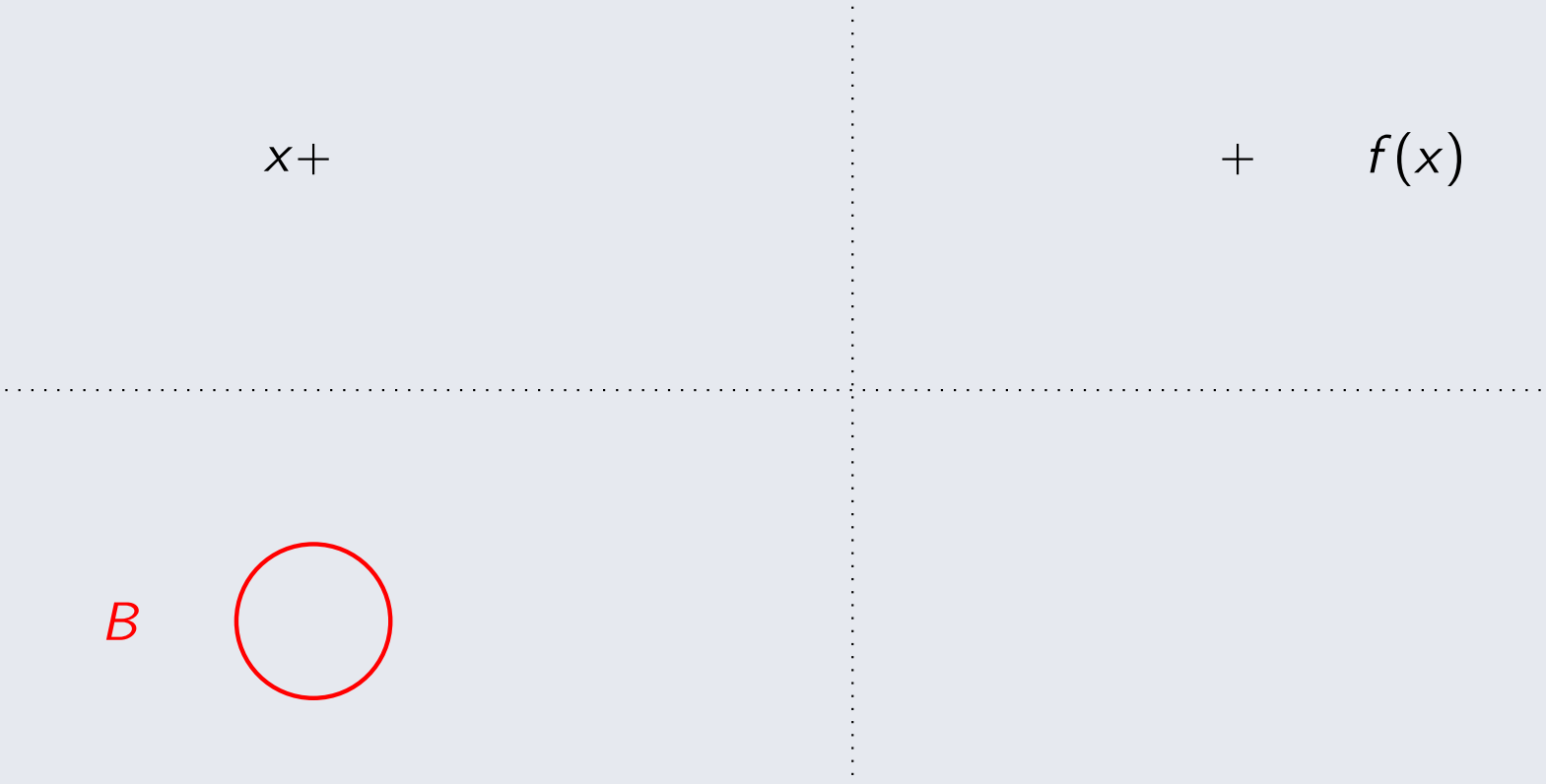
$$f(x + B) = f(x) + f'(x) \cdot B.$$

Geometrical meaning

Interpretation



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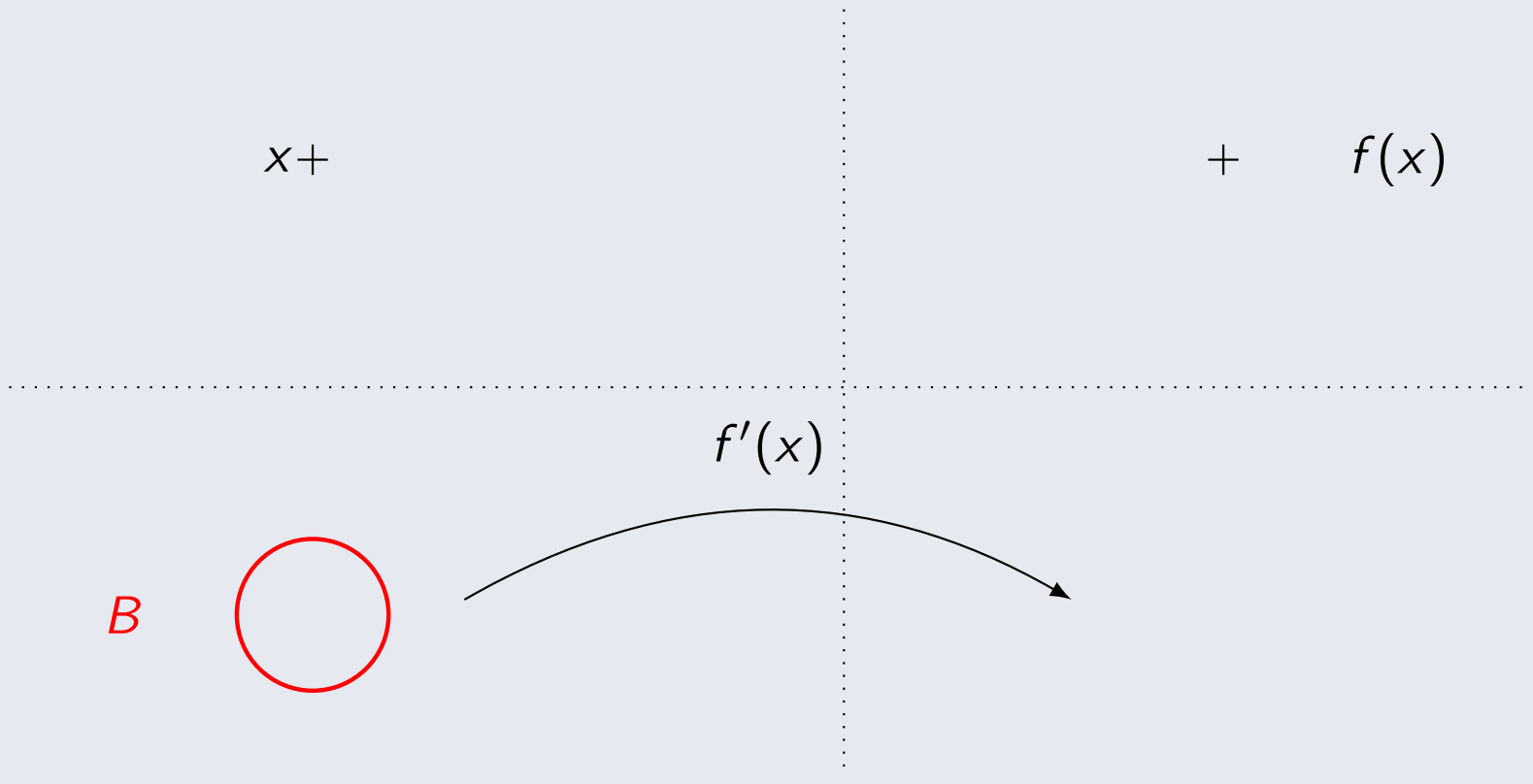

