A Constructive Algebraic Analysis Approach to the Equivalence of Multidimensional Linear Systems

Thomas Cluzeau
University of Limoges ; CNRS ; XLIM UMR 7252
123 avenue Albert Thomas, 87060 Limoges cedex, France
Email: thomas.cluzeau@unilim.fr

***Abstract*—There exist several models for writing the equations of a multidimensional (nD) linear system and equivalence transformations can be used to pass from one representation to another. Within the constructive algebraic analysis approach to nD linear systems theory, this equivalence issue is studied by means of isomorphisms of finitely presented modules. The present paper illustrates this general algebraic analysis approach by focussing on the equivalence problem for two frequently used 2D models, namely the generalized Fornasini-Marchesini models and the Roesser models.**

I. Introduction

Multidimensional (nD) systems are nowadays used to model phenomena appearing in image or signal processing, iterative learning control systems, distributed networks, ... Linear nD systems are defined by linear functional equations whose dependent variables are continuous or discrete functions of n independent variables. In the present paper, we focus on linear 2D systems satisfied by discrete functions of two independent variables denoted by i and j. In the literature, there exist many ways of writing the equations of such linear 2D discrete systems but two frequently used models are:

1) the model introduced by Fornasini-Marchesini [9] and generalized by Kurek [17] and Kaczorek [14], [15] which, in its more general form, can be written as

$$\begin{aligned} E\,x(i+1,j+1) &= F_1\,x(i+1,j) + F_2\,x(i,j+1) + F_3\,x(i,j) \\ &\quad + G_1\,u(i+1,j) + G_2\,u(i,j+1) + G_3\,u(i,j), \end{aligned} \tag{1}$$

$$y(i,j) = C\,x(i,j) + D\,u(i,j), \tag{2}$$

where x is the state vector of dimension d, u is the input vector of dimension d_u, y is the output vector, and E, F_i, G_i (for $i = 1,2,3$), C, D are matrices of appropriate dimensions with constant entries in a field k (e.g., $k = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$).

2) the Roesser model [23] which can be written as

$$E' \begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u'(i,j), \tag{3}$$

$$y'(i,j) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + D'\,u'(i,j), \tag{4}$$

where x^h (resp. x^v) is the horizontal (resp. vertical) state vector of dimension d_h (resp. d_v), u' is the input vector of dimension $d_{u'}$, y' is the output vector, and E', A_{ij}, B_i, C_i (for $i,j = 1,2$), D' are matrices of appropriate dimensions with constant entries in a field k (e.g., $k = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$).

When the square matrix E (resp. E') is non-singular, the model (1) (resp. (3)) is said to be *explicit* and we can always assume w.l.o.g. that $E = \mathrm{I}_d$ (resp. $E' = \mathrm{I}_{d_h+d_v}$), where I_n denotes the identity matrix of size n. When the matrix E (resp. E') is singular, the model (1) (resp. (3)) is said to be *implicit*. In what follows, for simplicity, we will not take into account the output equations (2) and (4).

Depending on the structural properties that we want to study, it can be more suitable to consider a model written as (1) or (3) so that people have been interested in the possibility to rewrite a model of the form (1) under an *equivalent* form (3) and vice versa: see, e.g., [13], [18], [11], [16] and references therein. Within this approach, the notion of equivalence used is crucial since it has to preserve the structural properties under consideration (for example, controllability, observability, stability, ...). Taking this into account, all the transformations can not be allowed and Rosenbrock [24] and Fuhrmann [10] have developed a notion of *strict equivalence* for 1D systems which has been generalized to 2D systems in [12], [21] (see also [1], [27] and references therein).

The purpose of the present paper is to use the constructive *algebraic analysis* (or *behavioral*) approach to nD systems theory [19], [20], [26], [2], [22] to study the equivalence problem between models (1) and (3) by means of isomorphisms of modules. Indeed, within this algebraic analysis framework, the equivalence of models is naturally related to the isomorphism of the associated modules. The general problem of deciding whether two systems/modules are equivalent/isomorphic is an important and difficult problem in systems/module theory. However, the constructive approach developed in [4], [6], [7] and implemented in the computer algebra systems Maple [3], [5] and Mathematica [8] can be used to exhibit isomorphisms between models (1) and (3).

The paper is organized as follows. In Section II, we briefly recall the algebraic analysis approach to nD linear systems theory and we define the relevant notion of equivalence in this context. We then recall the constructive characterization of homomorphisms and isomorphisms of finitely generated modules in Section III. Within this framework, the problems of finding a model (3) (resp. (1)) equivalent to a given model (1) (resp. (3)) is then considered in Section IV (resp. V).

This work was supported by the ANR-13-BS03-0005 (MSDOS)

978-1-4799-8739-9/15/$31.00

II. An algebraic analysis approach to the equivalence problem

The *algebraic analysis (or behavioral) approach to linear nD systems theory* can be viewed as a unified mathematical framework to study determined/overdetermined/underdetermined linear nD systems appearing in control theory, mathematical physics, engineering sciences, ... See [19], [20], [26], [2], [22] and the references therein. Indeed, the methods used and the resulting algorithms do not depend on the particular functional equations, e.g., ordinary differential (OD) or partial differential (PD) equations, difference equations, OD time-delay equations, ..., that we deal with.

