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Serre's reduction of linear partial differential systems with holonomic adjoints☆

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ABSTRACT

Given a linear functional system (e.g., an ordinary/partial differential system, a differential time-delay system, a difference system), Serre's reduction aims at finding an equivalent linear functional system which contains fewer equations and fewer unknowns. The purpose of this paper is to study Serre's reduction of underdetermined linear systems of partial differential equations with either polynomial, formal power series or locally convergent power series coefficients, and with holonomic adjoints in the sense of algebraic analysis. We prove that these linear partial differential systems can be defined by means of only one linear partial differential equation. In the case of polynomial coefficients, we give an algorithm to compute the corresponding equation.

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1. Introduction

One of the main goals of symbolic computation is the problem of rewriting linear/polynomial/algebraic/differential systems of equations in such a way that interesting information on the systems, which does not clearly appear in their original forms, can be easily extracted from their new forms (e.g., Gaussian elimination, Smith or Jacobson normal forms, Gröbner or Janet bases, triangular sets, formal integrability).

Serre's reduction problem aims at simplifying linear functional systems (e.g., ordinary (OD) or partial differential (PD) systems, time-delay systems, difference systems) in the sense of finding an equivalent representation of the linear functional system which contains fewer unknowns and fewer equations.

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Serre's reduction generally helps studying the structural properties of a linear functional system and it can sometimes be used to compute its closed-form solutions. This problem also finds applications in numerical analysis. In module theory, Serre's reduction is related to the problem of characterizing the minimal number of generators (and relations) of a finitely presented module. The efficient generation of ideals of commutative polynomial rings and its interpretation in terms of complete intersection of affine algebraic varieties of codimension 2 were the reasons for which Serre studied this problem (Serre, 1986).

The constructive study of Serre's reduction problem has recently been initiated in Boudellioua and Quadrat (2010). Despite a precise mathematical characterization of the existence of Serre's reduction (see Section 4), to effectively recognize it and compute its reduced form is quite an issue. Moreover, as indicated by Serre in Serre (1986), this problem is connected to the difficult problem of recognizing whether or not certain projective or stably free modules are free (e.g., Serre's conjecture for the commutative polynomial case, nowadays known as the Quillen–Suslin theorem (see, e.g., Rotman, 2009)). An important case where Serre's reduction can be constructively tested is the case of a full row rank matrix $R \in D^{q\times p}$, where $D = k[x_1, \ldots, x_n]$ and k is a computable field, for which the D-module $N = D^q/(R\,D^p)$ is *0-dimensional*, namely, is a finite-dimensional k-vector space (see Boudellioua and Quadrat, 2010). This condition can be checked by applying classical Gröbner basis techniques to the D-module $D^{1\times q}/(D^{1\times p}\,R^T) \cong N$. In this case, we can test whether or not two invertible square matrices V and W exist such that $V\,R\,W = \mathrm{diag}(I_r, S)$, where $\mathrm{diag}(A, B)$ stands for the block-diagonal matrix formed by the two matrices A and B, I_r is the $r \times r$ identity matrix, and $S \in D^{(q-r)\times(p-r)}$ is a certain matrix. In particular, the computation of the matrices V and W requires the computation of bases of finitely generated free D-modules using an implementation of the Quillen–Suslin theorem such as can be found in the QuillenSuslin package (Fabiańska and Quadrat, 2007). As explained in Boudellioua and Quadrat (2010), the class of controllable linear OD time-delay systems studied in control theory fits in the above situation, which explained why it was possible in Boudellioua and Quadrat (2010), Cluzeau and Quadrat (2010b) and Quadrat (2010) to exhibit Serre's reductions for many linear OD time-delay systems studied in the literature of control theory.

The purpose of this paper is to study Serre's reduction of underdetermined linear PD systems. More precisely, we focus on the situation which generalizes the case of a 0-dimensional $D = k[x_1, \ldots, x_n]$-module $D^q/(R\,D^p)$, namely, the case where the right $D = A_n(k)$-module $D^q/(R\,D^p)$ is *holonomic* in the sense of algebraic analysis (see Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993 and the references therein), where $A_n(k)$ denotes the *polynomial Weyl algebra*, i.e., the noncommutative ring of PD operators with polynomial coefficients over a base field k of characteristic 0. In this case, if $p - q \geq 1$, then, combining a classical result in algebraic analysis which asserts that a holonomic module is cyclic (see Section 3) with Stafford's theorem proving that finitely generated projective left $A_n(k)$-modules are free when their ranks are at least 2 (see Section 2), we prove in Section 5 that the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ always admits a Serre's reduction, i.e., there always exists a row vector $Q \in D^{1\times(p-q+1)}$ such that $M \cong D^{1\times(p-q+1)}/(D\,Q)$. If $\mathcal{F}$ is any left D-module (e.g., $\mathcal{F} = \mathbb{R}[x_1, \ldots, x_n]$, $\mathbb{R}(x_1, \ldots, x_n)$, $\mathbb{R}[\![x_1, \ldots, x_n]\!]$, $\mathbb{R}\{x_1, \ldots, x_n\}$, $C^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$), then this result shows that the solution space $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ of the linear PD system defined by the matrix $R \in D^{q\times p}$ of PD operators is equivalent to the solution space defined by a sole linear PD equation $\ker_{\mathcal{F}}(Q.) = \{\zeta \in \mathcal{F}^{(p-q+1)} \mid Q\,\zeta = 0\}$. In particular, the knowledge of $\ker_{\mathcal{F}}(Q.)$ fully characterizes $\ker_{\mathcal{F}}(R.)$ and conversely. In order to compute the matrix Q, we first need to compute a cyclic generator for the cyclic right D-module $D^q/(R\,D^p)$ using, for instance, an algorithm developed in Leykin (2004) and implemented in the packages *Dmodules* (Leykin et al., 2004) and Serre (Cluzeau and Quadrat, 2010b), and then compute bases of certain finitely generated free left D-modules using, for instance, the algorithm obtained in Quadrat and Robertz (2007) and implemented in the Stafford package (Quadrat and Robertz, 2007). Moreover, if $q \geq 3$, then we prove that the matrix R is equivalent to $\mathrm{diag}(I_{q-1}, Q)$, i.e., two square matrices V and W invertible over D exist such that $V\,R\,W = \mathrm{diag}(I_{q-1}, Q)$. The corresponding algorithms, described in Section 5, are implemented in the Serre package (Cluzeau and Quadrat, 2010b) built upon the OreModules package (Chyzak et al., 2007).

We then focus on the case of linear OD systems with either polynomial, formal power series or locally convergent power series coefficients, i.e., with coefficients in $k[t]$, $k[\![t]\!]$, where k is a field of characteristic 0, or in $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. Using the recent extension of Stafford's theorem to

the case of the ring D of OD operators with either formal power series or locally convergent power series coefficients (Quadrat and Robertz, 2010), we prove that every linear OD system with either polynomial, power series or locally convergent power series coefficients, and presented by a full row rank matrix $R \in D^{q\times p}$, where $p-q \geq 1$, can be defined by a row vector $Q \in D^{1\times(p-q+1)}$, which yields $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q.)$ for all left D-modules $\mathcal{F}$. Moreover, if $q \geq 3$, then the matrix R is equivalent to the matrix $\mathrm{diag}(I_{q-1}, Q)$. This result is particularly interesting in control theory where $p-q$ is the number of inputs of the system. We also point out that the above results allow us to avoid singularities which may appear in the Jacobson normal forms of the matrix R (see Levandovskyy and Schindelar, 2011; Zerz, 2006) computed over the principal left ideal domain of OD operators with either rational, formal Laurent series or germs of (real) meromorphic function coefficients at 0 (i.e., $k(t)$, $k[\![t]\!][t^{-1}]$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$). Finally, in Section 4, we explain the connections between Serre's reduction of first order linear OD systems or linear evolution PD systems and the concepts of observability and controllability developed in control theory (see, e.g., Kalman et al., 1969), and show how Serre's reduction extends the concept of *cyclic vectors* used in the literature of linear OD systems with coefficients in a differential field (e.g., $\mathbb{R}(t)$, $\mathbb{R}[\![t]\!][t^{-1}]$, $\mathbb{R}\{t\}[t^{-1}]$) (see, e.g., Churchill and Kovacic , 2002; Cope, 1936) to linear OD systems with coefficients in a differential ring (e.g., $\mathbb{R}[t]$, $\mathbb{R}[\![t]\!]$, $\mathbb{R}\{t\}$).

This paper is an extension of the congress paper (Cluzeau and Quadrat, 2010a).

2. Algebraic analysis approach to linear system theory

In this section, we recall the algebraic analysis approach to mathematical system theory that will be used in the next sections. Moreover, we introduce the main notations and state a few results which will be used in what follows.

We shall denote by $D^{1\times p}$ (resp., D^q) the left (resp., right) D-module formed by row (resp., column) vectors of length p (resp., q) with entries in D and by $R \in D^{q\times p}$ a $q \times p$ matrix with entries in D. The *general linear group of D of degree p*, namely,

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\ U\,V = V\,U = I_p\},$$

is the subgroup of the ring $D^{p\times p}$ formed by invertible (*unimodular*) matrices.

In what follows, we shall use the following notations:

$$\begin{array}{rcl} .R : D^{1\times q} & \longrightarrow & D^{1\times p}, \\ \mu & \longmapsto & \mu\,R, \end{array} \qquad \begin{array}{rcl} R. : D^{p} & \longrightarrow & D^{q} \\ \eta & \longmapsto & R\,\eta. \end{array}$$

Within algebraic analysis (see, e.g., Björk, 1979; Chyzak et al., 2005; Coutinho, 1995; Maisonobe and Sabbah, 1993), a linear functional system (e.g., a linear system of OD equations or PD equations, OD time-delay equations, difference equations) can be studied by means of module theory and homological algebra. More precisely, if D is a noncommutative polynomial ring of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators), $R \in D^{q\times p}$ and $\mathcal{F}$ a left D-module, then the *linear functional system*

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

i.e., the abelian group (i.e., $\mathbb{Z}$-module) formed by the $\mathcal{F}$-solutions of the linear system $R\,\eta = 0$, can be studied by means of the left D-module $M \triangleq D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented* by the matrix R. Indeed, Malgrange's remark (Malgrange, 1962) asserts the existence of the abelian group isomorphism

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}), \tag{1}$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group of left D-*homomorphisms* from M to $\mathcal{F}$ and $P \cong Q$ means that P and Q are *isomorphic* modules (Rotman, 2009).

Let us describe the isomorphism (1). To do that, we first give an explicit description of M in terms of *generators and relations*. Let $\pi : D^{1\times p} \longrightarrow M = D^{1\times p}/(D^{1\times q}\,R)$ be the canonical projection onto M, namely, the left D-homomorphism which sends a row vector of $D^{1\times p}$ to its residue class $\pi(\lambda)$ in M, $\{f_j\}_{j=1,\ldots,p}$ the *standard basis* of $D^{1\times p}$, namely, f_j is the row vector of length p defined by 1 at the *jth*

entry and 0 elsewhere, and $y_j = \pi(f_j)$ the residue class of f_j in M for $j = 1, \ldots, p$. Since every element $m \in M$ is the residue class of an element $\lambda = (\lambda_1 \ldots \lambda_p) \in D^{1\times p}$, then we get

$$m = \pi(\lambda) = \pi\left(\sum_{j=1}^{p} \lambda_j f_j\right) = \sum_{j=1}^{p} \lambda_j \pi(f_j) = \sum_{j=1}^{p} \lambda_j y_j,$$

which shows that $\{y_j\}_{j=1,\ldots,p}$ is a family of generators of the left D-module M. Now, if $R_{i\bullet}$ denotes the *ith* row of R, then $R_{i\bullet} \in D^{1\times q} R$, which yields $\pi(R_{i\bullet}) = 0$ and thus

$$\forall\, i = 1, \ldots, q, \quad \pi(R_{i\bullet}) = \pi\left(\sum_{j=1}^{p} R_{ij} f_j\right) = \sum_{j=1}^{p} R_{ij}\, \pi(f_j) = \sum_{j=1}^{p} R_{ij}\, y_j = 0, \tag{2}$$

which shows that the set of generators $\{y_j\}_{j=1,\ldots,p}$ of M satisfies the left D-linear relations (2) and their left D-linear combinations. Thus, if we set $y = (y_1 \ldots y_p)^T \in M^p$, then (2) becomes $R\,y = 0$.