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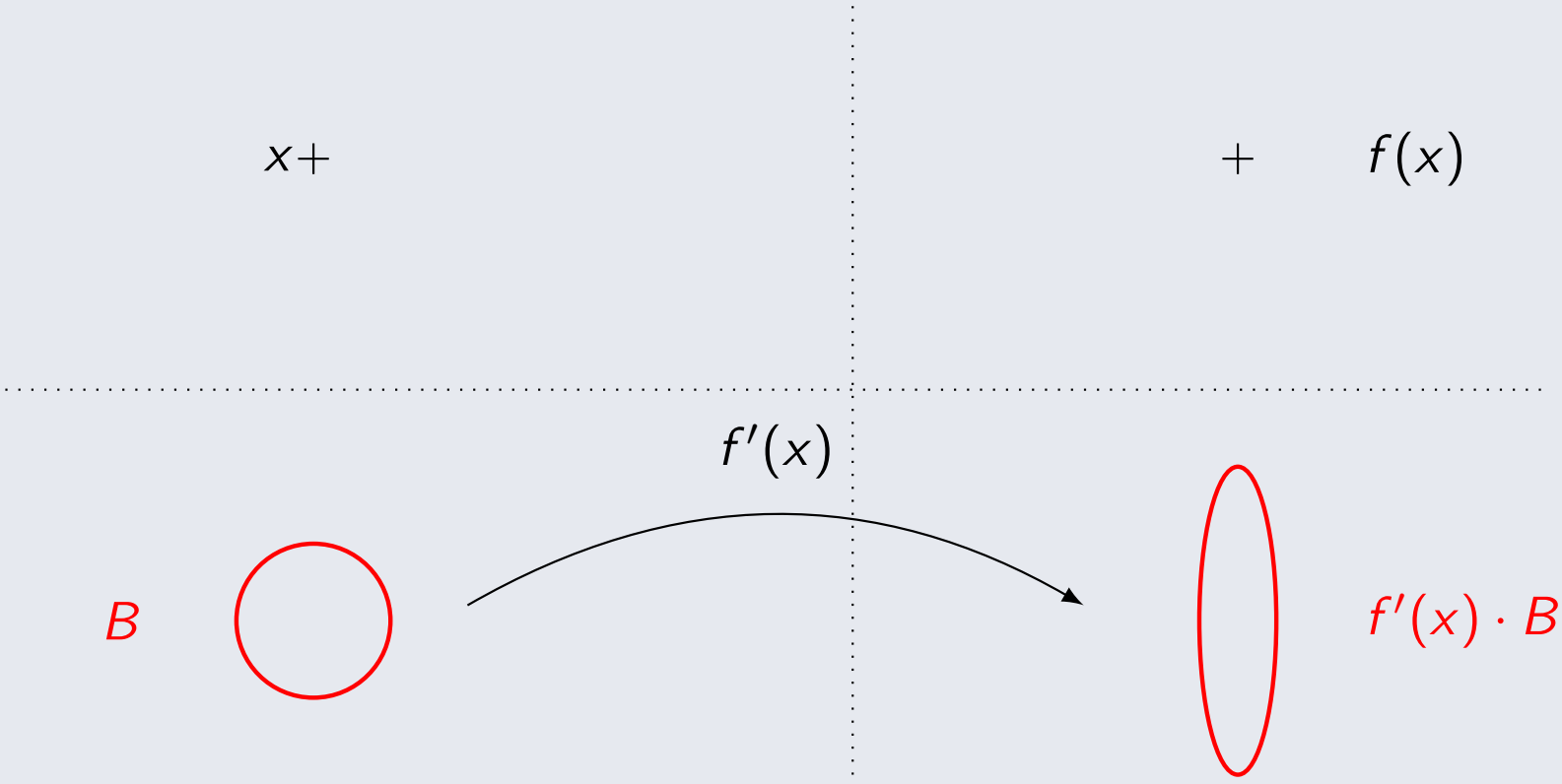