A linear nD system can be written as $R\,\eta = 0$, where R is a $q \times p$ matrix with entries in a (noncommutative) ring D of functional operators (e.g., OD or PD operators, shift operators, difference operators, OD time-delay operators) and η is a vector of unknown functions. The matrix $R \in D^{q\times p}$ induces the left D-homomorphism (i.e., left D-linear map)

$$.R\colon D^{1\times q} \longrightarrow D^{1\times p}, \quad \mu \longmapsto \mu\, R,$$

and we consider the left D-module

$$M := D^{1\times p}/(D^{1\times q}\, R),$$

which is defined as the *cokernel* of the map $.R$. It is the left D-module *finitely presented* by R and the following *exact sequence* (see [25]) holds:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

where $\pi \in \hom_D(D^{1\times p}, M)$ denotes the D-linear map such that $\pi(\lambda)$ is the *residue class* of $\lambda \in D^{1\times p}$ in M. The left D-module M can be defined by generators and relations as follows. If $\{f_j\}_{j=1,\ldots,p}$ is the *standard basis* of $D^{1\times p}$ (i.e., $f_j \in D^{1\times p}$ is the vector formed by 1 at the jth position and 0 elsewhere), then one can easily prove that $\{y_j := \pi(f_j)\}_{j=1,\ldots,p}$ is a *family of generators* of M which satisfies the *left D-linear relations* $\sum_{j=1}^{p} R_{ij}\, y_j = 0$, for $i = 1, \ldots, q$. For more details, see [2], [4], [22].

If $\mathcal{F}$ is a left D-module, then we can consider the *linear system* or *behavior* $\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$. The algebraic analysis approach to linear nD systems theory relies on the fact that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, finitely presented by the matrix R. This is based on Malgrange's isomorphism [19]:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \tag{5}$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group of left D-homomorphisms f (i.e., left D-linear maps) from M to $\mathcal{F}$, namely: for all $\delta_1, \delta_2 \in D$, for all $m_1, m_2 \in M$, $f(\delta_1\, m_1 + \delta_2\, m_2) = \delta_1\, f(m_1) + \delta_2\, f(m_2)$.

Module properties of M and $\mathcal{F}$ are then related to system properties of $\ker_{\mathcal{F}}(R.)$ via the isomorphism (5). To effectively check module properties of M, we use constructive homological algebra [25] for (noncommutative) rings D of functional operators (e.g., Ore algebras D) admitting Gröbner bases for admissible monomial orders (see [2], [4], [22]).

Example 1. *Let us consider an explicit Roesser model (3) with $E = I_{d_h+d_v}$. Let $D = k\langle \sigma_i, \sigma_j\rangle$ denote the commutative ring of partial shift operators with constant coefficients in k: an element $\delta \in D$ can thus be written as $\delta = \sum_{m,l} \gamma_{ml}\, \sigma_i^m\, \sigma_j^l$, where $\gamma_{ml} \in k$, the sum is finite, and δ acts on a function $f(i,j)$ as follows: $\delta\, f(i,j) = \sum_{m,l} \gamma_{ml}\, f(i+m, j+l)$. If we define $q = d_h + d_v$ and $p = d_h + d_v + d_{u'}$, then the Roesser model (3) can be written as $R\,\eta = 0$ with*

$$R = \begin{pmatrix} \sigma_i\, I_{d_h} - A_{11} & -A_{12} & -B_1 \\ -A_{21} & \sigma_j\, I_{d_v} - A_{22} & -B_2 \end{pmatrix} \in D^{q\times p},$$

and $\eta = \begin{pmatrix} x^h & x^v & u' \end{pmatrix}^T$. We then consider the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ finitely presented by the matrix R and the model (3) is then studied by means of M.

Definition 1 ([25])**.** *Two left D-modules M and M' are* isomorphic*, which is denoted by $M \cong M'$, if there exists an homomorphism $\phi \in \hom_D(M, M')$ which is an isomorphism, i.e., $\phi : M \to M'$ is both injective and surjective.*

The present paper is concerned with the *equivalence* of models (1) and (3). Within the algebraic analysis approach to linear nD systems theory, the term *equivalence* is naturally defined as follows:

Definition 2. *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ be two matrices and $M = D^{1\times p}/(D^{1\times q}\, R)$, $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ the associated left D-modules. If $\mathcal{F}$ is a left D-module, then the linear systems or behaviors $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ are* equivalent *if $M \cong M'$.*

Note that this notion of equivalence is more general that the notion of *strict equivalence* used in [24], [10], [12], [21], [1], [27] and has the advantage of being defined for any type of nD system. See also Corollary 1 below.

III. Isomorphisms of finitely presented left D-modules

We first recall the characterization of left D-homomorphisms of finitely presented left D-modules.