Let $\chi : \ker_{\mathcal{F}}(R.) \longrightarrow \hom_D(M, \mathcal{F})$ be the $\mathbb{Z}$-homomorphism defined by $\chi(\eta) = \phi_\eta$, where $\phi_\eta(\pi(\lambda)) = \lambda\,\eta \in \mathcal{F}$ for all $\lambda \in D^{1\times p}$. The $\mathbb{Z}$-homomorphism ϕ_η is well-defined since $\pi(\lambda) = \pi(\lambda')$ yields $\pi(\lambda - \lambda') = 0$, i.e., $\lambda - \lambda' = \mu\, R$ for a certain $\mu \in D^{1\times q}$, and thus $\phi_\eta(\pi(\lambda)) = \lambda\,\eta = \lambda'\,\eta + \mu\, R\,\eta = \lambda'\,\eta = \phi_\eta(\pi(\lambda'))$. Moreover, χ is injective since $\phi_\eta = 0$ yields $\lambda\,\eta = 0$ for all $\lambda \in D^{1\times p}$, and thus $\eta_j = f_j\,\eta = 0$ for all $j = 1, \ldots, p$, i.e., $\eta = 0$. It is also surjective since for all $\phi \in \hom_D(M, \mathcal{F})$, $\eta = (\phi(y_1)\ \ldots\ \phi(y_p))^T \in \mathcal{F}^p$ satisfies $\phi_\eta = \phi$ and

$$\forall\, i = 1, \ldots, q, \quad \sum_{j=1}^{p} R_{ij}\,\eta_j = \sum_{j=1}^{p} R_{ij}\,\phi(y_j) = \phi\left(\sum_{j=1}^{p} R_{ij}\, y_j\right) = \phi(0) = 0,$$

i.e., $\eta \in \ker_{\mathcal{F}}(R.)$, and thus $\chi(\eta) = \phi$. Thus, the $\mathbb{Z}$-homomorphism χ is an isomorphism.

Theorem 1 (*Malgrange, 1962*)**.** *Let D be a ring, $M = D^{1\times p}/(D^{1\times q} R)$ the left D-module finitely presented by the matrix $R \in D^{q\times p}$, $\pi : D^{1\times p} \longrightarrow M$ the canonical projection onto M, $\{f_j\}_{j=1,\ldots,p}$ the standard basis of $D^{1\times p}$, $y_j = \pi(f_j)$ for $j = 1, \ldots, p$, and $\mathcal{F}$ a left D-module. Then, we have the following $\mathbb{Z}$-isomorphism:*

$$\begin{aligned} \hom_D(M, \mathcal{F}) &\longrightarrow \ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \\ \phi &\longmapsto \eta = (\phi(y_1) \ldots \phi(y_p))^T. \end{aligned} \tag{3}$$

Hence, there is a one-to-one correspondence between the elements of $\hom_D(M, \mathcal{F})$ and the elements of $\ker_{\mathcal{F}}(R.)$.

Remark 2. Theorem 1 shows that $\ker_{\mathcal{F}}(R.)$ can be studied by means of the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q} R)$ and the left D-module $\mathcal{F}$: M intrinsically defines the linear system of equations defined by the matrix $R \in D^{q\times p}$ and $\mathcal{F}$ is the functional space where we seek the solutions of the linear functional system.

In what follows, D will denote a *noncommutative Noetherian domain*, namely, a unital ring satisfying that $d\,d'$ is not necessarily equal to $d'\,d$ for all $d, d' \in D$, containing no nontrivial zero-divisors, i.e., $d\,d' = 0$ yields $d = 0$ or $d' = 0$, and every left (resp., right) ideal of D is finitely generated, i.e., can be generated by a finite family of elements of D as a left (resp., right) D-module (see, e.g., Rotman, 2009).

A *differential ring* $(A, \{\delta_1, \ldots, \delta_n\})$ is a commutative ring A equipped with n *commuting derivations* $\delta_i : A \longrightarrow A$ for $i = 1, \ldots, n$, namely, maps satisfying:

$$\forall\, i, j = 1, \ldots, n, \quad \forall\, a_1, a_2 \in A, \quad \begin{cases} \delta_i \circ \delta_j = \delta_j \circ \delta_i, \\ \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \\ \delta_i(a_1\, a_2) = \delta_i(a_1)\, a_2 + a_1\, \delta_i(a_2). \end{cases}$$

If we take $a_1 = a_2 = 1$, then the above equality yields $\delta_i(1) = 2\,\delta_i(1)$, i.e., $\delta_i(1) = 0$. If A is a field and $a \in A \setminus \{0\}$, then $\delta_i(a)\, a^{-1} + a\,\delta_i(a^{-1}) = \delta_i(a\, a^{-1}) = \delta_i(1) = 0$, which yields $\delta_i(a^{-1}) = -a^{-2}\,\delta_i(a)$ and $(A, \{\delta_1, \ldots, \delta_n\})$ is then called a *differential field*.

In what follows, we shall mainly focus on the differential ring $\left(A, \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}\right)$, where $A = k[x_1, \ldots, x_n], k[\![x_1, \ldots, x_n]\!]$ (i.e., the ring of formal power series with coefficients in k), where k is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$), $k\{x_1, \ldots, x_n\}$ where $k = \mathbb{R}$ or $\mathbb{C}$ (i.e., the ring of locally convergent power series at 0 or equivalently, the ring of germs of real analytic or holomorphic functions at 0) or on the differential field $A = k$ or $k(x_1, \ldots, x_n)$, where k is a field.

The *ring* $D = A\langle\partial_1, \ldots, \partial_n\rangle$ *of PD operators* in $\partial_1, \ldots, \partial_n$ with coefficients in the differential ring $(A, \{\delta_1, \ldots, \delta_n\})$ is the noncommutative polynomial ring in the ∂_i's with coefficients in the ring A satisfying:

$$\forall\, i, j = 1, \ldots, n, \quad \forall\, a \in A, \quad \partial_i\, \partial_j = \partial_j\, \partial_i, \quad \partial_i\, a = a\, \partial_i + \delta_i(a).$$

An element $d \in D$ can uniquely be written as $d = \sum_{0 \leq |\nu| \leq r} a_\nu\, \partial^\nu$, with the notations $\nu = (\nu_1 \ldots \nu_n)^T \in \mathbb{N}^n$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\partial^\nu = \partial_1^{\nu_1} \ldots \partial_n^{\nu_n}$ and $a_\nu \in A$. If $n = 1$, then we shall simply use the notations $\delta = \frac{d}{dt}$ instead of δ_1, ∂ instead of ∂_1 and $k[t]$, $k(t)$, $k[\![t]\!]$ and $k\{t\}$ instead of $k[x_1]$, $k(x_1)$, $k[\![x_1]\!]$ and $k\{x_1\}$.

The *first* (polynomial) and the *second* (rational) *Weyl algebras* are defined by:

$$A_n(k) \triangleq k[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle, \quad B_n(k) \triangleq k(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle.$$

The ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$, where $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, and k is a field, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, is a Noetherian domain (see, e.g., McConnell and Robson, 2000).

Let us recall a few definitions of module theory.

Definition 3 (*Lam, 1999; McConnell and Robson, 2000; Rotman, 2009*)**.** Let D be a left Noetherian domain and M a *finitely generated* left D-module, namely, M can be generated by a finite family of elements of M as a left D-module.

(1) M is *free* if there exists $r \in \mathbb{N}$ such that $M \cong D^{1\times r}$. Then, r is called the *rank* of the free left D-module M and is denoted by $\mathrm{rank}_D(M)$.
(2) M is *stably free* if there exist $r, s \in \mathbb{N}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$. Then, $r - s$ is called the *rank* of the stably free left D-module M.
(3) M is *projective* if there exist $r \in \mathbb{N}$ and a left D-module N such that $M \oplus N \cong D^{1\times r}$, where $\oplus$ denotes the direct sum of left D-modules.
(4) M is *torsion-free* if the *torsion left D-submodule of* M, namely,

$$t(M) = \{m \in M \mid \exists\, d \in D \setminus \{0\} :\ d\, m = 0\},$$

is reduced to 0, i.e., if $t(M) = 0$. The elements of $t(M)$ are called the *torsion elements* of M.
(5) M is *torsion* if $t(M) = M$, i.e., if every element of M is a torsion element.
(6) M is *cyclic* if there exists $m \in M$ such that $M = D\, m \triangleq \{d\, m \mid d \in D\}$. The element m is then called a *cyclic generator* of M.

A free module is clearly stably free (take $s = 0$ in 2 of Definition 3), a stably free module is projective (take $N = D^{1\times s}$ in 3 of Definition 3) and a projective module is torsion-free (since it can be embedded into a free, and thus into a torsion-free module).

The converses of the previous results are generally not true. However, some of them hold in the following interesting situations for mathematical system theory (Lam, 1999; McConnell and Robson, 2000; Quadrat and Robertz, 2010; Rotman, 2009).

Theorem 4. (1) *If D is a* principal left ideal domain*, namely, every left ideal of the domain D is principal (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A such as $A = k$, $k(t)$ and $k[\![t]\!][t^{-1}]$, where k is a field of characteristic 0, or $A = k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then every finitely generated torsion-free left D-module is free.*
(2) *If $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring with coefficients in a field k, then every finitely generated projective D-module is free (*Quillen–Suslin theorem*).*
(3) *If D is the Weyl algebra $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, then every finitely generated projective left D-module is stably free and every finitely generated stably free left D-module of rank at least 2 is free (Stafford's theorem).*

(4) *If $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in $A = k[\![t]\!]$, where k is a field of characteristic 0, or in $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then every finitely generated projective left D-module is stably free and every finitely generated stably free left D-module of rank at least 2 is free.*

If the matrix R has *full row rank*, namely, $\ker_D(.R) \triangleq \{\mu \in D^{1\times q} \mid \mu R = 0\} = 0$, i.e., the rows of R are left D-linearly independent, then the next proposition characterizes when the left D-module $M = D^{1\times p}/(D^{1\times q} R)$ is a stably free or free module.

Theorem 5 (*see, e.g., Fabiańska and Quadrat, 2007; Quadrat and Robertz, 2007*)**.** *Let D be a Noetherian domain, $R \in D^{q\times p}$ a full row rank matrix, i.e., $\ker_D(.R) = 0$, and $M = D^{1\times p}/(D^{1\times q} R)$ the left D-module finitely presented by R.*

(1) *M is a projective left D-module iff M is a stably free left D-module.*
(2) *M is a stably free left D-module of rank $p - q$ iff R admits a* right inverse over *D, namely, iff there exists a matrix $S \in D^{p\times q}$ satisfying $RS = I_q$.*
(3) *M is a free left D-module of rank $p - q$ iff there exists $U \in \mathrm{GL}_p(D)$ such that:*

$$RU = (I_q \quad 0).$$

If $U = (S \;\; Q)$, where $S \in D^{p\times q}$ and $Q \in D^{p\times(p-q)}$, then we have

$$\begin{array}{lll} \psi : M \longrightarrow D^{1\times(p-q)} & & \psi^{-1} : D^{1\times(p-q)} \longrightarrow M \\ \pi(\lambda) \longmapsto \lambda Q, & & \mu \longmapsto \pi(\mu T), \end{array}$$

where the matrix $T \in D^{(p-q)\times p}$ is defined by:

$$U^{-1} = \begin{pmatrix} R \\ T \end{pmatrix} \in D^{p\times p}.$$

In particular, $M \cong D^{1\times p} Q = D^{1\times(p-q)}$. The matrix Q is then called an injective parametrization *of the free left D-module M. Finally, $\{\pi(T_{i\bullet})\}_{i=1,\ldots,p-q}$ is a basis of the free left D-module M of rank $p - q$.*

The Quillen–Suslin theorem (resp., the Stafford's theorem) has recently been implemented in the QuillenSuslin package (Fabiańska and Quadrat, 2007) (resp., Stafford package Quadrat and Robertz, 2007). Hence, we can compute bases and injective parametrizations of finitely generated free (left) D-modules, where $D = k[x_1, \ldots, x_n]$ and $k = \mathbb{Q}$ or $\mathbb{F}_p$ (p is a prime number), $A_n(\mathbb{Q})$ or $B_n(\mathbb{Q})$.

3. Holonomic *D*-modules

In this section, we consider the ring $D = A\langle\partial_1, \ldots, \partial_n\rangle$ of PD operators with coefficients in the differential ring $A = k$, $k[x_1, \ldots, x_n]$, $k(x_1, \ldots, x_n)$ or $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$. The purpose of this section is to briefly recall two important results on the so-called *holonomic D-modules*, namely, the forthcoming Theorems 11 and 12, which will play a central role in Section 5.

The ring D is endowed with the *order filtration* defined by:

$$\forall\, r \in \mathbb{N}, \quad D_r = \left\{ \sum_{0\leq|\alpha|\leq r} a_\alpha\, \partial^\alpha \mid a_\alpha \in A \right\}.$$

We can check that the filtration properties hold, namely:

(1) $\forall\, r, s \in \mathbb{N}, r \leq s \;\Rightarrow\; D_r \subseteq D_s$.
(2) $D = \bigcup_{r\in\mathbb{N}} D_r$.
(3) $\forall\, r, s \in \mathbb{N}$: $D_r\, D_s \subseteq D_{r+s}$.

The ring D is called a *filtered ring* and an element of D_r has a *degree* less than or equal to r. We can easily check that $D_0 = A$ and D_r is a finitely generated A-module.