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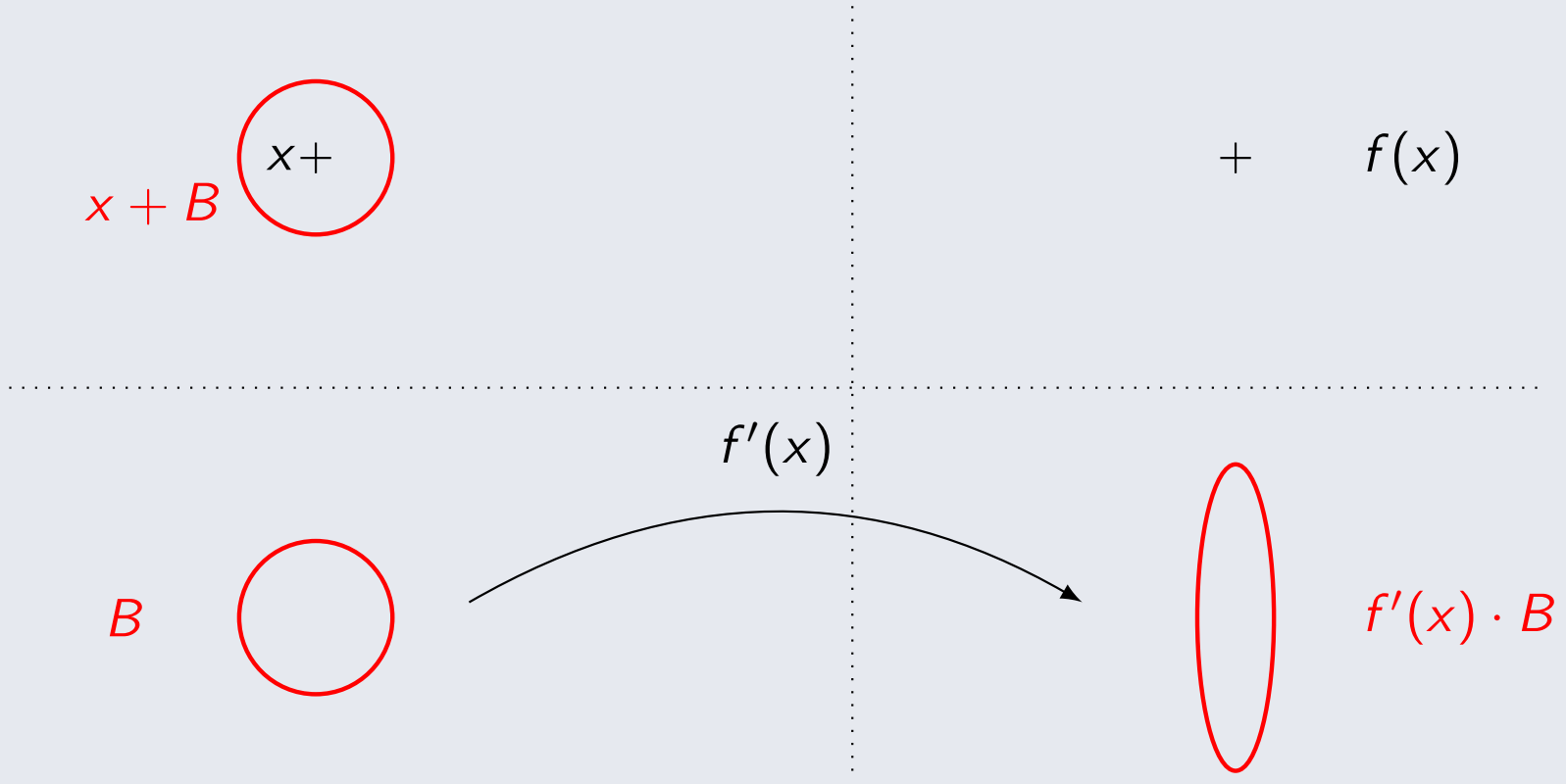

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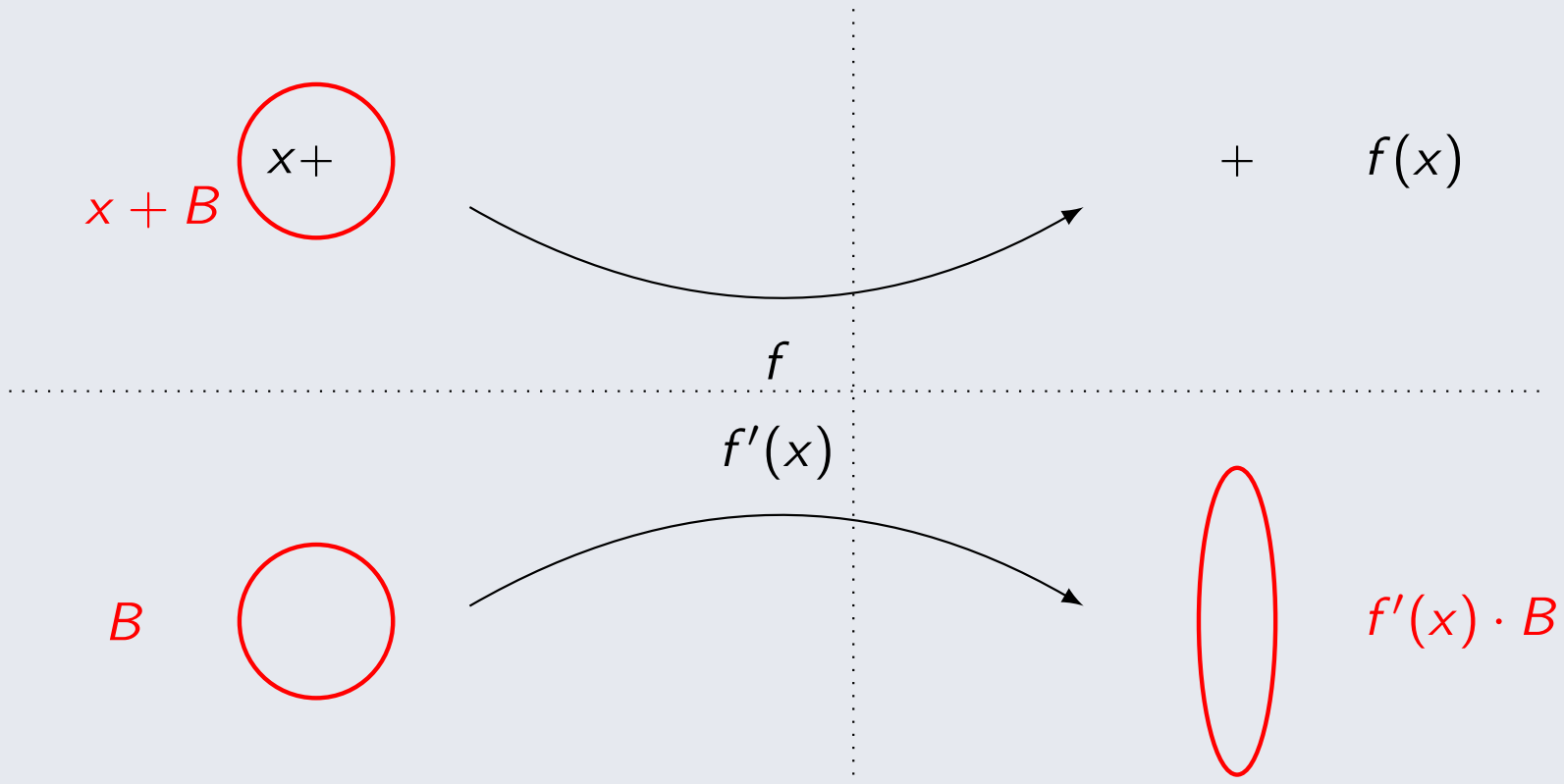