Lemma 1 ([25], [4])**.** *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ be two matrices and consider the finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$.*

1) *The existence of $f \in \hom_D(M, M')$ is equivalent to the existence of $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying*

$$R\,P = Q\,R'.$$

Then, the commutative exact diagram *(see [25])*

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle .Q} & & \downarrow{\scriptstyle .P} & & \downarrow{\scriptstyle f} & & \\ D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \end{array}$$

holds, where $f \in \hom_D(M, M')$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P).$$

2) *Let $R'_2 \in D^{q'_2 \times q'}$ be such that $\ker_D(.R') = D^{1\times q'_2}\, R'_2$ and let $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ be two matrices satisfying $R\, P = Q\, R'$. Then, the matrices*

$$\overline{P} := P + Z\, R', \quad \overline{Q} := Q + R\, Z + Z_2\, R'_2,$$

where $Z \in D^{p\times q'}$ and $Z_2 \in D^{q\times q'_2}$ are two arbitrary matrices, satisfy the relation $R\,\overline{P} = \overline{Q}\, R'$ and:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P) = \pi'(\lambda\, \overline{P}).$$

Algorithms for computing homomorphisms of finitely presented left D-modules are given in [4] and have been implemented both in the Maple package OREMORPHISMS [5] based on OREMODULES [3] and in the Mathematica package OREALGEBRAICANALYSIS [8]. Moreover, when D is commutative, $\hom_D(M, M')$ inherits a D-module structure and, using the Kronecker product to solve the equation $R\, P = Q\, R'$ for P and Q and Gröbner bases computations, we have algorithms (and implementations) for computing generators and relations of $\hom_D(M, M')$. This means that when D is commutative, which is the case for the systems considered in the present paper (see Example 1), we can compute a representation of all D-homomorphisms between finitely generated D-modules. For more details, see [4].

Let $f : M \longrightarrow M'$ be a left D-homomorphism of left D-modules. Then, we can define the kernel, image, coimage and cokernel of f as the following left D-modules:

$$\begin{cases} \ker f := \{m \in M \mid f(m) = 0\}, \\ \operatorname{im} f := \{m' \in M' \mid \exists\, m \in M :\ m' = f(m)\}, \\ \operatorname{coim} f := M/\ker f, \\ \operatorname{coker} f := M'/\operatorname{im} f. \end{cases}$$

Moreover we have that f is *a left D-isomorphism of left D-modules*, i.e., $M \cong M'$ if and only if $\ker f = 0$ (f is injective) and $\operatorname{coker} f = 0$ (f is surjective). In order to explicitly decide when a given left D-homomorphism is an isomorphism, we recall the characterizations of $\ker f$ and $\operatorname{coker} f$ when M and M' are two finitely presented left D-modules.

Lemma 2 ([4])**.** *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ be two matrices and consider the finitely presented left D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$. Let $f \in \hom_D(M, M')$ be defined by $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying $R\, P = Q\, R'$. If we define the following matrices:*

1) *$S \in D^{r\times p}$ and $T \in D^{r\times q'}$ such that*

$$\ker_D\left(.\begin{pmatrix} P \\ R' \end{pmatrix}\right) = D^{1\times r}\,(S \quad -T),$$

2) *$L \in D^{q\times r}$ such that $R = L\, S$,*
3) *$S_2 \in D^{r_2\times r}$ such that $\ker_D(.S) = D^{1\times r_2}\, S_2$,*

then we have:

$$\ker f \cong D^{1\times r}/\left(D^{1\times(q+r_2)}\begin{pmatrix} L \\ S_2 \end{pmatrix}\right),$$

$$\operatorname{coker} f = D^{1\times p'}/\left(D^{1\times(p+q')}\begin{pmatrix} P \\ R' \end{pmatrix}\right).$$

Corollary 1 ([4])**.** *With the notation of Lemma 2, the morphism $f \in \hom_D(M, M')$ is an isomorphism, i.e., $M \cong M'$, if and only if the matrices $(L^T \quad S_2^T)^T$ (injectivity) and $(P^T \quad R'^T)^T$ (surjectivity) admit a left inverse.*

The two conditions of Corollary 1 can be proved to be equivalent to the conditions defining the strict equivalence of the systems $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ in the particular case when the strict equivalence is defined, e.g., full row rank and dimensions conditions on the matrices R and R' (see [1] and references therein). This result is more general in the sense that it applies to all types of linear systems.

Algorithms for deciding whether a given homomorphism of finitely presented left D-modules is injective, surjective or defines an isomorphism (and if so, compute its inverse) are implemented in the Maple package OREMORPHISMS [5].