If $d_1, d_2 \in D$, then we can define the *bracket* of d_1 and d_2 by $[d_1, d_2] \triangleq d_1\, d_2 - d_2\, d_1$. If $d_1 \in D_r$ and $d_2 \in D_s$, then $d_1\, d_2$ and $d_2\, d_1$ belong to D_{r+s} since $D_r\, D_s \subseteq D_{r+s}$ and $D_s\, D_r \subseteq D_{r+s}$, and we can check that $[d_1, d_2] \in D_{r+s-1}$, i.e., $[D_r, D_s] \subseteq D_{r+s-1}$.

Let us now introduce the following A-module

$$\mathrm{gr}(D) = \bigoplus_{r\in\mathbb{N}} D_r/D_{r-1},$$

where we have set $D_{-1} = 0$. If $\pi_r : D_r \longrightarrow D_r/D_{r-1}$ is the canonical projection for all $r \in \mathbb{N}$, then the A-module $\mathrm{gr}(D)$ inherits a ring structure defined by

$$\forall\, d_1 \in D_r, \quad \forall\, d_2 \in D_s, \quad \begin{cases} \pi_r(d_1) + \pi_s(d_2) \triangleq \pi_t(d_1 + d_2) \in D_t/D_{t-1}, \\ \pi_r(d_1)\,\pi_s(d_2) \triangleq \pi_{r+s}(d_1\,d_2) \in D_{r+s}/D_{r+s-1}, \end{cases}$$

where $t = \max(r, s)$. The ring $\mathrm{gr}(D)$ is called the *graded ring* associated with the order filtration of D.

Let $\chi_i \triangleq \pi_1(\partial_i) \in D_1/D_0$ for all $i = 1, \ldots, n$. Then, $\pi_1([\partial_i, \partial_j]) = 0$ and $\pi_1([\partial_i, a]) = 0$ for all $a \in A$ and all $i, j = 1, \ldots, n$ since $[\partial_i, \partial_j] = 0$ and $[\partial_i, a] \in D_0$, which shows that

$$\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$$

is the commutative polynomial ring in $\chi_1, \ldots, \chi_n$ with coefficients in the commutative Noetherian ring A.

We can now extend the concepts of filtered and graded rings to modules.

Definition 6 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module.

(1) A *filtration* of M is a sequence $\{M_q\}_{q\in\mathbb{N}}$ of A-submodules of M (with the convention that $M_{-1} = 0$) such that:
   (a) $\forall\, q, r \in \mathbb{N}, q \leq r \;\Rightarrow\; M_q \subseteq M_r$.
   (b) $M = \bigcup_{q\in\mathbb{N}} M_q$.
   (c) $\forall\, q, r \in \mathbb{N}$: $D_r\, M_q \subseteq M_{q+r}$.
   The left D-module M is then called a *filtered module*.

(2) The associated *graded* $\mathrm{gr}(D)$-module $\mathrm{gr}(M)$ is defined by:
   (a) $\mathrm{gr}(M) = \bigoplus_{q\in\mathbb{N}} M_q/M_{q-1}$.
   (b) For every $d \in D_r$ and every $m \in M_q$, we set
   $$\pi_r(d)\,\sigma_q(m) \triangleq \sigma_{q+r}(d\,m) \in M_{q+r}/M_{q+r-1},$$
   where $\sigma_q : M_q \longrightarrow M_q/M_{q-1}$ is the canonical projection for all $q \in \mathbb{N}$.

(3) A filtration $\{M_q\}_{q\in\mathbb{N}}$ is called a *good filtration* if one of the two following equivalent conditions is satisfied:
   (a) M_q is a finitely generated A-module for all $q \in \mathbb{N}$ and there exists $p \in \mathbb{N}$ such that $D_r\, M_p = M_{p+r}$ for all $r \in \mathbb{N}$.
   (b) $\mathrm{gr}(M) = \bigoplus_{q\in\mathbb{N}} M_q/M_{q-1}$ is a finitely generated $\mathrm{gr}(D)$-module.

If M is a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module, then $\mathrm{gr}(M)$ is a finitely generated module over the commutative polynomial ring $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$. Hence, we are back to the realm of commutative algebra. Based on techniques of algebraic geometry and commutative algebra, one can then use invariants of $\mathrm{gr}(M)$ (e.g., dimension, multiplicity) to characterize properties of the left D-module M (see, e.g., Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993).

We denote by $\mathrm{spec}(A)$ the set of all *prime ideals* of the commutative ring A endowed with the *Zariski topology* (see, e.g., Eisenbud, 1995).

We can now introduce the concept of a *characteristic variety* of a differential module.

Theorem 7 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** *Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module and $G = \mathrm{gr}(M)$ the associated graded $\mathrm{gr}(D) = A[\chi_1, \ldots, \chi_n]$-module for a good filtration of M. Then, the* characteristic ideal *$I(M)$ of M is the ideal of the commutative polynomial ring $\mathrm{gr}(D)$ defined by:*

$$I(M) = \sqrt{\mathrm{ann}(G)} \triangleq \{a \in \mathrm{gr}(D) \mid \exists\, n \in \mathbb{N} :\; a^n\, G = 0\}.$$

The ideal $I(M)$ of $\mathrm{gr}(D)$ does not depend on the good filtration of M. The characteristic variety *of M is then the subset of $\mathrm{spec}(\mathrm{gr}(D))$ defined by:*

$$\mathrm{char}_D(M) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \mid \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}.$$

We can easily prove that every finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module M admits a good filtration (e.g., if $M = \sum_{j=1}^{p} D\, y_j$, then $M_q = \sum_{i=1}^{p} D_q\, y_i$ is a good filtration of M since $\mathrm{gr}(M) = \sum_{i=1}^{p} \mathrm{gr}(D)\, y_i$ is a finitely generated $\mathrm{gr}(D)$-module) and thus a characteristic variety (see, e.g., Björk, 1979; Maisonobe and Sabbah, 1993).

The *dimension* of the left D-module M can be defined as the geometric dimension (*Krull dimension*) of the characteristic variety $\mathrm{char}_D(M)$ of M.

Definition 8 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** Let M be a finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module. Then, the *dimension* of M is the supremum of the lengths d of chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d$ of distinct prime ideals in the commutative ring $A[\chi_1, \ldots, \chi_n]/I(M)$. If $M = 0$, then we set $\dim_D(M) = -1$.

In what follows, we shall simply write $\dim(D)$ instead of $\dim_D(D)$.

Example 9 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** We have

$$\dim(k[x_1, \ldots, x_n]) = n, \quad \dim(B_n(k')) = n, \quad \dim(A\langle\partial_1, \ldots, \partial_n\rangle) = 2\,n,$$

where k is a field, k' is a field of characteristic 0, and $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$.

Definition 10 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** Let M be a non-zero finitely generated left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module. If $\dim_D(M) = n$ then M is called a *holonomic* left D-module.

The next two theorems will play important roles in Section 5.

Theorem 11 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** *If $D = A\langle\partial\rangle$ is the ring of OD operators with coefficients in $A = k[t]$, $k[\![t]\!]$, where k is a field of characteristic 0, or $k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then a finitely generated left (resp., right) D-module M is holonomic iff M is a finitely generated torsion left (resp., right) D-module.*

Theorem 12 (*Björk, 1979; Coutinho, 1995; Maisonobe and Sabbah, 1993*)**.** *If $A = k[x_1, \ldots, x_n]$, $k[\![x_1, \ldots, x_n]\!]$, where k is a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then a holonomic left $D = A\langle\partial_1, \ldots, \partial_n\rangle$-module M is cyclic, i.e., M can be generated by one element as a left D-module. More precisely, if $\{y_j\}_{j=1,\ldots,p}$ is a set of generators of the holonomic left D-module M, then there exist $d_2, \ldots, d_p \in D$ such that M is generated by $y_1 + d_2\, y_2 + \cdots + d_p\, y_p$. Similar results hold for holonomic right D-modules.*

Remark 13. For $D = A_n(k)$, where k is a computable field of characteristic 0 (e.g., $k = \mathbb{Q}$), a constructive algorithm for the computation of a cyclic generator of a finitely presented holonomic left D-module M is given in Leykin (2004). The corresponding algorithm is implemented in the *Dmodules* package of *Macaulay 2* (Leykin et al., 2004) and in the Serre package (Cluzeau and Quadrat, 2010b).

4. Serre's reduction of linear systems

In this section, we first study when a finitely presented left (resp., right) D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ (resp., $N = D^q/(R\, D^p)$) can be generated by less than p (resp., q) generators. Then, we recall recent results on *Serre's reduction* (Serre, 1986) obtained in Boudellioua and Quadrat (2010). They will be used in Section 5 to study Serre's reduction of the class of linear PD systems with holonomic *adjoints*.

If $R \in D^{q\times p}$ and $\{f_j\}_{j=1,\ldots,p}$ is the standard basis of $D^{1\times p}$, then the beginning of Section 2 shows that $\{y_j = \pi(f_j)\}_{j=1,\ldots,p}$ is a family of generators of the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, where $\pi : D^{1\times p} \longrightarrow M$ is the canonical projection. Moreover, $\{y_j\}_{j=1,\ldots,p}$ satisfies the relations $R\, y = 0$, where $y = (y_1 \ldots y_p)^T \in M^p$.

Let us first investigate when the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ can be generated by less than p generators. Let $0 \leq r \leq p-1$ and $\Lambda \in D^{(p-r)\times p}$ be such that

$$P \triangleq \begin{pmatrix} R \\ \Lambda \end{pmatrix} \in D^{(q+p-r)\times p}$$

admits a *left inverse* $S = (S_1 \;\; S_2) \in D^{p\times(q+p-r)}$, where $S_1 \in D^{p\times q}$ and $S_2 \in D^{p\times(p-r)}$, i.e., $S\,P = I_p$. We note that this result always holds if we take $\Lambda = I_p$, which shows that the interesting case starts with $r \geq 1$. Then, we have $D^{1\times(q+p-r)}\,P = D^{1\times p}$, which yields

$$M = D^{1\times p}/(D^{1\times q}\,R) = (D^{1\times(q+p-r)}\,P)/(D^{1\times q}\,R), \tag{4}$$

and shows that $\{z_k = \pi(\Lambda_{k\bullet}) = \sum_{j=1}^{p}\Lambda_{kj}\,y_j\}_{k=1,\ldots,p-r}$ is a family of generators of M. Let us give another way to understand this result. The identity $S_1\,R + S_2\,\Lambda = I_p$ yields

$$y = S_1\,(R\,y) + S_2\,(\Lambda\,y) = S_2\,(\Lambda\,y),$$

which shows that $z \triangleq \Lambda\,y \in M^{(p-r)}$ satisfies $y = S_2\,z$, i.e., $\{z_k = \sum_{j=1}^{p}\Lambda_{kj}\,y_j\}_{k=1,\ldots,p-r}$ is a family of generators of M. In particular, if $r = p-1$, then $\Lambda \in D^{1\times p}$ and M is generated by $\pi(\Lambda)$, i.e., $M = D\,\pi(\Lambda)$ is a cyclic left D-module and $\pi(\Lambda)$ is a cyclic generator of M.

Moreover, let $Q = (Q_1 \;\; Q_2) \in D^{s\times(q+p-r)}$, where $Q_1 \in D^{s\times q}$ and $Q_2 \in D^{s\times(p-r)}$, be a matrix such that $\ker_D(.P) = D^{1\times s}\,Q$. Using the identity $R = (I_q \;\; 0)\,P$, Lemma 3.1 of Cluzeau and Quadrat (2008) then yields

$$\begin{aligned} M = (D^{1\times(q+p-r)}\,P)/(D^{1\times q}\,R) &\cong D^{1\times(q+p-r)}\Big/\left(D^{1\times(q+s)}\begin{pmatrix} I_q & 0\\ Q_1 & Q_2\end{pmatrix}\right)\\ &\cong L \triangleq D^{1\times(p-r)}/(D^{1\times s}\,Q_2), \end{aligned}$$

where the left D-isomorphism $\phi : M \longrightarrow L$ is defined by $\phi(\pi(\mu\,P)) = \sigma(\mu_2)$ for all $\mu = (\mu_1 \;\; \mu_2)$, where $\mu_1 \in D^{1\times q}$, $\mu_2 \in D^{1\times(p-r)}$, and $\sigma : D^{1\times(p-r)} \longrightarrow L$ is the canonical projection onto L. Moreover, $\phi^{-1} : L \longrightarrow M$ is defined by $\phi^{-1}(\sigma(\mu_2)) = \pi(\mu_2\,\Lambda)$ for all $\mu_2 \in D^{1\times(p-r)}$. If $\{g_k\}_{k=1,\ldots,p-r}$ is the standard basis of $D^{1\times(p-r)}$, then the generators $\{v_k = \sigma(g_k)\}_{k=1,\ldots,p-r}$ of the left D-module L satisfy the relations $Q_2\,v = 0$, where $v = (v_1 \ldots v_{p-r})^T$. We also have $\phi^{-1}(v_k) = \pi(g_k\,\Lambda) = \pi(\Lambda_{k\bullet}) = z_k$, i.e., $\phi(z_k) = v_k$ for all $k = 1, \ldots, p-r$, which shows that, up to isomorphism, M can be generated by $p-r$ elements satisfying s relations. Finally, we can interpret this result by noticing that $Q_2\,z = 0$ generates the compatibility conditions of the inhomogeneous linear system:

$$\begin{cases} R\,y = 0,\\ \Lambda\,y = z. \end{cases}$$

Conversely, let us suppose that the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ can be generated by a family of $p-r$ generators $\{z_k = \pi(\Lambda_{k\bullet}) = \sum_{j=1}^{p}\Lambda_{kj}\,y_j\}_{k=1,\ldots,p-r}$ for a certain r satisfying $0 \leq r \leq p-1$ and a certain matrix $\Lambda \in D^{(p-r)\times p}$. Then, there exists a matrix $U \in D^{p\times(p-r)}$ such that $y_j = \sum_{k=1}^{p-r}U_{jk}\,z_k$ for all $j = 1, \ldots, p$, which yields $(I_p - U\,\Lambda)\,y = 0$, and thus there exists a matrix $V \in D^{p\times q}$ such that $I_p = U\,\Lambda + V\,R$, which proves that $P \triangleq (R^T \;\; \Lambda^T)^T \in D^{(q+p-r)\times p}$ admits a left inverse over D.