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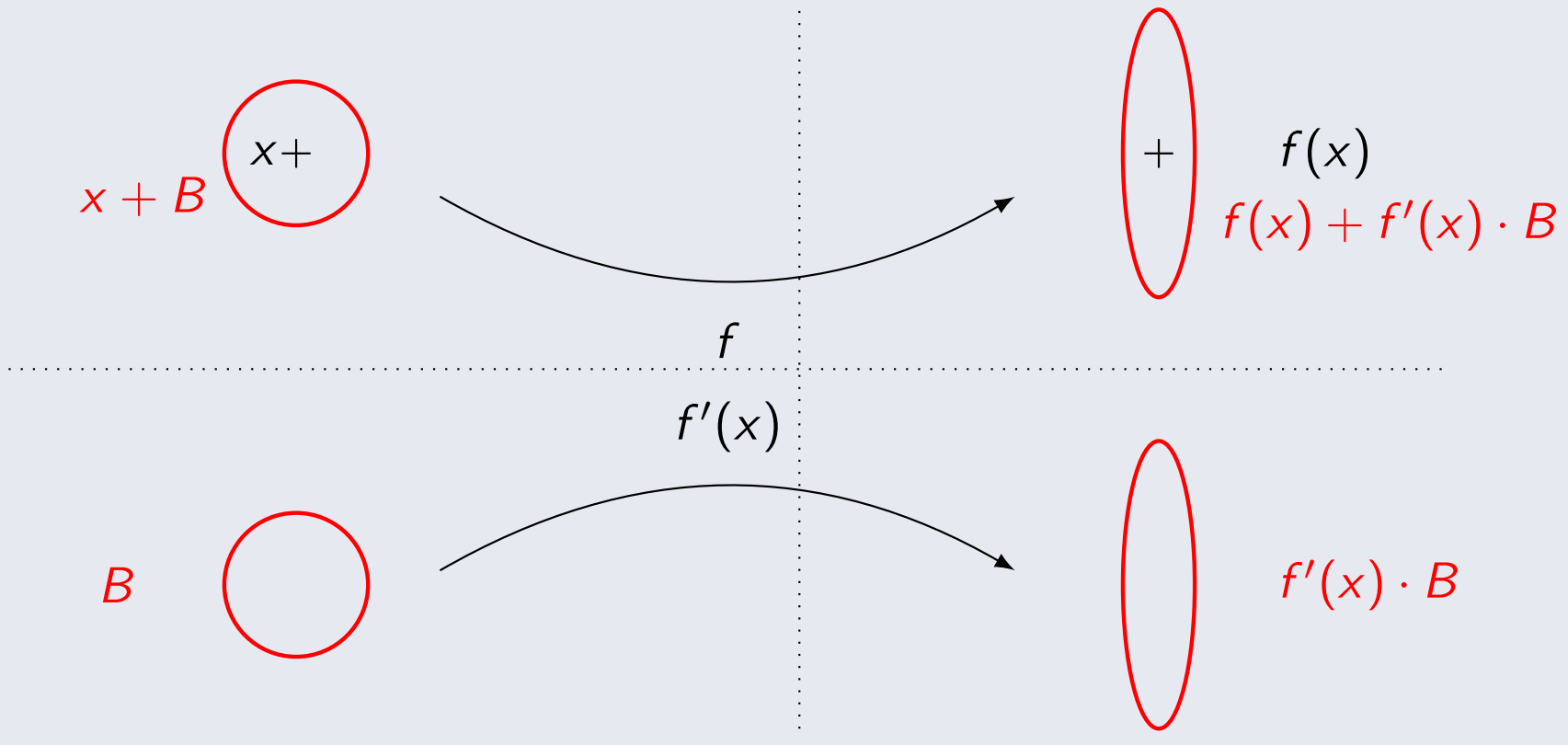

Higher differentials

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What is **small enough** ?

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This can be determined with **Newton-polygon** techniques.