IV. FROM MODEL (1) TO MODEL (3)

A. Explicit case

Starting from the general explicit model (1) with $E = \mathrm{I}_d$, if we define the vectors

$$x^h(i,j) := x(i,j+1) - F_1\, x(i,j) - G_1\, u(i,j),$$
$$x^v(i,j) := (x(i,j)^T \quad u(i,j)^T)^T,$$
$$u'(i,j) := u(i,j+1),$$

of respective dimensions d, $d + d_u$, and d_u, then we get:

$$\begin{aligned} x^h(i+1,j) = {} & F_2\, x^h(i,j) + (F_2\, F_1 + F_3)\, x(i,j) \\ & + (F_2\, G_1 + G_3)\, u(i,j) + G_2\, u'(i,j) \end{aligned}$$

so that (1) with $E = \mathrm{I}_d$ can be written in the form (3) with

$$\begin{cases} E' = \mathrm{I}_{2\,d+d_u},\ A_{11} = F_2,\ A_{12} = (F_2\, F_1 + F_3 \quad F_2\, G_1 + G_3), \\ A_{21} = (\mathrm{I}_d \quad 0)^T,\ A_{22} = \begin{pmatrix} F_1 & G_1 \\ 0 & 0 \end{pmatrix},\ B_1 = G_2,\ B_2 = (0 \quad \mathrm{I}_{d_u})^T. \end{cases} \tag{6}$$

Theorem 1. *The explicit model (1) with $E = I_d$ is equivalent in the sense of Definition 2 to the explicit model (3) defined by (6).*

Proof. Let $D = k\langle \sigma_i, \sigma_j \rangle$. The model (1) with $E = \mathrm{I}_d$ can be written as $F\, \eta = 0$ with $\eta = \left(x^T \quad u^T\right)^T$, and $F \in D^{d\times(d+d_u)}$ is given by:

$$F = (\mathrm{I}_d\, \sigma_i\, \sigma_j - F_1\, \sigma_i - F_2\, \sigma_j - F_3 \quad -G_1\, \sigma_i - G_2\, \sigma_j - G_3).$$

Similarly, the explicit model (3) defined by (6) can be written as $R\, \zeta = 0$ with $\zeta = \left((x^h)^T \quad (x^v)^T \quad u'^T\right)^T$, and $R \in D^{(2\,d+d_u)\times(2\,d+2\,d_u)}$ is given by:

$$R = \begin{pmatrix} \mathrm{I}_d\, \sigma_i - F_2 & -(F_2\, F_1 + F_3) & -(F_2\, G_1 + G_3) & -G_2 \\ -\mathrm{I}_d & \mathrm{I}_d\, \sigma_j - F_1 & -G_1 & 0 \\ 0 & 0 & \mathrm{I}_{d_u}\, \sigma_j & -\mathrm{I}_{d_u} \end{pmatrix}.$$

We shall prove that $M_F \cong M_R$ where:

$$M_F := D^{1\times(d+d_u)}/(D^{1\times d}\, F),$$

$$M_R := D^{1\times(2\,d+2\,d_u)}/(D^{1\times(2\,d+d_u)}\,R),$$

and to achieve this we are going to exhibit an isomorphism between M_F and M_R. A morphism $f \in \hom_D(M_F, M_R)$ is given by $f(\pi_F(\lambda)) = \pi_R(\lambda\,P)$, for all $\lambda \in D^{1\times(d+d_u)}$, where the matrix $P \in D^{(d+d_u)\times(2\,d+2\,d_u)}$ satisfies that there exists $Q \in D^{d\times(2\,d+d_u)}$ such that $F\,P = Q\,R$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times d} & \xrightarrow{.F} & D^{1\times(d+d_u)} & \xrightarrow{\pi_F} & M_F & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle .Q} & & \downarrow{\scriptstyle .P} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & D^{1\times(2\,d+d_u)} & \xrightarrow{.R} & D^{1\times(2\,d+2\,d_u)} & \xrightarrow{\pi_R} & M_R & \longrightarrow & 0.
\end{array}$$

Using OREMORPHISMS [5], we can effectively compute a representation of the D-module $\hom_D(M_F, M_R)$ given by generators and relations and we find that

$$P = \begin{pmatrix} 0 & \mathrm{I}_d & 0 & 0 \\ 0 & 0 & \mathrm{I}_{d_u} & 0 \end{pmatrix}, \tag{7}$$

defines a particular morphism, i.e., we have $F\,P = Q\,R$ where $Q = (\mathrm{I}_d \quad \mathrm{I}_d\,\sigma_i - F_2 \quad G_2)$. We now use Corollary 1 to check that P defines an isomorphism. Using OREMODULES [3], we compute the matrices defined in Lemma 2. We first find that $S = F$ which is a full row rank matrix, i.e., $\ker_D(.S) = 0$, so that $L = \mathrm{I}_d$ and the morphism defined by P is clearly injective. Finally we can check that $(P^T \quad R^T)^T$ admits the following left inverse:

$$\begin{pmatrix} \mathrm{I}_d\,\sigma_j - F_1 & -G_1 & 0 & -\mathrm{I}_d & 0 \\ \mathrm{I}_d & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I}_{d_u} & 0 & 0 & 0 \\ 0 & \mathrm{I}_{d_u}\,\sigma_j & 0 & 0 & -\mathrm{I}_{d_u} \end{pmatrix}.$$