Lemma 14. *The finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ can be generated by $p-r$ elements, where r satisfies $0 \leq r \leq p-1$, iff there exists a matrix $\Lambda \in D^{(p-r)\times p}$ such that $P = (R^T \;\; \Lambda^T)^T \in D^{(q+p-r)\times p}$ admits a left inverse over D. Then, $\{\pi(\Lambda_{k\bullet})\}_{k=1,\ldots,p-r}$ is a family of generators of M, where $\pi : D^{1\times p} \longrightarrow M$ is the canonical projection onto M.*

Remark 15. If D is a commutative ring and the D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ can be generated by $p-r$ elements, then $\mathrm{Fitt}_{p-r}(M) = D$, where $\mathrm{Fitt}_{p-r}(M)$ denotes the $(p-r)$*th Fitting ideal* of M, namely, the ideal of D generated by the $r \times r$ minors of the matrix R. For more details, see Eisenbud (1995). Hence, if $\mathrm{Fitt}_{p-r}(M) \subsetneq D$, then M cannot be generated by $p-r$ elements.

We point out that checking the existence of $\Lambda \in D^{(p-r)\times p}$, where $r \geq 1$, such that $P = (R^T \;\; \Lambda^T)^T$ admits a left inverse over D is generally a difficult issue. However, if M is a *0-dimensional* $D = k[x_1, \ldots, x_n]$-module, namely, a finite-dimensional k-vector space, then an algorithm which tests the existence of Λ is given in Boudellioua and Quadrat (2010). Moreover, in Section 5, we shall prove that there always exists $\Lambda \in D^{1\times p}$ such that $P = (R^T \;\; \Lambda^T)^T$ admits a left inverse over D when the left

$D = A\langle\partial_1, \ldots, \partial_n\rangle$-module M is holonomic, where $A = k[x_1, \ldots, x_n]$ or $k[\![x_1, \ldots, x_n]\!]$ and k a field of characteristic 0, or $k\{x_1, \ldots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$.

Let us illustrate Lemma 14 with the class of linear time-varying first order OD systems. Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in a Noetherian differential integral domain $\left(A, \frac{d}{dt}\right)$, $F \in A^{n\times n}$, $R = \partial\, I_n - F \in D^{n\times n}$. Using Lemma 14, the left D-module $M = D^{1\times n}/(D^{1\times n}\, R)$ can be generated by $p \triangleq n - r$ elements $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$, where $0 \leq r \leq n-1$ and $H \in D^{p\times n}$, iff $P = ((\partial\, I_n - F)^T \;\; H^T)^T$ admits a left inverse over D, i.e., iff the left D-module $E \triangleq D^{1\times n}/(D^{1\times(n+p)}\, P)$ is reduced to 0. In terms of generators and relations, the left D-module E is generated by $\{x_i\}_{i=1,\ldots,n}$ which satisfies:

$$\begin{cases} \partial\, x - F\, x = 0, \\ H\, x = 0. \end{cases} \tag{5}$$

Since $\partial\, x = F\, x$ and $F \in A^{n\times n}$, without loss of generality, we can take $H \in A^{p\times n}$. Pre-multiplying the second equation of (5) by ∂ and taking into account the first equation of (5), we obtain $H\,\partial\, x + \dot{H}\, x = 0$, where $\dot{H}$ is the matrix obtained by applying $\frac{d}{dt}$ to each entry of H, i.e., $\dot{H} = (\dot{H}_{ij})_{i=1,\ldots,p, j=1,\ldots,n}$, and thus $(H\, F + \dot{H})\, x = 0$. We can now repeat the same operations with this new zero-order equation and so on. We obtain:

$$(5) \quad \Leftrightarrow \quad \begin{cases} \partial\, x - F\, x = 0, \\ H_i\, x = 0, \quad \forall\, i \in \mathbb{N}, \end{cases}$$

where the matrices H_i's are inductively defined by:

$$\begin{cases} H_0 = H, \\ H_{i+1} = H_i\, F + \dot{H}_i. \end{cases} \tag{6}$$

Let $\mathcal{L}_j \triangleq \sum_{i=0}^{j} A^{1\times p}\, H_i$ be the A-submodule of the left A-module $A^{1\times n}$ generated by the H_i's for $i = 1, \ldots, j$. Since $\mathcal{L}_j \subseteq \mathcal{L}_{j+1}$ for all $j \in \mathbb{N}$, the sequence $(\mathcal{L}_j)_{j\in\mathbb{N}}$ of A-submodules of the Noetherian A-module $A^{1\times n}$ stabilizes, namely, there exists $s \in \mathbb{N}$ such that $\mathcal{L}_{s+j} = \mathcal{L}_s = \sum_{i=0}^{s} A^{1\times p}\, H_i$ for all $j \in \mathbb{N}$. Therefore, we get:

$$(5) \quad \Leftrightarrow \quad \begin{cases} \partial\, x - F\, x = 0, \\ H_0\, x = 0, \\ \quad \vdots \\ H_s\, x = 0. \end{cases}$$

If $\mathcal{L}_s = A^{1\times n}$, i.e., if $(H_0^T \ldots H_s^T)^T$ admits a left inverse over A, then (5) yields $x = 0$, i.e., $E = D^{1\times n}/(D^{1\times(n+p)}\, P) = 0$, which shows that P admits a left inverse over D, and thus M is generated by $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$ by Lemma 14.

Conversely, let us suppose that there exists a left inverse $(X \;\; Y)$ of the matrix P, where $X \in D^{n\times n}$ and $Y \in D^{n\times p}$, i.e., $X\,(\partial\, I_n - F) + Y\, H = I_n$. Using (6), we have:

$$\forall\, i \in \mathbb{N}, \quad \partial\, H_i = H_i\,\partial + \dot{H}_i = H_i\,\partial + H_{i+1} - H_i\, F = H_i\,(\partial\, I_n - F) + H_{i+1}.$$

Moreover, we have:

$$\begin{aligned} \forall\, i \in \mathbb{N}, \quad \partial^2\, H_i &= \partial\,(\partial\, H_i) = \partial\,(H_i\,(\partial\, I_n - F) + H_{i+1}) = (\partial\, H_i)\,(\partial\, I_n - F) + \partial\, H_{i+1} \\ &= (H_i\,(\partial\, I_n - F) + H_{i+1})\,(\partial\, I_n - F) + H_{i+1}\,(\partial\, I_n - F) + H_{i+2} \\ &= H_i\,(\partial\, I_n - F)^2 + 2\, H_{i+1}\,(\partial\, I_n - F) + H_{i+2}. \end{aligned}$$

More generally, we can inductively prove:

$$\forall\, i \in \mathbb{N}, \quad \forall\, l \in \mathbb{N}, \quad \partial^l\, H_i = \sum_{j=0}^{l} \frac{l!}{j!\,(l-j)!}\, H_{i+j}\,(\partial\, I_n - F)^{l-j}. \tag{7}$$

If $Y = \sum_{l=0}^{s} C_l\,\partial^l$, where $C_l \in A^{n\times p}$, then using $H_0 = H$ and (7) with $i = 0$, we obtain

$$\forall\, l \in \mathbb{N}, \quad Y\,H = \sum_{l=0}^{s} C_l\,\partial^l H = \sum_{l=0}^{s}\sum_{j=0}^{l} \frac{l!}{j!\,(l-j)!}\,C_l\,H_j\,(\partial\,I_n - F)^{l-j}$$
$$= \sum_{l=0}^{s} C_l\,H_l + Z\,(\partial\,I_n - F),$$

for a certain matrix Z, which implies that $X\,(\partial\,I_n - F) + Y\,H = I_n$ is equivalent to:

$$(X+Z)\,(\partial\,I_n - F) + \sum_{l=0}^{s} C_l\,H_l = I_n. \tag{8}$$

Since each entry of the matrix $\partial\,I_n - F$ has degree 1 in ∂ and $C_l\,H_l$ has degree 0 in ∂ for $l = 1,\ldots,s$, the identity (8) can only hold when $X = -Z$, which yields $\sum_{l=0}^{s} C_l\,H_l = I_n$ and shows that the existence of a left inverse of P over D implies the existence of a left inverse of the matrix $(H_0^T \ldots H_s^T)^T$ over A.

Corollary 16. *Let $\left(A,\,\frac{d}{dt}\right)$ be a Noetherian differential integral domain, $F \in A^{n\times n}$, $D = A\langle\partial\rangle$ the ring of OD operators with coefficients in A and $R = \partial\,I_n - F \in D^{n\times n}$. Then, the finitely presented left D-module $M = D^{1\times n}/(D^{1\times n}\,R)$ can be generated by p elements $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$, where $H \in A^{p\times n}$, iff the matrix $P = (R^T \;\; H^T)^T$ admits a left inverse over D, i.e., iff there exists $s \in \mathbb{N}$ such that the matrix $(H_0^T \ldots H_s^T)^T$ admits a left inverse over A, where the matrices H_i's are defined by (6).*

If we want to check whether or not the left D-module M can be generated by $\{\pi(H_{k\bullet})\}_{k=1,\ldots,p}$, then we first have to saturate (5) (*formal integrability* of (5)), namely, find $s \in \mathbb{N}$ such that $\mathcal{L}_{s+j} = \mathcal{L}_s$ for all $j \in \mathbb{N}$, and then test whether or not the matrix $(H_0^T \ldots H_s^T)^T$ admits a left inverse over A. For instance, if $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then the last step can be achieved by checking whether or not $\mathrm{rank}_k((H_0^T \ldots H_s^T)^T(0))$ is equal to n (an element a of the local ring A is invertible iff $a \notin (t)$, i.e., $a(0) \neq 0$).

In control theory (see, e.g., Kalman et al., 1969; Sontag, 1998), the state x of $\dot{x} = F\,x$ is said to be *observable* from the output $y \triangleq H\,x$ if x can be expressed by means of y and its derivatives, which means that $P = ((\partial\,I_n - F)^T \;\; H^T)^T$ admits a left inverse over D. Using the above results, x is observable by $y = H\,x$ iff $M = D^{1\times n}/(D^{1\times n}\,R)$ can be generated by $\{y_k = \pi(H_{k\bullet}) = \sum_{j=1}^{n} H_{kj}\,x_j\}_{k=1,\ldots,p}$, i.e., $M = \sum_{k=1}^{p} D\,y_k$. Moreover, in control theory, the sequence of matrices H_i's is called the *observability distribution*, and the condition that there exists $s \in \mathbb{N}$ such that the matrix $(H_0^T \ldots H_s^T)^T$ admits a left inverse over A is the *observability condition*. Hence, the search for a generating set of M can be interpreted as the search for outputs $y = H\,x$ of the linear system $\dot{x} = F\,x$ such that x is observable. Finally, if $A = k[t]$ or $k[\![t]\!]$, where k is a field of characteristic 0, or $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, then Theorem 12 shows that the state x of $\dot{x} = F\,x$ can always be observed by a single output $y = H\,x$, where $H \in A^{1\times n}$ is a row vector which can be computed by means of Algorithm 3 of Leykin (2004) in the case of $A = k[t]$, where k is a computable field of characteristic 0 (see Remark 13).

If A is now a differential field (e.g., $A = k\{t\}[t^{-1}]$, where $k = \mathbb{R}$ or $\mathbb{C}$), then, in the literature of linear OD systems, a vector $H \in A^{1\times n}$ is called a *cyclic vector* of $\dot{x} = F\,x$ if $\det(H_0^T \ldots H_{n-1}^T)^T \neq 0$, where the row vectors H_i's are defined by (6). In particular, if $A = k(t)$, where k is a field of characteristic 0, then a cyclic vector always exists for $\dot{x} = F\,x$, where $F \in A^{n\times n}$ (Churchill and Kovacic , 2002; Cope, 1936).