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Differential and differential equation

Theorem

Let $\Phi : (g, h) \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$. *Then,*

$$\Phi'(g, h) \cdot (\delta g, \delta h) = h(y) \int \delta g + \frac{g \delta h(y)}{h(y)}.$$

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In our case, $p \neq 2$, $y, g, h \in \mathbb{Z}_p[\![x]\!]$, $g(0) = h(0) = 1$. *If* $\delta g = \delta h = O(p^k)$, *then*

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$$\Phi'(y) \cdot (\delta g, \delta h) \mod x^{2^N+1} \in \frac{O(p^k)}{p^N} \mathbb{Z}_p[\![x]\!].$$

First conclusion on the application of the lemma

Proposition

$\Phi(g,h) \mod (p, t^{2^n})$ *is determined by* $g, h \mod (p^{1+\log_p 2^n}, t^{2^n})$. *In other words, we have a* logarithmic loss *in precision.*

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Remark

To prove this result, we compute the norms of the differential of higher order. It is possible to give explicit bounds and apply our framework.

What happens in practice ?

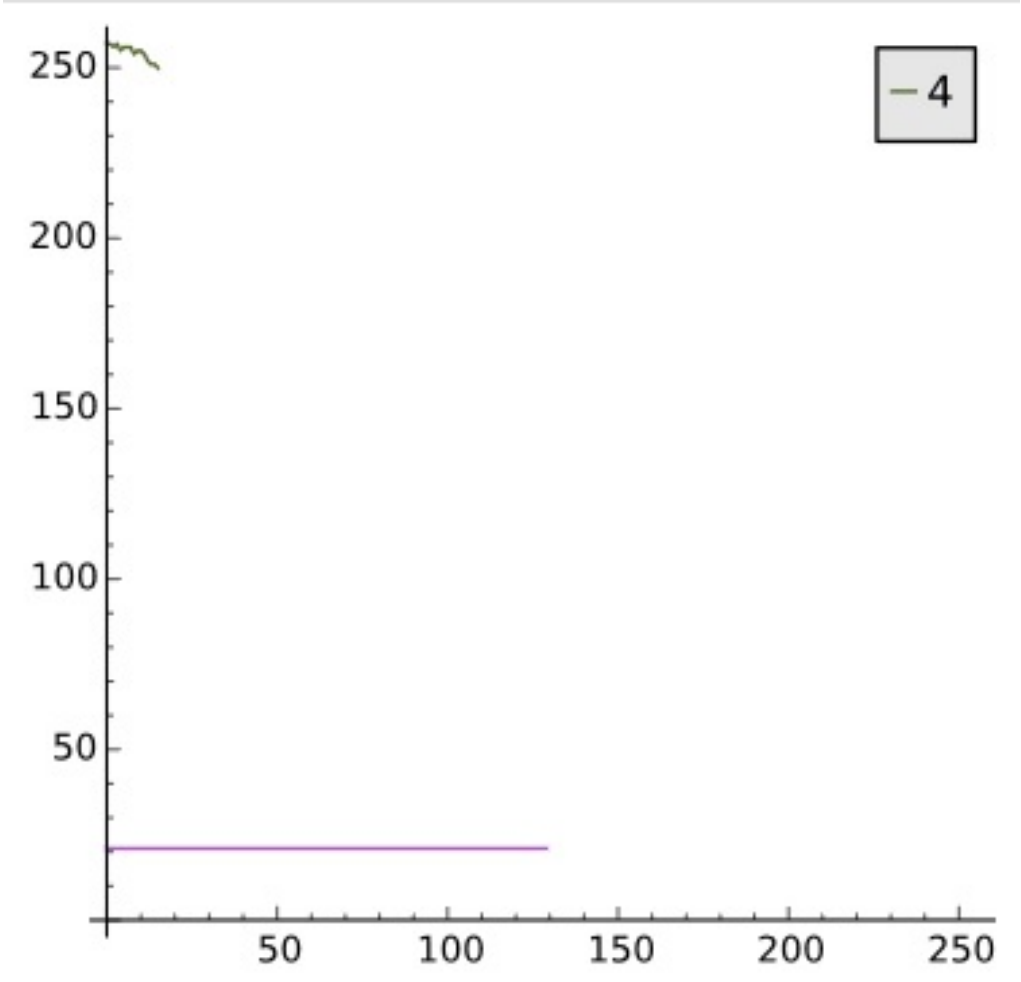

Figure: Precision over the output

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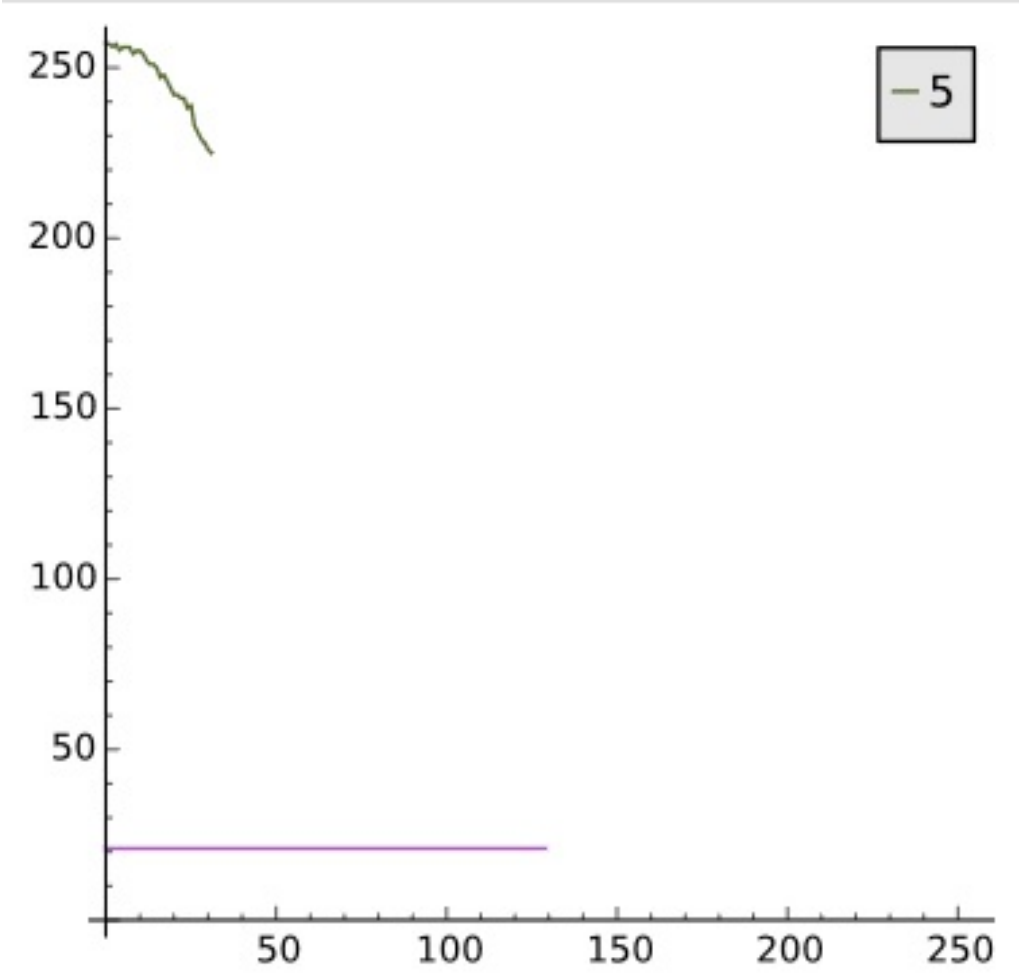