□

B. Implicit case

If we now consider an implicit model (1) where E is a singular matrix, then proceeding as in the previous subsection, we define the vectors

$$\begin{aligned} x^h(i,j) &:= E\,x(i,j+1) - F_1\,x(i,j) - G_1\,u(i,j), \\ x^v(i,j) &= (x(i,j)^T \quad u(i,j)^T)^T, \\ u'(i,j) &= u(i,j+1), \end{aligned}$$

of respective dimensions d, $d + d_u$, and d_u. Then we get:

$$x^h(i+1,j) - F_2\,x(i,j+1) = F_3\,x(i,j) + G_3\,u(i,j) + G_2\,u'(i,j),$$

so that the implicit model (1) can be written in the form of the implicit model (3) with

$$\begin{cases} E' = \begin{pmatrix} \mathrm{I}_d & -F_2 & 0 \\ 0 & E & 0 \\ 0 & 0 & \mathrm{I}_{d_u} \end{pmatrix},\ A_{11} = 0,\ A_{12} = (F_3 \quad G_3),\ B_1 = G_2, \\ A_{21} = (\mathrm{I}_d \quad 0)^T,\ A_{22} = \begin{pmatrix} F_1 & G_1 \\ 0 & 0 \end{pmatrix},\ B_2 = (0 \quad \mathrm{I}_{d_u})^T. \end{cases} \tag{8}$$

Theorem 2. *The implicit model (1) is equivalent in the sense of Definition 2 to the implicit model (3) defined by (8).*

Proof. We proceed as in the proof of Theorem 1. The matrices F and R are now given by:

$$F = (E\,\sigma_i\,\sigma_j - F_1\,\sigma_i - F_2\,\sigma_j - F_3 \quad -G_1\,\sigma_i - G_2\,\sigma_j - G_3),$$

$$R = \begin{pmatrix} \sigma_i\,\mathrm{I}_d & -F_2\,\sigma_j - F_3 & -G_3 & -G_2 \\ -\mathrm{I}_d & E\,\sigma_j - F_1 & -G_1 & 0 \\ 0 & 0 & \sigma_j\,\mathrm{I}_{d_u} & -\mathrm{I}_{d_u} \end{pmatrix},$$

and the matrix P defined by (7) still defines a homomorphism between the D-modules M_F and M_R respectively finitely presented by F and R. Then, similar arguments as in the proof of Theorem 1 hold and prove that P defines an isomorphism. □

V. FROM MODEL (3) TO MODEL (1)

A. Explicit case

We now start from an explicit model (3) with $E' = \mathrm{I}_{d_h+d_v}$ and try to construct an equivalent model (1) in the sense of Definition 2. In many papers (see, for example [11], [1]), the authors explain that such a model (3) can be written into the form of an explicit model (1) by setting $E = E' = \mathrm{I}_{d_h+d_v}$, $x = ((x^h)^T \quad (x^v)^T)^T$ of dimension $d_h + d_v$, $u = u'$ and

$$\begin{cases} F_1 = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix},\ F_2 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix},\ F_3 = 0, \\ G_1 = (0 \quad B_2^T)^T,\ G_2 = (B_1^T \quad 0)^T,\ G_3 = 0. \end{cases}$$

However, as it is already noticed in [1], such a way of rewriting a model (3) into a model (1) is not a strict equivalence and it is clearly not an equivalence in the sense of Definition 2. Indeed having a D-isomorphism of modules will imply that one can recover the equations of the original model (3) starting from the equivalent model (1) by working over D, i.e., without using the inverse shifts σ_i^{-1} and σ_j^{-1}. Note also that in Section IV, starting from a model (1), we have found an equivalent model (3) with a state vector of higher dimension (namely, $2\,d + d_u$ instead of d), so that, started from a model (3), we may be interested in finding an equivalent model (1) with a state vector of smaller dimension.

Two cases have to be distinguished:

1) Case 1: A_{12} or A_{21} admits a left inverse: Let us assume w.l.o.g. that the matrix $A_{12} \in k^{d_h\times d_v}$ admits a left inverse, i.e., there exists $A_{12}^{\mathrm{linv}} \in k^{d_v\times d_h}$ such that $A_{12}^{\mathrm{linv}}\,A_{12} = \mathrm{I}_{d_v}$. Then we have:

$$x^v(i,j) = A_{12}^{\mathrm{linv}}\,x^h(i+1,j) - A_{12}^{\mathrm{linv}}\,A_{11}\,x^h(i,j) - A_{12}^{\mathrm{linv}}\,B_1\,u'(i,j),$$

so that a straightforward calculation leads to:

$$\begin{aligned} x^h(i+1,j+1) = {} & A_{12}\,A_{22}\,A_{12}^{\mathrm{linv}}\,x^h(i+1,j) + A_{11}\,x^h(i,j+1) \\ & + (A_{12}\,A_{21} - A_{12}\,A_{22}\,A_{12}^{\mathrm{linv}}\,A_{11})\,x^h(i,j) \\ & + B_1\,u'(i,j+1) \\ & + (A_{12}\,B_2 - A_{12}\,A_{22}\,A_{12}^{\mathrm{linv}}\,B_1)\,u'(i,j), \end{aligned}$$

and setting $x = x^h$, $u = u'$, we get the explicit model (1) defined by:

$$\begin{cases} E = \mathrm{I}_{d_h},\ F_1 = A_{12}\,A_{22}\,A_{12}^{\mathrm{linv}},\ F_3 = A_{12}\,(A_{21} - A_{22}\,A_{12}^{\mathrm{linv}}\,A_{11}), \\ F_2 = A_{11},\ G_1 = 0,\ G_2 = B_1,\ G_3 = A_{12}\,(B_2 - A_{22}\,A_{12}^{\mathrm{linv}}\,B_1). \end{cases} \tag{9}$$