Finally, if $\frac{d}{dt}$ is a trivial derivation of A, i.e., $\frac{d}{dt}\,a = \dot{a} = 0$ for all $a \in A$ (e.g., $A = \mathbb{R}$), then (6) yields $H_i = H\,F^i$ for all $i \in \mathbb{N}$, and the Cayley–Hamilton theorem (see, e.g., Rotman, 2009) for a commutative ring A then shows that $F^n = \sum_{i=0}^{n-1} a_i\,F^i$ for certain $a_i \in A$, and thus $\mathcal{L}_{n-1+i} = \mathcal{L}_{n-1}$ for all $i \in \mathbb{N}$, so that we can take $s = n-1$.

Let us state a right module analogue of Lemma 14: the right D-module $N = D^q/(R\,D^p)$ can be generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$, where $\Lambda \in D^{q\times(q-r)}$ and $\Lambda_{\bullet i}$ denotes the *ith* column of Λ, iff $P \triangleq (R \quad -\Lambda)$ admits a right inverse over D, i.e., iff the right D-module $D^q/(P\,D^{(p+q-r)}) = 0$. If R has full row rank, then 2 of Theorem 5 shows that the left D-module $E \triangleq D^{1\times(p+q-r)}/(D^{1\times q}\,P)$ is stably free iff P admits a right inverse over D. Moreover, the right D-module N depends only on M since we

can easily prove that N is equal to the right *extension* D-module $\mathrm{ext}^1_D(M, D)$ (up to isomorphism) (see, e.g., Rotman, 2009). The right D-module N is called the *Auslander transpose* of M (see, e.g., Chyzak et al., 2005). The above results are particular instances of the following result.

Theorem 17 (*Boudellioua and Quadrat, 2010*). *Let $R \in D^{q\times p}$ be a full row rank matrix with entries in a Noetherian domain D, $0 \leq r \leq q-1$, $\Lambda \in D^{q\times(q-r)}$, $P = (R \quad -\Lambda)$ and $M = D^{1\times p}/(D^{1\times q}R)$ (resp., $E = D^{1\times(p+q-r)}/(D^{1\times q}P)$) the left D-module finitely presented by R (resp., P) defining the following* short exact sequence

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{\alpha} E \xrightarrow{\beta} M \longrightarrow 0,$$

namely, α is injective, β is surjective and $\ker\beta = \mathrm{im}\,\alpha$. *Then, the following results are equivalent:*

(1) *The left D-module E is stably free of rank $p-r$.*
(2) *The matrix $P = (R \quad -\Lambda) \in D^{q\times(p+q-r)}$ admits a right inverse over D, i.e., there exists a matrix $S \in D^{(p+q-r)\times q}$ such that $PS = I_q$.*
(3) $D^q/\left(PD^{(p+q-r)}\right) = 0$.
(4) *$\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(RD^p)$, where the right D-homomorphism $\tau : D^q \longrightarrow D^q/(RD^p)$ is the canonical projection onto $\mathrm{ext}^1_D(M, D)$ and $\Lambda_{\bullet i}$ is the ith column of the matrix Λ.*

Finally, the previous results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in

$$\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right) \triangleq D^{q\times(q-r)}/\left(RD^{p\times(q-r)}\right), \tag{9}$$

i.e., they depend only on the following row vector:

$$(\tau(\Lambda_{\bullet 1}) \ldots \tau(\Lambda_{\bullet(q-r)})) \in \mathrm{ext}^1_D(M, D)^{1\times(q-r)}.$$

One of the main points of Theorem 17 is that the equivalences depend only on the residue class $\rho(\Lambda)$ of Λ in the right D-module $\mathrm{ext}^1_D\left(M, D^{1\times(q-r)}\right)$, i.e., $\overline{\Lambda} \triangleq \Lambda + RX$ can be taken instead of Λ for all matrices $X \in D^{p\times(q-r)}$.

Remark 18. If we take $r = q-1$, i.e., $\Lambda \in D^q$, then Theorem 17 shows that the left D-module $E = D^{1\times(p+1)}/(D^{1\times q}P)$, where $P = (R \quad -\Lambda) \in D^{q\times(p+1)}$, is stably free of rank $p-q+1$ iff $\tau(\Lambda)$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(RD^{p+1})$, i.e., iff $\mathrm{ext}^1_D(M, D)$ is a cyclic right D-module. This result was first pointed out by Serre (1986).

Remark 19. If the ring D admits an *involution* θ, namely, a map $\theta : D \longrightarrow D$ satisfying

$$\forall\, d_1, d_2 \in D, \quad \theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \quad \theta(d_1 d_2) = \theta(d_2)\theta(d_1), \quad \theta^2 = \mathrm{id}_D,$$

then $(R \quad -\Lambda)$ admits a right inverse over D iff $(\widetilde{R}^T \quad -\widetilde{\Lambda}^T)^T$ admits a left inverse over D, where $\widetilde{R} \triangleq (\theta(R_{ij}))^T \in D^{p\times q}$ and $\widetilde{\Lambda} \triangleq (\theta(\Lambda_{ij}))^T \in D^{(q-r)\times q}$ (see Chyzak et al., 2005). Hence, the right D-module $N = D^q/(RD^p)$ can be generated by $\{\tau(\Lambda_{\bullet i})\}_{i=1,\ldots,q-r}$ iff the left D-module $\widetilde{N} \triangleq D^{1\times q}/(D^{1\times p}\widetilde{R})$ can be generated by $\left\{\kappa\left(\widetilde{\Lambda}_{i\bullet}\right)\right\}_{i=1,\ldots,q-r}$, where $\kappa : D^{1\times q} \longrightarrow \widetilde{N}$ is the canonical projection onto $\widetilde{N}$. For instance, if D is a commutative ring, then $\theta = \mathrm{id}_D$ is the trivial involution of D. An involution θ of $D = A\langle\partial_1, \ldots, \partial_n\rangle$ is:

$$\forall\, a \in A, \quad \theta(a) = a, \qquad \forall\, i = 1, \ldots, n, \quad \theta(\partial_i) = -\partial_i.$$

The matrix $\widetilde{R}$ is then called the *formal adjoint* of R and the finitely presented left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\widetilde{R})$ is the *adjoint* of the left D-module $M = D^{1\times p}/(D^{1\times q}R)$.

Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in a Noetherian differential integral domain $\left(A, \frac{d}{dt}\right)$, $F \in A^{n\times n}$, $G \in A^{n\times m}$, $R = \partial I_n - F$ and $M = D^{1\times n}/(D^{1\times n}R)$. Theorem 17 then shows that the right D-module $\mathrm{ext}^1_D(M, D) = D^n/(RD^n)$ is generated by $\{\tau(G_{\bullet i})\}_{i=1,\ldots,m}$, where $\tau : D^n \longrightarrow D^n/(RD^n)$ is the canonical projection, iff the matrix $P \triangleq (\partial I_n - F \quad -G) \in D^{n\times(n+m)}$ admits a right inverse over D, i.e., iff the left D-module $E = D^{1\times(n+m)}/(D^{1\times n}P)$ is stably free. Using the involution θ of D defined by $\theta(a) = a$ for all $a \in A$ and $\theta(\partial) = -\partial$, Remark 19 then shows that

P admits a right inverse over D iff $\widetilde{P} = -(\partial\, I_n + F^T \quad G^T)^T$ admits a left inverse over D, i.e., iff the left D-module $\widetilde{E} = D^{1\times n}/(D^{1\times(n+m)}\,(-\widetilde{P}))$ defined by

$$\begin{cases} \partial\, x = -F^T x, \\ G^T x = 0, \end{cases}$$

is reduced to 0. Using the results obtained at the beginning of the section, the right D-module $\mathrm{ext}^1_D(M, D) = D^n/(R\,D^n)$ is generated by $\{\tau(G_{\bullet i})\}_{i=1,\ldots,m}$ iff the increasing sequence $\mathcal{L}_j = \sum_{i=0}^{j} A^{1\times m} H_i$ of A-submodules of the Noetherian A-module $A^{1\times n}$ satisfies $\mathcal{L}_s = A^{1\times n}$ for a certain $s \in \mathbb{N}$, where the matrices H_i's are defined by

$$\begin{cases} H_0 = G^T, \\ H_{i+1} = H_i\, F^T - \dot{H}_i, \quad \forall\, i \in \mathbb{N}, \end{cases}$$

i.e., iff the matrix $(H_0^T \ldots H_s^T)^T$ admits a left inverse over A, and thus iff the matrix $(G_0 \ldots G_s)$ admits a right inverse over A, where the matrices G_i's are defined by:

$$\begin{cases} G_0 = G, \\ G_{i+1} = F\, G_i - \dot{G}_i, \quad \forall\, i \in \mathbb{N}. \end{cases} \tag{10}$$

Corollary 20. *Let $\left(A, \frac{d}{dt}\right)$ be a Noetherian differential integral domain, $F \in A^{n\times n}$, $D = A\langle\partial\rangle$ the ring of OD operators with coefficients in A and $R = \partial\, I_n - F \in D^{n\times n}$. The finitely presented right D-module $N = D^n/(R\,D^n)$ can be generated by m elements $\{\pi(G_{\bullet k})\}_{k=1,\ldots,m}$, where $G \in A^{n\times m}$, iff the matrix $P = (R \quad G)$ admits a right inverse over D, i.e., iff there exists $s \in \mathbb{N}$ such that the matrix $(G_0 \ldots G_s)$ admits a right inverse over A, where the matrices G_i's are defined by (10).*

In control theory (see Kalman et al., 1969; Sontag, 1998), a linear OD system

$$\dot{x} = F\, x + G\, u \tag{11}$$

is called *controllable* on $[t_0, t_1]$ if for every $x_1 \in \mathbb{R}^n$, there exists an essentially bounded function $u : [t_0, t_1] \longrightarrow \mathbb{R}^m$ such that $x(t_1) = x_1$, where x satisfies (11) with the initial condition $x(t_0) = 0$. The sequence of matrices G_i's is called the *controllability distribution* and the condition that there exists $s \in \mathbb{N}$ such that the matrix $(G_0 \ldots G_s)$ admits a right inverse is the *controllability condition*.

If $\frac{d}{dt}$ is a trivial derivation of A, i.e., $\frac{d}{dt}\, a = \dot{a} = 0$ for all $a \in A$, then (10) yields $G_{i+1} = F^{i+1}\, G$ for all $i \in \mathbb{N}$. Using the Cayley–Hamilton theorem, the controllability condition is then equivalent to the existence of a right inverse of the following matrix:

$$\Omega \triangleq (G \quad F\,G \quad F^2\,G \quad \ldots \quad F^{n-1}\,G) \in A^{n\times n\,m}. \tag{12}$$

Example 21. Let us consider the trivial derivation $\frac{\partial}{\partial t}$ of $A = \mathbb{Q}\langle\partial_x\rangle = \mathbb{Q}[\partial_x]$, the commutative polynomial ring $D = A\langle\partial_t\rangle = \mathbb{Q}[\partial_x, \partial_t]$ of PD operators in ∂_t and ∂_x,

$$F = \begin{pmatrix} 0 & \partial_x + 1 \\ \partial_x^2 & 0 \end{pmatrix} \in A^{2\times 2},$$

$R = \partial_t\, I_2 - F \in D^{2\times 2}$ (Lee and Zak, 1983) and the D-module $M = D^{1\times 2}/(D^{1\times 2}\,R)$. Then, by Theorem 17, $N = D^2/(R\,D^2)$ is a cyclic D-module iff there exists a column vector $G = (G_1 \quad G_2)^T \in A^2$ such that $P = (R \quad -G)$ admits a right inverse over A. Using (12), we then get:

$$\Omega = \begin{pmatrix} G_1 & (\partial_x + 1)\, G_2 \\ G_2 & \partial_x^2\, G_1 \end{pmatrix}.$$

If $\det \Omega = (\partial_x\, G_1)^2 - (\partial_x + 1)\, G_2^2$ is different from zero, then $\det \Omega$ is a polynomial in ∂_x of degree at least 1, and thus Ω is not invertible in A, which proves that the D-module N is not cyclic. Now, if we consider

$$F' = \begin{pmatrix} 0 & 1 \\ \partial_x^2\,(\partial_x + 1) & 0 \end{pmatrix} \in A^{2\times 2}, \tag{13}$$

$R' = \partial_t\, I_2 - F' \in D^{2\times 2}$ (Lee and Zak, 1983) and $M' = D^{1\times 2}/(D^{1\times 2}\, R')$, then

$$\Omega' = \begin{pmatrix} G_1 & G_2 \\ G_2 & \partial_x^2\,(\partial_x + 1)\, G_1 \end{pmatrix},$$

and thus $\det \Omega' = \partial_x^2\,(\partial_x + 1)\, G_1^2 - G_2^2$, which shows that $N' = D^2/(R'\, D^2)$ is a cyclic D-module generated by $\tau((0 \;\; 1)^T)$. Finally, since $N = \text{ext}_D^1(M, D)$ and $N' = \text{ext}_D^1(M', D)$, we deduce that M and M' are not isomorphic D-modules.