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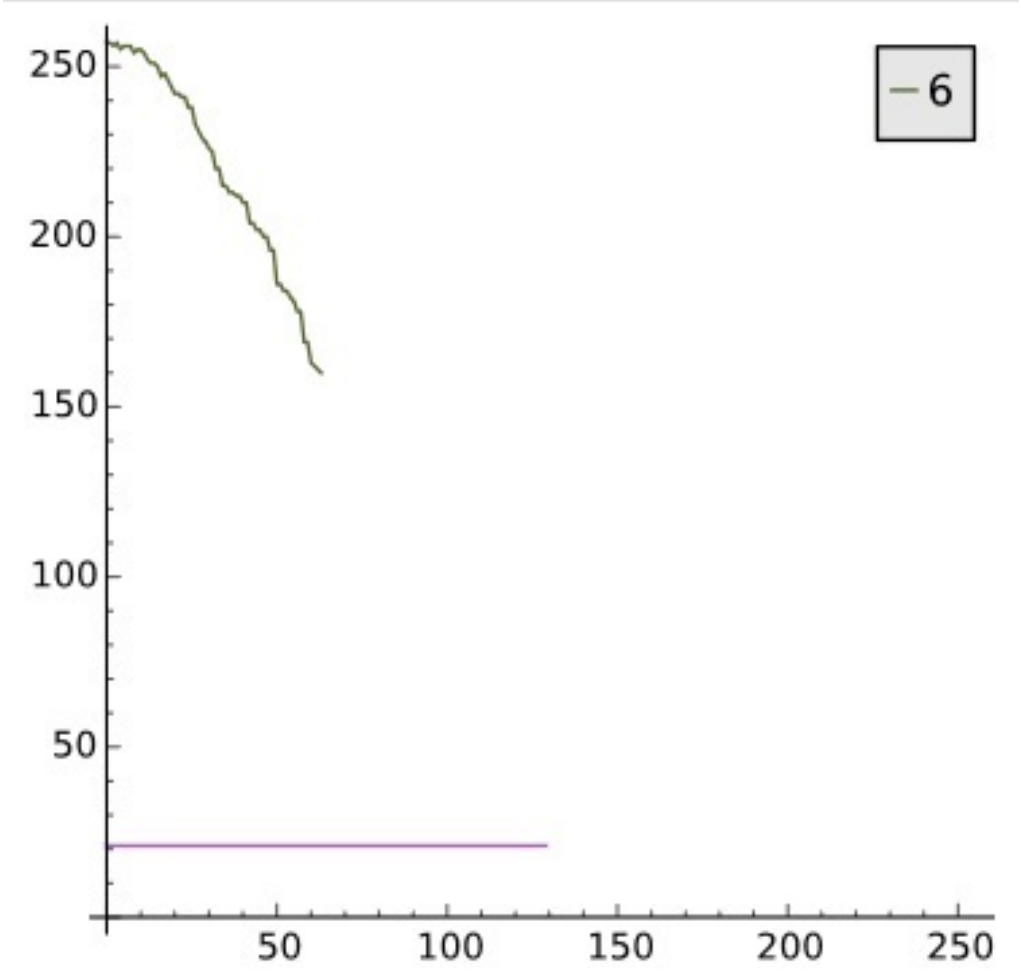