Theorem 3. *If the matrix $A_{12} \in k^{d_h \times d_v}$ admits a left inverse A_{12}^{linv}, then the explicit model (3) with $E' = I_{d_h+d_v}$ is equivalent in the sense of Definition 2 to the explicit model (1) defined by (9).*

Proof. We consider the matrices $R \in D^{(d_h+d_v)\times(d_h+d_v+d_{u'})}$ and $F \in D^{d_h \times (d_h+d_{u'})}$ given by:

$$R = \begin{pmatrix} \mathrm{I}_{d_h}\,\sigma_i - A_{11} & -A_{12} & -B_1 \\ -A_{21} & \mathrm{I}_{d_v}\,\sigma_j - A_{22} & -B_2 \end{pmatrix},$$

$$F = (\mathrm{I}_{d_h}\,\sigma_i\,\sigma_j - F_1\,\sigma_i - F_2\,\sigma_j - F_3 \quad -G_1\,\sigma_i - G_2\,\sigma_j - G_3),$$

where the F_i's and G_i's, $i = 1, 2, 3$ are defined by (9).

We shall prove that $M_R \cong M_F$ where:

$$M_R := D^{1\times(d_h+d_v+d_{u'})}/(D^{1\times(d_h+d_v)}\,R),$$

and

$$M_F := D^{1\times(d_h+d_{u'})}/(D^{1\times d_h}\,F).$$

To achieve this we are going to exhibit an isomorphism between M_R and M_F. A morphism $f \in \hom_D(M_R, M_F)$ is given by $f(\pi_R(\lambda)) = \pi_F(\lambda\,P)$, for all $\lambda \in D^{1\times(d_h+d_v+d_{u'})}$, where the matrix $P \in D^{(d_h+d_v+d_{u'})\times(d_h+d_{u'})}$ satisfies that there exists $Q \in D^{(d_h+d_v)\times d_h}$ such that $R\,P = Q\,F$. Using OreMorphisms [5], we can effectively compute a representation of the D-module $\hom_D(M_R, M_F)$ given by generators and relations and we find that

$$P = \begin{pmatrix} \mathrm{I}_{d_h} & 0 \\ A_{12}^{\text{linv}}\,\sigma_i - A_{12}^{\text{linv}}\,A_{11} & -A_{12}^{\text{linv}}\,B_1 \\ 0 & \mathrm{I}_{d_{u'}} \end{pmatrix},$$

defines a particular morphism, i.e., we have $R\,P = Q\,F$ with $Q = \begin{pmatrix} 0^T & A_{12}^{\text{linv}\,T} \end{pmatrix}^T$. We now use Corollary 1 to check that P defines an isomorphism. Using OreModules [3], we compute the matrices defined in Lemma 2. We first find that $S = R$ which is a full row rank matrix, i.e., $\ker_D(.S) = 0$, so that $L = \mathrm{I}_{d_h+d_v}$ and the morphism defined by P is clearly injective. Finally, we can check that $(P^T \quad F^T)^T$ admits the following left inverse:

$$\begin{pmatrix} \mathrm{I}_{d_h} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I}_{d_{u'}} & 0 \end{pmatrix}.$$

□

Note that if A_{21} admits a left inverse, then the same technique can be applied to get an equivalent model (1) with a state variable of dimension d_v.

2) Case 2: A_{12} and A_{21} do not admit left inverses: If A_{12} does not admit a left inverse, we can compute two unimodular matrices $U \in \mathrm{GL}_{d_h}(k)$ and $V \in \mathrm{GL}_{d_v}(k)$ such that

$$S := U\,A_{12}\,V = \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix}.$$

This can be performed by computing the *Smith* form of A_{12}. Consequently, setting $x_1^h := U\,x^h$ and $x_1^v := V^{-1}\,x^v$, the model (3) is equivalent to another model of the same form (3) for the new state vectors x_1^h and x_1^v where the matrix A_{12} is replaced by its Smith form S. To simplify, we still denote by A_{ij} and B_i, $i = 1, 2$ the matrices defining this model. Note that these two models are clearly equivalent in the sense of Definition 2. Then partitioning the new vertical state vector x_1^v into two sub-vectors s_1 and s_2 respectively of size r and $d_v - r$, we get the three equations

$$x^h(i+1,j) = A_{11}\,x^h(i,j) + \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} s_1(i,j) + B_1\,u'(i,j), \quad (10)$$

$$s_1(i,j+1) = A_{21}^1\,x^h(i,j) + A_{22}^{11}\,s_1(i,j) + A_{22}^{12}\,s_2(i,j) + B_2^1\,u'(i,j), \quad (11)$$

$$s_2(i,j+1) = A_{21}^2\,x^h(i,j) + A_{22}^{21}\,s_1(i,j) + A_{22}^{22}\,s_2(i,j) + B_2^2\,u'(i,j), \quad (12)$$

with the following appropriate partitions of A_{21}, A_{22} and B_2:

$$A_{21} = \begin{pmatrix} A_{21}^1 \\ A_{21}^2 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} A_{22}^{11} & A_{22}^{12} \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} B_2^1 \\ B_2^2 \end{pmatrix}.$$