Let us give a necessary and sufficient condition for the existence of Serre's reduction.

Theorem 22 (Boudellioua and Quadrat, 2010)**.** *Let $R \in D^{q\times p}$ be a full row rank matrix with entries in a Noetherian domain D, $0 \leq r \leq q-1$ and $\Lambda \in D^{q\times(q-r)}$ such that there exists $U \in \text{GL}_{p+q-r}(D)$ satisfying $(R \;\; -\Lambda)\, U = (I_q \;\; 0)$. If*

$$U = \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix}, \tag{14}$$

where $S_1 \in D^{p\times q}$, $S_2 \in D^{(q-r)\times q}$, $Q_1 \in D^{p\times(p-r)}$ and $Q_2 \in D^{(q-r)\times(p-r)}$, and if we introduce the left D-module $L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2)$ finitely presented by the full row rank matrix Q_2, i.e., defined by the following short exact sequence

$$0 \longrightarrow D^{1\times(q-r)} \xrightarrow{.Q_2} D^{1\times(p-r)} \xrightarrow{\kappa} L \longrightarrow 0, \tag{15}$$

then we have:

$$M = D^{1\times p}/(D^{1\times q}\, R) \;\cong\; L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2). \tag{16}$$

Conversely, if M is isomorphic to a left D-module L defined by the short exact sequence (15), then there exist two matrices $\Lambda \in D^{q\times(q-r)}$ and $U \in \text{GL}_{p+q-r}(D)$ satisfying:

$$(R \;\; -\Lambda)\, U = (I_q \;\; 0).$$

Corollary 23 (Boudellioua and Quadrat, 2010)**.** *With the notations of Theorem 22, the left D-isomorphism (16) obtained in Theorem 22 is defined by*

$$\begin{aligned} M = D^{1\times p}/(D^{1\times q}\, R) &\xrightarrow{\varphi} L = D^{1\times(p-r)}/(D^{1\times(q-r)}\, Q_2) \\ \pi(\lambda) &\longmapsto \kappa(\lambda\, Q_1), \end{aligned}$$

and its inverse $\varphi^{-1} : L \longrightarrow M$ is defined by $\varphi^{-1}(\kappa(\mu)) = \pi(\mu\, T_1)$, where

$$U^{-1} = \begin{pmatrix} R & -\Lambda \\ T_1 & -T_2 \end{pmatrix} \in \text{GL}_{p+q-r}(D),$$

$T_1 \in D^{(p-r)\times p}$ and $T_2 \in D^{(p-r)\times(q-r)}$. These results depend only on the residue class $\rho(\Lambda)$ of $\Lambda \in D^{q\times(q-r)}$ in the right D-module $\text{ext}_D^1\left(M, D^{1\times(q-r)}\right)$ defined by (9).

A straightforward consequence of Corollary 23 is the following result.

Corollary 24 (Boudellioua and Quadrat, 2010)**.** *Let $\mathcal{F}$ be a left D-module and:*

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}, \quad \ker_{\mathcal{F}}(Q_2.) = \{\zeta \in \mathcal{F}^{(p-r)} \mid Q_2\,\zeta = 0\}.$$

Then, we have $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(Q_2.)$ and, more precisely:

$$\ker_{\mathcal{F}}(R.) = Q_1\, \ker_{\mathcal{F}}(Q_2.), \qquad \ker_{\mathcal{F}}(Q_2.) = T_1\, \ker_{\mathcal{F}}(R.).$$

Corollary 25 (Boudellioua and Quadrat, 2010)**.** *Let $R \in D^{q\times p}$ be a full row rank matrix and $\Lambda \in D^{q\times(q-r)}$ such that $P = (R \;\; -\Lambda)$ admits a right inverse over D. Then, Theorem 22 and Corollary 23 hold when D satisfies one of the following properties:*

(1) *D is a principal left ideal domain (e.g., the ring $A\langle\partial\rangle$ of OD operators with coefficients in a differential field A).*
(2) *$D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a field k.*

(3) *D is $A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, and $p - r \geq 2$.*
(4) *$D = A\langle\partial\rangle$ is the ring of OD operators, where $A = k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, and $p - r \geq 2$.*

Example 26. Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in a Noetherian differential integral domain $\left(A, \frac{d}{dt}\right)$, $F \in A^{n\times n}$, $G \in A^{n\times m}$, $R = \partial\, I_n - F$ and the D-module $M = D^{1\times n}/(D^{1\times n}\, R)$. If $P = (\partial\, I_n - F \quad -G)$ admits a right inverse over D, then the left $D = A\langle\partial\rangle$-module $E = D^{1\times(n+m)}/(D^{1\times n}\, P)$ is stably free by 2 of Theorem 5.

(1) If A is a differential field (e.g., $A = \mathbb{R}$, $\mathbb{R}(t)$, $\mathbb{R}\{t\}[t^{-1}]$), then 1 of Corollary 25 shows that there exists a matrix $Q_2 \in D^{m\times m}$ such that:

$$M = D^{1\times n}/(D^{1\times n}\,(\partial\, I_n - F)) \cong L \triangleq D^{1\times m}/(D^{1\times m}\, Q_2). \tag{17}$$

(2) If $A = k[t]$ or $k[\![t]\!]$, where k is a field of characteristic 0, or $A = k\{t\}$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $m = n - r \geq 2$, then there exists a matrix $Q_2 \in D^{m\times m}$ such that (17) holds by 4 of Corollary 25.
(3) If $\frac{d}{dt}$ is a trivial derivation of the ring $A = k[x_1, \ldots, x_n]$, where k is a field, then, by 2 of Corollary 25, there exists $Q_2 \in D^{m\times m}$ such that (17) holds. For instance, if we consider again the $D = A[\partial_t] = \mathbb{Q}[\partial_x, \partial_t]$-module $M' = D^{1\times 2}/(D^{1\times 2}\, R')$, where $R' = \partial_t\, I_2 - F'$ and F' is defined by (13), then, in Example 21, we proved that the D-module $\mathrm{ext}^1_D(M', D) = N' = D^2/(R'\, D^2)$ is cyclic and generated by $\tau((0 \quad 1)^T)$. Using Theorem 17 and 2 of Theorem 4, we obtain that the stably free D-module $E = D^{1\times 3}/(D^{1\times 2}\, P)$, where $P = (R' \quad -\Lambda)$ and $\Lambda = (0 \quad 1)^T$, is free of rank 1. Using a constructive proof of the Quillen–Suslin theorem implemented in the QuillenSuslin package (Fabiańska and Quadrat, 2007), we get:

$$\begin{pmatrix} \partial_t & -1 & 0 \\ -\partial_x^2\,(\partial_x + 1) & \partial_t & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & \partial_t \\ -\partial_t & -1 & \partial_t^2 - \partial_x^2\,(\partial_x + 1) \end{pmatrix} = I_3. \tag{18}$$

Thus, we have $M \cong D/(D\, Q_2)$, where $Q_2 = \partial_t^2 - \partial_x^2\,(\partial_x + 1)$. Moreover, if we consider the D-module $\mathcal{F} = C^\infty(\mathbb{R}^2)$ or $\mathcal{D}'(\mathbb{R}^2)$, then:

$$\ker_{\mathcal{F}}(R'.) = \begin{pmatrix} 1 \\ \partial_t \end{pmatrix} \ker_{\mathcal{F}}(Q_2.), \qquad \ker_{\mathcal{F}}(Q_2.) = (1 \quad 0)\ \ker_{\mathcal{F}}(R'.).$$

Example 27. Let $D = A[\partial]$ be a commutative polynomial ring in ∂ with coefficients in a commutative ring A, $F \in A^{n\times n}$, $R = \partial\, I_n - F \in D^{n\times n}$ and $M = D^{1\times n}/(D^{1\times n}\, R)$. Let us suppose that there exists $G \in A^n$ such that the finitely presented D-module $E = D^{1\times(n+1)}/(D^{1\times n}\, P)$ is free of rank 1, where $P = (\partial\, I_n - F \quad -G)$. For $i = 1, \ldots, n+1$, let $m_i \triangleq \det \widehat{P}_i$ be the determinant of the $n \times n$ submatrix of P obtained by removing the *ith* column of P (e.g., $m_{n+1} = \det(\partial\, I_n - F)$). Since P admits a right inverse over D, we can prove that there exist $d_1, \ldots, d_{n+1} \in D$ such that $\sum_{i=1}^{n+1} d_i\, m_i = 1$ (see, e.g., Fabiańska and Quadrat, 2007). If $T_1 = (-1)^n\,(d_1 - d_2 \quad d_3 \ldots (-1)^{n+1}\, d_n)$ and $T_2 = -d_{n+1}$, then developing the determinant of the following matrix

$$\begin{pmatrix} \partial\, I_n - F & -G \\ T_1 & -T_2 \end{pmatrix} \in D^{(n+1)\times(n+1)}$$

along its last row and using Laplace's formula, we get that its determinant is 1, and the above unimodular matrix corresponds to the matrix U^{-1} defined in Corollary 23. Moreover, we have $Q_2 = \det(\partial\, I_n - F)$ and $Q_1 = (-1)^n\,(m_1 - m_2 \quad m_3 \ldots (-1)^{n+1}\, m_n)^T$. For more details, see, e.g., Fabiańska and Quadrat (2007). For instance, if we consider again 3 of Example 26, we get $m_1 = 1$, $m_2 = -\partial_t$, $m_3 = \det(\partial\, I_2 - F') = \partial_t^2 - \partial_x^2\,(\partial_x + 1)$, $d_1 = 1$, $d_2 = 0$ and $d_3 = 0$, and we find again (18).

Corollary 28 (*Boudellioua and Quadrat, 2010*)**.** *With the notations of Theorem 22 and Corollary 23, if the matrix $\Lambda \in D^{q\times(q-r)}$ admits a left inverse $\Gamma \in D^{(q-r)\times q}$, i.e., $\Gamma\, \Lambda = I_{q-r}$, then Q_1 admits the left inverse $T_1 - T_2\, \Gamma\, R \in D^{(p-r)\times p}$, and the left D-module $\ker_D(.Q_1)$ is stably free of rank r.*

Moreover, if the left D-module $\ker_D(.Q_1)$ *is free of rank r, then there exists a matrix* $Q_3 \in D^{p\times r}$ *such that* $W = (Q_3 \;\; Q_1) \in \mathrm{GL}_p(D)$. *If we write* $W^{-1} = (Y_3^T \;\; Y_1^T)^T$, *where* $Y_3 \in D^{r\times p}$ *and* $Y_1 \in D^{(p-r)\times p}$, *then the matrix* $X = (R\,Q_3 \;\; \Lambda)$ *is unimodular, i.e.,* $X \in \mathrm{GL}_q(D)$ *and:*

$$V = X^{-1} = \begin{pmatrix} Y_3 \\ Q_2\,Y_1 - Q_2\,Y_1\,S_1 - S_2 \end{pmatrix}.$$

Then, we have $R = X\,\mathrm{diag}(I_r, Q_2)\,W^{-1}$, *i.e., the matrix R is then equivalent to:*

$$V\,R\,W = \begin{pmatrix} I_r & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Finally, the left D-module $\ker_D(.Q_1)$ *is free when D satisfies* 1 *or* 2 *of Corollary 25 or if* $D = A_n(k)$ *or* $B_n(k)$, *where k is a field of characteristic 0, and* $r \geq 2$, *or if* $D = A\langle\partial\rangle$ *is the ring of OD operators with coefficients in* $A = k[\![t]\!]$, *where k a field of characteristic 0, or in* $A = k\{t\}$ *and* $k = \mathbb{R}$ *or* $\mathbb{C}$, *and* $r \geq 2$.

Example 29. Let us consider again the PD linear system defined by $R' = \partial_t\,I_2 - F'$, where F' is given by (13). In Example 26, we showed that we could take $\Lambda = (0 \;\; 1)^T$. Since Λ admits the left inverse $\Gamma = (0 \;\; 1)$ and $D = \mathbb{Q}[\partial_x, \partial_t]$ is a commutative polynomial ring over the field $\mathbb{Q}$, Corollary 25 shows that R' is equivalent to the diagonal matrix $\mathrm{diag}(1, Q_2)$, where $Q_2 = \partial_t^2 - \partial_x^2\,(\partial_x + 1)$. Let us compute two matrices $V, W \in \mathrm{GL}_2(D)$ such that $V\,R'\,W = \mathrm{diag}(1, Q_2)$. We have $\ker_D(.Q_1) = D\,K$, where $K = (\partial_t \;\; -1)$, and K admits the right inverse $Q_3 = (0 \;\; -1)^T$. Hence, if we introduce the following matrices

$$W = (Q_3 \;\; Q_1) = \begin{pmatrix} 0 & 1 \\ -1 & \partial_t \end{pmatrix} \in \mathrm{GL}_2(D), \qquad X = (R'\,Q_3 \;\; \Lambda) = \begin{pmatrix} 1 & 0 \\ -\partial_t & 1 \end{pmatrix} \in \mathrm{GL}_2(D),$$

then the matrix R' is equivalent to the following diagonal matrix:

$$X^{-1}\,R'\,W = \begin{pmatrix} 1 & 0 \\ 0 & \partial_t^2 - \partial_x^2\,(\partial_x + 1) \end{pmatrix}.$$

For more sophisticated examples, see Boudellioua and Quadrat (2010), Cluzeau and Quadrat (2010b) and Quadrat (2010).