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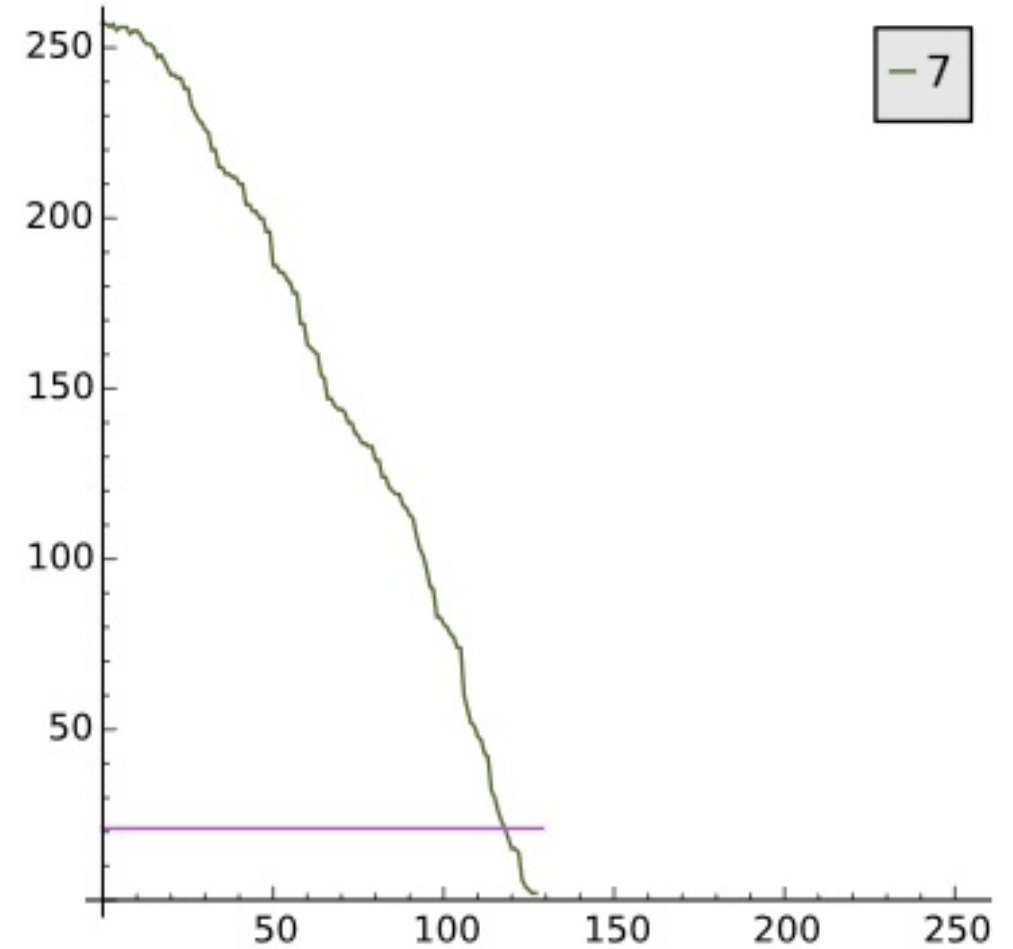

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Different way of representing the p-adics

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- In the previous computation, we start with some given approximations of g, h, u_0 and try **to follow** the algorithm for the exact counterparts of g, h, u_0.

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- Another way is then to modify the current g, h, u_0 **at each step**, in a consistent way, so as to keep on getting better approximate solutions.

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- Another way is then to modify the current g, h, u_0 **at each step**, in a consistent way, so as to keep on getting better approximate solutions.
- A third way here will be to work entirely in $\mathbb{Z}/p^\kappa\mathbb{Z}$.

Table of contents

New framework

In this new computation, we consider h as given, and not varying for the lemma.

Lemma

Let $Y : g \mapsto y$ *such that* $y(0) = 0$ *and* $y' = gh(y)$. *Then,*

$$Y'(g) \cdot (\delta g) = h(y) \int \delta g.$$

Consequence

Small modification of g corresponds to small modification of y (and conversely).

In concrete terms

A new take on the iteration

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- $u_0' = g_0 h(u_0) \mod (p^k, t^1)$
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In the end

- $u'_l = gh(u_l) \mod (p^k, t^{2^l})$

- $g_l = g \mod p^k$

Final take on the Newton scheme

As a consequence, we can prove that it is harmless to work in $\mathbb{Z}/p^k\mathbb{Z}$ for our computation.

Proposition

We can obtain the solution $\Phi(g,h) \mod (p, t^{n+1})$ *knowing* $g, h \mod (p^{\lfloor \log_p n \rfloor + 1}, t^{n+1})$ *and applying the following iteration:*

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modulo $p^{\lfloor \log_p n \rfloor + 1}$ *and growing order of truncation.*

In other words

We achieve **optimal loss** in precision: **linear** in the number of iteration for the Newton method.