From the first equation, we can express:

$$s_1(i,j) = \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} \Big(x^h(i+1,j) - A_{11}\,x^h(i,j) - B_1\,u'(i,j) \Big).$$

Now applying σ_j to (10), replacing $s_1(i,j+1)$ using (11) and plugging the latter formula for $s_1(i,j)$, we get an expression of $x^h(i+1,j+1)$ in terms of x^h, s_2 and u'. Finally, one can eliminate s_1 from (12) which yields $s_2(i,j+1)$ in terms of x^h, s_2 and u'. Setting $x = (x^{h^T} \quad s_2^T)^T$, $u = u'$, the explicit model (3) can be written as the *implicit* model (1) defined by:

$$\begin{cases} E = \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} A_{22}^{11} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} & 0 \\ A_{22}^{22} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} & 0 \end{pmatrix}, \\ F_3 = \begin{pmatrix} \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} A_{21}^1 - \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} A_{22}^{11} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} A_{11} & \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} A_{22}^{12} \\ A_{21}^2 - A_{22}^{21} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} A_{11} & A_{22}^{22} \end{pmatrix}, \\ F_2 = \begin{pmatrix} A_{11} & 0 \\ 0 & -\mathrm{I}_{d_v-r} \end{pmatrix}, \; G_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; G_2 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \\ G_3 = \begin{pmatrix} \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} B_2^1 - \begin{pmatrix} \mathrm{I}_r \\ 0 \end{pmatrix} A_{22}^{11} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} B_1 \\ B_2^2 - A_{22}^{21} \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} B_1 \end{pmatrix}. \end{cases} \quad (13)$$

Theorem 4. *If the matrix $A_{12} \in k^{d_h \times d_v}$ does not admit a left inverse, then the explicit model (3) with $E' = I_{d_h+d_v}$ is equivalent in the sense of Definition 2 to the implicit model (1) defined by (13).*

Proof. Proceeding as in the proof of Theorem 3, we can prove that the D-module M_R associated with the explicit model defined by Equations (10), (11) and (12) is isomorphic to the D-module M_F associated with the implicit model defined by (13). The isomorphism is given by the matrix

$$P = \begin{pmatrix} \mathrm{I}_{d_h} & 0 & 0 \\ \begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} (\sigma_i - A_{11}) & 0 & -\begin{pmatrix} \mathrm{I}_r & 0 \end{pmatrix} B_1 \\ 0 & \mathrm{I}_{d_v-r} & 0 \\ 0 & 0 & \mathrm{I}_{d'_u} \end{pmatrix},$$

i.e., $f(\pi_R(\lambda)) = \pi_F(\lambda\,P)$, for all $\lambda \in D^{1\times(d_h+d_v+d_{u'})}$. □

Theorems 3 and 4 imply that an *explicit* model (3) is always equivalent in the sense of Definition 2 to a model (1) but in general, it does not seem to be always possible to get an equivalent *explicit* model (1).

B. Implicit case

Starting from an implicit model of the form (3) with

$$E' = \begin{pmatrix} E'_{11} & E'_{12} \\ E'_{21} & E'_{22} \end{pmatrix},$$

one can always find an equivalent model (1) by setting:

$$\begin{cases} E = 0,\ F_1 = -\begin{pmatrix} E'_{11} & 0 \\ E'_{21} & 0 \end{pmatrix},\ F_2 = -\begin{pmatrix} 0 & E'_{12} \\ 0 & E'_{22} \end{pmatrix}, \\ F_3 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ G_1 = G_2 = 0,\ G_3 = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \end{cases}$$

See, for example, [16]. In general, finding an equivalent model of the form (1) with $E \neq 0$ and with a state vector x of dimension smaller than $d_h + d_v$ seems to be more involved.

VI. Conclusion

Within the constructive algebraic analysis approach to nD systems theory, we have studied the equivalence of models (1) and (3) in the sense of Definition 2. Our results can be summarized as follows:

1) Starting from an explicit (resp. implicit) model of the form (1), one can find an equivalent explicit (resp. implicit) model of the form (3),
2) Starting from an explicit model of the form (3) where either A_{12} or A_{21} admits a left inverse, one can find an equivalent explicit model of the form (1),
3) Starting from an explicit model of the form (3) where neither A_{12} nor A_{21} admits a left inverse, one can find an equivalent implicit model of the form (1),
4) Starting from an implicit model of the form (3), one can find an equivalent implicit model of the form (1) with the same state vector and $E = 0$.

In the future, we shall investigate the input/output properties preserved by the notion of equivalence given by Definition 2 and use it for studying stability issues for nD systems.

Acknowledgment

The author would like to thank all the members of the ANR project MSDOS for discussions about this subject, and in particular O. Bachelier and A. Quadrat.