As explained in Boudellioua and Quadrat (2010), the existence of Serre's reduction can be constructively checked when $\mathrm{ext}_D^1(M, D) = D^q/(R\,D^p)$ is a 0-dimensional $D = k[x_1, \ldots, x_n]$-module (k a computable field), namely, when $D^q/(R\,D^p)$ is a finite-dimensional k-vector space. The purpose of Section 5 is to study the corresponding case when $D = A_n(k)$ or $A\langle\partial_1, \ldots, \partial_n\rangle$, where $A = k[\![x_1, \ldots, x_n]\!]$ and k a field of characteristic 0, or $A = k\{x_1, \ldots, x_n\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, i.e., the case where the Auslander right D-module $N = D^q/(R\,D^p)$ of M is holonomic or, equivalently, the case where the adjoint left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$ of M is holonomic.

5. Serre's reduction of PD linear systems based on holonomy

We are now in position to state the main result of this paper.

Theorem 30. *Let* $D = A\langle\partial_1, \ldots, \partial_n\rangle$ *be the ring of PD operators with coefficients in* $A = k[x_1, \ldots, x_n]$ *or* $k[\![x_1, \ldots, x_n]\!]$, *where k is a field of characteristic 0, or in* $A = k\{x_1, \ldots, x_n\}$ *and* $k = \mathbb{R}$ *or* $\mathbb{C}$, $R \in D^{q\times p}$ *a full row rank matrix and* $M = D^{1\times p}/(D^{1\times q}\,R)$ *the left D-module finitely presented by R. If* $\mathrm{ext}_D^1(M, D) = D^q/(R\,D^p)$ *is a holonomic right D-module, then Theorem 17 holds and we can choose a column vector* $\Lambda \in D^q$ *which admits a left inverse over D and is such that* $\tau(\Lambda)$ *generates the right D-module* $D^q/(R\,D^p)$, *where* $\tau : D^q \longrightarrow D^q/(R\,D^p)$ *is the canonical projection onto* $\mathrm{ext}_D^1(M, D)$. *If* $A = k[x_1, \ldots, x_n]$ *and* $p - q \geq 1$, *then Theorem 22 and Corollaries 23 and 24 hold, i.e.,* $M \cong L = D^{1\times(p-q+1)}/(D\,Q_2)$ *for a certain row vector* $Q_2 \in D^{1\times(p-q+1)}$. *Finally, if* $q \geq 3$, *then Corollary 28 holds, i.e., the matrix R is equivalent to* $\mathrm{diag}(I_{q-1}, Q_2)$.

Proof. Since by hypothesis, $\mathrm{ext}^1_D(M, D)$ is a holonomic right D-module, Theorem 12 proves that $\mathrm{ext}^1_D(M, D)$ is cyclic and it can be generated by $\tau(\Lambda)$, where $\Lambda = (1 \ d_2 \ \ldots \ d_q)^T$, for certain d_i's in D. Using Remark 18, we obtain that $E = D^{1\times(p+1)}/(D^{1\times q}\, P)$, where $P = (R \ \ -\Lambda) \in D^{q\times(p+1)}$, is stably free of rank $p+1-q$. If $A = k[x_1, \ldots, x_n]$, i.e., $D = A_n(k)$, and $p+1-q \geq 2$, i.e., $p-q \geq 1$, then 3 of Theorem 4 shows that E is a free left D-module of rank $p+1-q$, and using 3 of Theorem 5, Theorem 22 holds. Moreover, $\Gamma = (1 \ 0 \ldots 0)$ is a left inverse of Λ, and thus Corollary 28 holds. Finally, if $r = q-1 \geq 2$, i.e., $q \geq 3$, then the stably free left D-module $\ker_D(.Q_1)$ of rank r is free by Stafford's theorem (see 3 of Theorem 4), and Corollary 28 proves that R is equivalent to $\mathrm{diag}(I_{q-1}, Q_2)$ for a certain row vector $Q_2 \in D^{1\times(p-q+1)}$. □

In the case of $D = A_n(k)$, where k is a field of characteristic 0, we can use Leykin (2004) and Quadrat and Robertz (2007) to obtain the following algorithm implemented in the Serre package (Cluzeau and Quadrat, 2010b).

Algorithm 1.
- **Input:** A full row rank matrix $R \in D^{q\times p}$ such that $p-q \geq 1$ and $N = D^q/(R\,D^p)$ is a holonomic right $D = A_n(k)$-module (k a field of characteristic 0).
- **Output:** A matrix $Q_2 \in D^{1\times(p-q+1)}$ such that $M \cong D^{1\times(p-q+1)}/(D\,Q_2)$. Moreover, if $q \geq 3$, then two more matrices $V \in \mathrm{GL}_q(D)$ and $W \in \mathrm{GL}_p(D)$ are returned such that:

$$V\,R\,W = \mathrm{diag}(I_{q-1}, Q_2).$$

(1) Use Algorithm 3 of Leykin (2004) to compute a vector $\Lambda = (1 \ \ d_2 \ \ \ldots \ d_q)^T$ such that $N = D^q/(R\,D^p) = \tau(\Lambda)\,D$, where $\tau : D^q \longrightarrow N$ is the canonical projection onto N, i.e., such that $N = (P\,D^{p+1})/(R\,D^p)$, where $P = (R \ \ -\Lambda) \in D^{q\times(p+1)}$.
(2) Using Algorithm 3 of Quadrat and Robertz (2007), compute two matrices

$$\begin{cases} Q = (Q_1^T \ \ Q_2^T)^T \in D^{(p+1)\times(p-q+1)}, \\ T = (T_1 \ \ T_2) \in D^{(p-q+1)\times(p+1)}, \end{cases}$$

where $Q_1 \in D^{p\times(p-q+1)}$, $Q_2 \in D^{1\times(p-q+1)}$, $T_1 \in D^{(p-q+1)\times p}$ and $T_2 \in D^{(p-q+1)\times 1}$, such that $\ker_D(.Q) = D^{1\times q}\,P$ and $T\,Q = I_{p-q+1}$ (i.e., compute an injective parametrization and a basis of the free left D-module $E = D^{1\times(p+1)}/(D^{1\times q}\,P)$ of rank $p+1-q \geq 2$).
(3) If $q \leq 2$, then return the matrix Q_2.
(4) Else, compute a matrix $K \in D^{r\times p}$ such that $\ker_D(.Q_1) = D^{1\times r}\,K$.
(5) Compute a matrix $L \in D^{s\times r}$ such that $\ker_D(.K) = D^{1\times s}\,L$.
(6) If $L = 0$, i.e., $\ker_D(.K) = 0$, then $r = q-1$.
  (a) Compute a right inverse $Q_3 \in D^{p\times(q-1)}$ of the matrix $K \in D^{(q-1)\times p}$.
  (b) Form the matrices $X = (R\,Q_3 \ \ \Lambda) \in \mathrm{GL}_q(D)$ and $W = (Q_3 \ Q_1) \in \mathrm{GL}_p(D)$.
  (c) Compute $V = X^{-1}$.
  (d) Return the matrices Q_2, V and W.
(7) Else, i.e., $L \neq 0$, then:
  (a) Using Algorithm 4 of Quadrat and Robertz (2007), compute $F \in D^{r\times(q-1)}$ and $G \in D^{(q-1)\times r}$ such that $\ker_D(.F) = D^{1\times s}\,L$ and $G\,F = I_{q-1}$ (i.e., compute an injective parametrization and a basis of the free left D-module $D^{1\times r}/(D^{1\times s}\,L) \cong \mathrm{im}_D(.K) = \ker_D(.Q_1)$ of rank $q-1 \geq 2$).
  (b) Form the full row rank matrix $G\,K \in D^{(q-1)\times p}$.
  (c) Compute a right inverse $Q_3 \in D^{p\times(q-1)}$ of the matrix $G\,K \in D^{(q-1)\times q}$.
  (d) Form the matrices $X = (R\,Q_3 \ \ \Lambda) \in \mathrm{GL}_q(D)$ and $W = (Q_3 \ \ Q_1) \in \mathrm{GL}_p(D)$.
  (e) Compute $V = X^{-1}$.
  (f) Return the matrices Q_2, V and W.

Example 31. Let us consider the commutative polynomial ring $D = \mathbb{Q}\left[\partial_x, \partial_y\right]$ of PD operators and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$ finitely presented by R defined by:

$$R = \begin{pmatrix} \partial_x & \partial_y & 0 \\ 0 & \partial_x & \partial_y \end{pmatrix} \in D^{2\times 3}. \tag{19}$$

The matrix R defines the equation $R\,\sigma = 0$ of the equilibrium of the *stress tensor* in $\mathbb{R}^2$:

$$\begin{cases} \partial_x\,\sigma^{11} + \partial_y\,\sigma^{12} = 0, \\ \partial_x\,\sigma^{12} + \partial_y\,\sigma^{22} = 0. \end{cases} \tag{20}$$

We can check that $\mathrm{ext}^1_D(M, D) = D^2/\left(R\,D^3\right)$ is a $\mathbb{Q}$-vector space of dimension 3 and a basis of $\mathrm{ext}^1_D(M, D)$ is defined by $\tau((1 \quad 0)^T)$, $\tau((0 \quad 1)^T)$ and $\tau((0 \quad \partial_x)^T)$, where $\tau : D^2 \longrightarrow D^2/(R\,D^3)$ is the canonical projection. Hence, without loss of generality, we can assume that Λ has the form $\Lambda = (a \quad b + c\,\partial_x)^T$, where the three indeterminates a, b and c have to be substituted by suitable constants. Considering the new ring $D' = \mathbb{Q}[a, b, c]\left[\partial_x, \partial_y\right]$, $P = (R \quad -\Lambda)$ and the D'-module $E = D'^{1\times 4}/(D'^{1\times 2}\,P)$, then, using Gröbner basis techniques, we can check that the matrix P does not admit a right inverse over D'. According to Theorem 5, we obtain that the D'-module E is not a stably free D'-module, which proves that (20) cannot be defined by a sole PD equation with constant coefficients, and the minimal number of generators $\mu(M)$ of the D-module M is 3.

Let $M' = B^{1\times 3}/(B^{1\times 2}\,R)$ be the left $B = A_2(\mathbb{Q})$-module finitely presented by R. The right B-module $\mathrm{ext}^1_B(M', B) = B^2/(R\,B^3)$ is holonomic and thus cyclic by Proposition 12. For instance, the element $\tau(\Lambda)$ of $\mathrm{ext}^1_B(M', B)$, where $\Lambda = (1 \quad x)^T$, generates $\mathrm{ext}^1_B(M', B)$ since the matrix $P = (R \quad -\Lambda) \in B^{2\times 4}$ admits the following right inverse:

$$S = \begin{pmatrix} -x & 1 \\ -x^2 & x \\ -x^3 & x^2 \\ -x\,(x\,\partial_y + \partial_x) - 2 & \partial_x + x\,\partial_y \end{pmatrix}.$$

Then, the left B-module $E' = B^{1\times 4}/(B^{1\times 2}\,P)$ is stably free of rank 2 (see Remark 18), i.e., free by Stafford's theorem (see 3 of Theorem 4). Using the Stafford package (Quadrat and Robertz, 2007), an injective parametrization of E' is defined by

$$Q = \begin{pmatrix} \partial_y & \partial_x \\ x\,\partial_y & x\,\partial_x - 1 \\ x^2\,\partial_y - 1 & x^2\,\partial_x - x \\ (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which yields $M' \cong B^{1\times 2}/(B\,((\partial_x + x\,\partial_y)\,\partial_y \quad (\partial_x + x\,\partial_y)\,\partial_x - \partial_y))$.