Timings

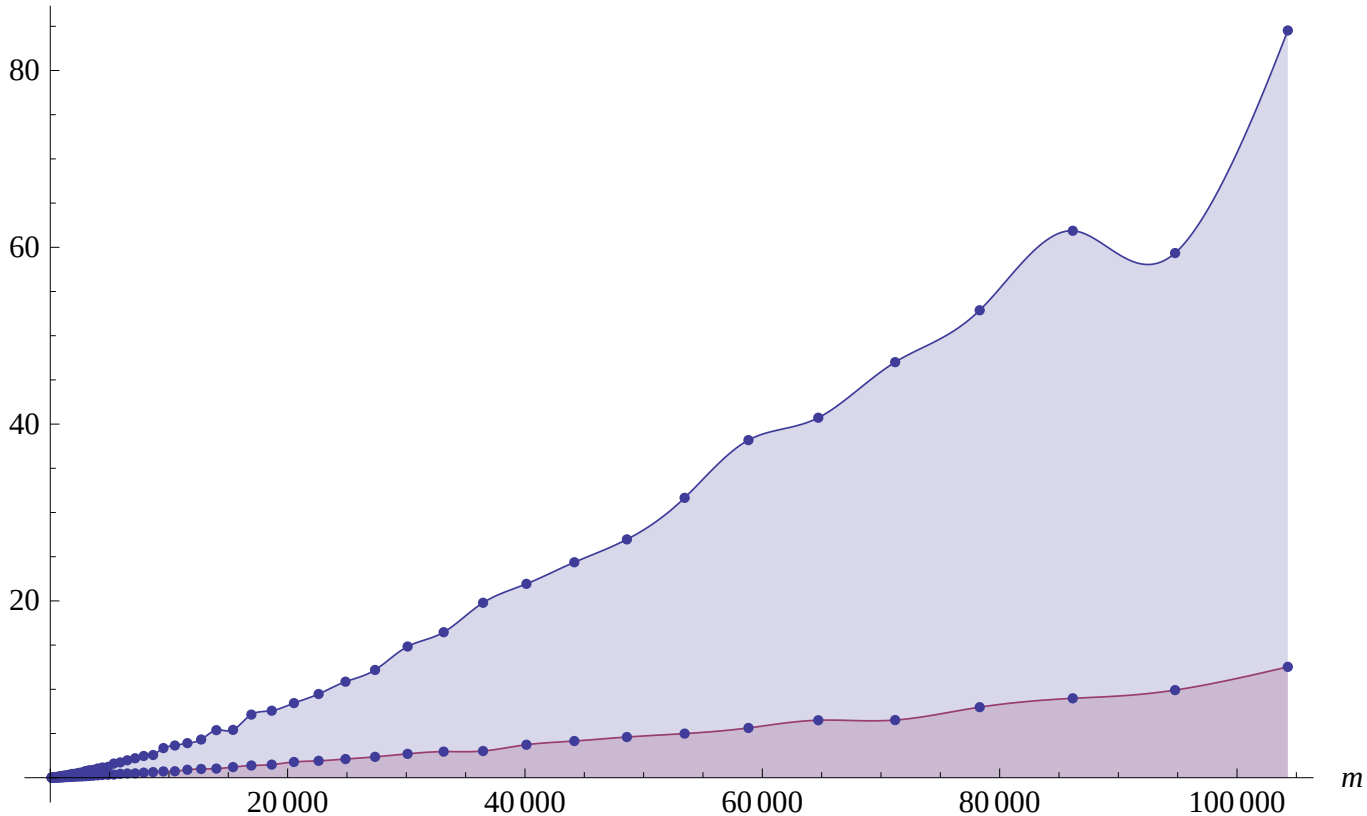

Figure: Timings in seconds, measured on a laptop, of our Algorithm run at precision λ_{old} (upper curve) and λ_{new} (lower curve) in order to compute an approximation modulo $(5, t^{4m+1})$ of some given m-isogenies.

Speedup

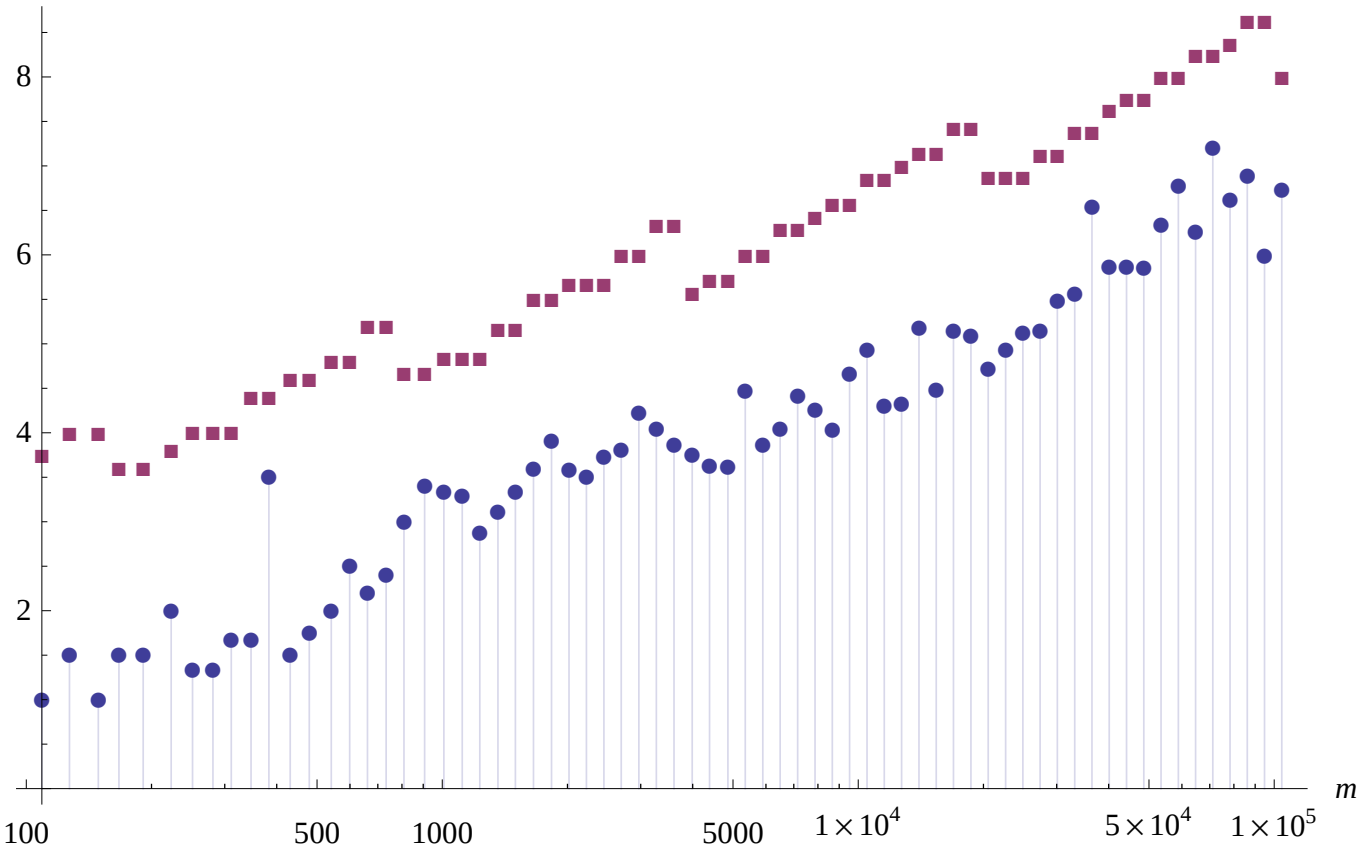

Figure: Practical speedup obtained with the new precision analysis compared with the theoretical improvement (m-axis in logarithmic scale). (■) is the ratio on precisions, (•) is the actual speedup.

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- Can attain **optimal loss** in precision for differential equations with separation of variables.
- Future works: higher order and $p = 2$.

References

Initial article

- Xavier Caruso, David Roe and Tristan Vaccon Tracking p-adic precision, ANTS XI, 2014.

Linear Algebra

- Xavier Caruso, David Roe and Tristan Vaccon p-adic stability in linear algebra, ISSAC 2015.

Differential equations, this current work

- Pierre Lairez and Tristan Vaccon On p-adic differential equations with separation of variables, ISSAC 2016.

Thank you for your attention

Thanks