References

[1] M. S. Boudellioua. *Strict system equivalence of 2D linear discrete state space models.* Journal of Control Science and Engineering, vol. 2012, 6 pages, 2012.

[2] F. Chyzak, A. Quadrat, and D. Robertz. *Effective algorithms for parametrizing linear control systems over Ore algebras.* Appl. Algebra Engrg. Comm. Comput., 16, 319-376, 2005.

[3] F. Chyzak, A. Quadrat, and D. Robertz. OreModules*: A symbolic package for the study of multidimensional linear systems.* Springer, Lecture Notes in Control and Inform. Sci., 352, 233-264, 2007. http://pages.saclay.inria.fr/alban.quadrat/OreModules.html

[4] T. Cluzeau, A. Quadrat. *Factoring and decomposing a class of linear functional systems.* Linear Algebra Appl., 428, 324-381, 2008.

[5] T. Cluzeau, A. Quadrat. OreMorphisms*: A homological algebraic package for factoring, reducing and decomposing linear functional systems.* Springer, Lecture Notes in Control and Inform. Sci., 388, 179-194, 2009. http://pages.saclay.inria.fr/alban.quadrat/OreMorphisms.html

[6] T. Cluzeau, A. Quadrat. *A constructive version of Fitting's theorem on isomorphisms and equivalences of linear systems.* Proceedings of the 7th international Workshop on Multidimensional (nD) Systems (nDS'11), Poitiers (France), 05-07/09/2011.

[7] T. Cluzeau, A. Quadrat. *Isomorphisms and Serre's reduction of linear systems.* Proceedings of the 8th international Workshop on Multidimensional (nD) Systems (nDS'13), Erlangen (Germany), 09-11/09/2013.

[8] T. Cluzeau, A. Quadrat, M. Tõnso. OreAlgebraicAnalysis*: A Mathematica package for the algorithmic study of linear functional systems.* OreAlgebraicAnalysis project, 2015. http://pages.saclay.inria.fr/alban.quadrat/OreAlgebraicAnalysis.html

[9] E. Fornasini, G. Marchesini. *Doubly indexed dynamical systems: state space models and structural properties.* Math. Systems Theory, 12, 59-72, 1970.

[10] P. A. Fuhrmann. *On strict system equivalence and similarity.* International Journal of Control, 25(1), 5-10, 1977.

[11] K. Gałkowski. *The Fornasini-Marchesini and the Roesser model: algebraic methods for recasting.* IEEE Trans. on Automatic Control, 41(1), 107-112, 1996.

[12] D. S. Johnson. *Coprimeness in multidimensional system theory and symbolic computation.* PhD thesis, Loughborough University of Technology, Leicestershire, UK, 1993.

[13] T. Kaczorek. *Equivalence and similarity for singular 2-D linear systems.* New Trends in Systems Theory, Birkhauser, 448-455, 1991.

[14] T. Kaczorek. *The singular general model of 2-D systems and its solution.* IEEE Trans. on Automatic Control, 33(11), 1060-1061, 1988.

[15] T. Kaczorek. *Two-Dimensional Linear Systems.* Springer Verlag, 1985.

[16] N. P. Karampetakis. *On the solution of the implicit Roesser model.* Bull. of the Polish Academy of Sciences, Tech. Sciences, 55(4), 365-378, 2007.

[17] J. E. Kurek. *The general state-space model for a two-dimensional linear digital system.* IEEE Trans. on Automatic Control, 30, 600-602, 1985.

[18] F. L. Lewis. *A review of 2-D implicit systems.* Automatica, 28(2), 345-354, 1992.

[19] B. Malgrange. *Systèmes différentiels à coefficients constants.* Séminaire Bourbaki 1962/63, 1-11, 1962.

[20] U. Oberst. *Multidimensional constant linear systems.* Acta Appl. Math., 20, 1-175, 1990.

[21] A. C. Pugh, S. J. McInerney, M. Hou, and G. E. Hayton. *A transformation for 2-D systems and its invariants.* Proc. of IEEE 35th conference on Decision and Control, 2157-2158, Kobe, Japan, 1996.

[22] A. Quadrat. *An introduction to constructive algebraic analysis and its applications.* Les cours du CIRM, Journées Nationales de Calcul Formel (2010), 1(2), 281-471, 2010.

[23] R. P. Roesser. *A discrete state-space model for linear image processing.* IEEE Trans. on Automatic Control, 20(1), 1-10,1975.

[24] H. H. Rosenbrock. *State Space and Multivariable Theory.* Nelson-Wiley, London, UK,1970.

[25] J. J. Rotman. *An Introduction to Homological Algebra.* Academic Press, 1979.

[26] J. Wood, E. Rogers, and D. H. Owens. *Behaviours, Modules, and Duality.* Multidimensional Signals, Circuits and Systems, K. Galkowski, J. Wood, Eds., Taylor & Francis, Ch. 3, 45-56, 2001.

[27] E. Zerz. *On strict system equivalence for multidimensional systems.* International Journal of Control, 73(6), 495-504, 2000.