Since $\Gamma = (1 \quad 0)$ is a left inverse of Λ, using Corollary 28, we obtain the following unimodular matrices:

$$W = \begin{pmatrix} -1 & \partial_y & \partial_x \\ -x & x\,\partial_y & x\,\partial_x - 1 \\ -x^2 & x^2\,\partial_y - 1 & x\,(x\,\partial_x - 1) \end{pmatrix}, \qquad W^{-1} = \begin{pmatrix} x\,\partial_x & x\,\partial_y - \partial_x & -\partial_y \\ 0 & x & -1 \\ x & -1 & 0 \end{pmatrix},$$

$$X = \begin{pmatrix} -(\partial_x + x\,\partial_y) & 1 \\ -x\,(\partial_x + x\,\partial_y) - 1 & x \end{pmatrix}, \qquad V = X^{-1} = \begin{pmatrix} x & -1 \\ x^2\,\partial_y + x\,\partial_x + 2 & -(\partial_x + x\,\partial_y) \end{pmatrix}.$$

Then, the matrix R defined by (19) is equivalent to the following block-diagonal matrix

$$V\,R\,W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\partial_x + x\,\partial_y)\,\partial_y & (\partial_x + x\,\partial_y)\,\partial_x - \partial_y \end{pmatrix},$$

which proves that (20) is equivalent to

$$\tau_1 = 0, \quad (\partial_x + x\,\partial_y)\,\partial_y\,\tau_2 + (\partial_x + x\,\partial_y)\,\partial_x\,\tau_3 - \partial_y\,\tau_3 = 0,$$

under the following invertible transformations:

$$\begin{cases} \sigma^{11} = -\tau_1 + \partial_y\,\tau_2 + \partial_x\,\tau_3, \\ \sigma^{12} = -x\,\tau_1 + x\,\partial_y\,\tau_2 + x\,\partial_x\,\tau_3 - \tau_3, \\ \sigma^{22} = -x^2\,\tau_1 + x^2\,\partial_y\,\tau_2 - \tau_2 + x^2\,\partial_x\,\tau_3 - x\,\tau_3, \end{cases}$$

$$\begin{cases} \tau_1 = x\,(\partial_x\,\sigma^{11} + \partial_y\,\sigma^{12}) - (\partial_x\,\sigma^{12} + \partial_y\,\sigma^{22}) = 0, \\ \tau_2 = x\,\sigma^{12} - \sigma^{22}, \\ \tau_3 = x\,\sigma^{11} - \sigma^{12}. \end{cases}$$

We note that we have lost the symmetry of (20). It would be interesting to get a more symmetric equivalent linear PD equation by considering another cyclic vector of $\mathrm{ext}^1_B(M', B)$ and/or another injective parametrization of the free left D-module E' of rank 2.

If D is a Noetherian domain and $M = D^{1\times p}/(D^{1\times q}R)$ a left D-module finitely presented by a full row rank matrix R, then we can prove that the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(RD^p)$ is torsion (Chyzak et al., 2005). Then, using Theorem 11 and 4 of Corollary 25, we obtain the following corollary of Theorem 30.

Corollary 32. *Let $D = A\langle\partial\rangle$ be the ring of OD operators with coefficients in $A = k[t]$ or $k[\![t]\!]$ and k is a field of characteristic 0, or in $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, $R \in D^{q\times p}$ a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}R)$. Then, Theorem 17 holds and $\Lambda \in D^q$ can be chosen so that it admits a left inverse over D and $\tau(\Lambda)$ generates the right D-module $\mathrm{ext}^1_D(M, D) = D^q/(RD^p)$. Moreover, if $p - q \geq 1$, then Theorem 22 and Corollaries 23 and 24 hold. Finally, if $q \geq 3$, then Corollary 28 holds.*

In particular, Corollary 32 shows every real analytic/holomorphic linear OD system defined by a full row rank matrix with at least one column more than rows is isomorphic to a real analytic/holomorphic linear OD equation. Moreover, if the system has at least 3 equations, then the system is equivalent to a sole OD equation. These results are particularly meaningful in control theory. For instance, if we consider

$$\dot{x} = Fx + Gu, \tag{21}$$

where $F \in A^{n\times n}$ and $G \in A^{n\times m}$, $A = k[t]$ or $k[\![t]\!]$ and k is a field of characteristic 0, or $A = k\{t\}$ and $k = \mathbb{R}$ or $\mathbb{C}$, then we get $M = D^{1\times(n+m)}/(D^{1\times n}R)$, where $D = A\langle\partial\rangle$ and $R = (\partial I_n - F \quad -G) \in D^{n\times(n+m)}$. In particular, we have $p = n + m$, $q = n$ and $\mathrm{rank}_D(M) = p - q = m$. Therefore, if $m \geq 1$, i.e., if the dimension of the input vector u of (21) is at least 1, i.e., $G \neq 0$, then $M \cong D^{1\times(m+1)}/(DQ_2)$ for a certain row vector $Q_2 \in D^{1\times(m+1)}$. Moreover, if $n \geq 3$, i.e., if the dimension of the state vector x of (21) is at least 3, then (21) is equivalent to the linear OD equation $Q_2\,\zeta = 0$.

Since the rings $D = B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$, where k is a field of characteristic 0, or $k\{t\}[t^{-1}]\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, are *simple* principal left ideal domains (see, e.g., Björk, 1979; McConnell and Robson, 2000), using the concept of *Jacobson normal form*, namely, a generalization of the *Smith normal form* to principal left or right ideal domains (see, e.g., Culianez, 2005; Levandovskyy and Schindelar, 2011; Zerz, 2006), for every matrix $R \in D^{q\times p}$, there exist $V \in \mathrm{GL}_q(D)$, $W \in \mathrm{GL}_p(D)$ and $d \in D$ such that

$$VRW = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0),$$

i.e., R is equivalent to the diagonal matrix $\overline{R} = \mathrm{diag}(1, \ldots, 1, d, 0, \ldots, 0)$, for a certain $d \in D \setminus \{0\}$. In particular, if R has full row rank, then R is equivalent to $\mathrm{diag}(1, \ldots, 1, d)$.

Now, if $D = A_1(k)$, $k[\![t]\!]\langle\partial\rangle$, where k is a field of characteristic 0, or $D = k\{t\}\langle\partial\rangle$, where $k = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q\times p}$, then the Jacobson normal form of R can be computed by considering the injection of D into the simple principal left ideal domain D', where D' is respectively $B_1(k)$, $k[\![t]\!][t^{-1}]\langle\partial\rangle$ and $k\{t\}[t^{-1}]\langle\partial\rangle$. Therefore, there exist $V \in \mathrm{GL}_q(D')$, $W \in \mathrm{GL}_p(D')$ and $e \in D'$ such that $VRW = \mathrm{diag}(1, \ldots, 1, e, 0, \ldots, 0)$. However, singularities may have been introduced in e, V and W. Corollary 32 shows that there always exist three matrices $Q_2 \in D^{1\times(p-q+1)}$, $X \in \mathrm{GL}_q(D)$ and $Y \in \mathrm{GL}_p(D)$ such that $XRY = \mathrm{diag}(I_{q-1}, Q_2)$. Since the entries of Q_2, X, Y, X^{-1} and Y^{-1} belong to D, no singularity can appear. For the computation of Jacobson normal forms and implementations, see Blinkov et al. (2003), Culianez (2005) and Levandovskyy and Schindelar (2011).

Example 33. Let $M = D^{1\times 4}/(D^{1\times 3}R)$ be the left $D = A_1(\mathbb{Q})$-module finitely presented by the following matrix of OD operators with polynomial coefficients:

$$R = \begin{pmatrix} t\,\partial + 2 & 0 & \partial & \partial \\ \partial + t^2 & 0 & \partial^2 + 1 & t \\ t^2(\partial^2 - 1) + 2t\,\partial & -t\,\partial & t\,\partial & t\,\partial^2 - t \end{pmatrix} \in D^{3\times 4}. \tag{22}$$

Using Algorithm 3 of Leykin (2004), we obtain that the column vector $\Lambda = (0 \;\; 0 \;\; 1)^T$ is such that the matrix $P = (R \;\; -\Lambda)$ admits the following right inverse:

$$S = \begin{pmatrix} \partial^2+1 & -\partial & -\partial & -t\,(\partial^2+1) & 0 \\ -\partial & 1 & 1 & t\,\partial & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}^T.$$

In other words, the right D-module $\mathrm{ext}^1_D(M, D) = D^3/(R\,D^4)$ is cyclic and is generated by $\tau(\Lambda)$, and thus the left D-module $E = D^{1\times 5}/(D^{1\times 3}\,P)$ is stably free of rank 2, i.e., is free of rank 2 by Stafford's theorem (see 3 of Theorem 4). Computing an injective parametrization of E by means of the StafFord package, we obtain that the matrix $Q = (Q_1^T \;\; Q_2^T)^T \in D^{5\times 2}$, where

$$Q_1 = \begin{pmatrix} \partial^3 + (1-t)\,\partial - 1 & 0 \\ -\partial^2 - \partial + t & 1 \\ -\partial^2 + t & 0 \\ -t\,\partial^3 + t\,(t-1)\,\partial + t - 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} t & -t\,\partial \end{pmatrix},$$

satisfies $\ker_D(.Q) = D^{1\times 3}\,P$ and $T\,Q = I_2$, where:

$$T = \begin{pmatrix} -t & 0 & 0 & -1 & 0 \\ 1 & 1 & \partial - 1 & 0 & 0 \end{pmatrix}.$$

Thus, we get $M \cong D^{1\times 2}/(D\,Q_2)$. Moreover, since Λ admits the left inverse $\Gamma = (0 \;\; 0 \;\; 1)$, R is equivalent to $\mathrm{diag}(I_2, Q_2)$. More precisely, we have $\ker_D(.Q_1) = D^{1\times 2}\,K$, where

$$K = \begin{pmatrix} t\,\partial + 2 & 0 & \partial & \partial \\ \partial + t^2 & 0 & \partial^2 + 1 & t \end{pmatrix},$$

and the right inverse Q_3 of K, defined by

$$Q_3 = \begin{pmatrix} \partial^2+1 & 0 & -\partial & -t\,(\partial^2+1) \\ -\partial & 0 & 1 & t\,\partial \end{pmatrix}^T,$$

is such that $W = (Q_3 \;\; Q_1) \in \mathrm{GL}_4(D)$. Then, we have:

$$X = (R\,Q_3 \;\; \Lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t\,\partial^2 & t\,\partial & 1 \end{pmatrix}, \qquad V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t\,\partial^2 & -t\,\partial & 1 \end{pmatrix}.$$

We obtain that the matrix R is equivalent to the following block-diagonal matrix:

$$V\,R\,W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & -t\,\partial \end{pmatrix}.$$

If we compute a Jacobson normal form J of the matrix R, then we get

$$J = Y\,R\,Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

for certain large matrices Y and Z with entries in $B_1(\mathbb{Q})$, i.e., $Y \in \mathrm{GL}_3(B_1(\mathbb{Q}))$ and $Z \in \mathrm{GL}_4(B_1(\mathbb{Q}))$. Finally, we note that the left D-module $L = D^{1\times 2}/(D\,Q_2)$ has the non-trivial torsion element $z = z_1 - \partial\,z_2$, where $z_1 = \kappa((1 \;\; 0))$ and $z_2 = \kappa((0 \;\; 1))$ are the generators of L, which satisfies $t\,(z_1 - \partial\,z_2) = 0$, whereas the left $D' \triangleq B_1(\mathbb{Q})$-module $D'^{1\times 2}/(D'\,Q_2)$ is free of rank 1.

Finally, we note that the conditions $p - q \geq 1$ and $q \geq 3$ in Corollary 32 are only here to ensure that the stably free left $D = A\langle\partial\rangle$-modules $E = D^{1\times(p+1)}/(D^{1\times q}\,(R \;\; -\Lambda))$ and $\ker_D(.Q_1)$ are free of rank respectively $p - q + 1$ and $r = q - 1$ by 3 and 4 of Theorem 4. But, it can happen that the stably free left $D = A\langle\partial\rangle$-modules E and $\ker_D(.Q_1)$ are both free when these conditions are not fulfilled (e.g., the square case $p = q$).

Example 34. Let us consider Example 5.3 of Levandovskyy and Schindelar (2011):

$$R = \begin{pmatrix} \partial^2 & \partial+1 & 0 \\ \partial+1 & 0 & \partial^3 - t^2\,\partial \\ 2\,\partial+1 & \partial^3+\partial^2 & \partial^2 \end{pmatrix} \in A_1(\mathbb{Q})^{3\times 3}.$$

Let $D = A_1(\mathbb{Q})$, $\Lambda = (0 \quad 1 \quad 0)^T$, $P = (R \quad -\Lambda) \in D^{3\times 4}$ and the left D-module $E = D^{1\times 4}/(D^{1\times 3}\,P)$. Using the OreModules package (Chyzak et al., 2007), we can check that P admits a right inverse over D, which proves that E is a stably free left D-module of rank 1. Hence, Stafford's theorem (3 of Theorem 4) cannot be applied here to conclude that E is free. However, using techniques developed in Chyzak et al. (2005) and Pommaret and Quadrat (1999) on *minimal parametrizations*, we can check that

$$Q = \begin{pmatrix} \partial^2\,(\partial+1) \\ -\partial^4 \\ (\partial+1)\,(\partial^4 - 2\,\partial - 1) \\ (\partial^6 - t^2\,\partial^4 - 2\,\partial^3 + (2\,t^2+1)\,\partial + t^2)\,(\partial+1)\,\partial \end{pmatrix}$$

is such that $\ker_D(.Q) = D^{1\times 3}\,P$ and $T\,Q = 1$, where:

$$T = (-\partial + 7 \quad 3\,\partial^2 + 2\,\partial - 2 \quad 3\,\partial - 1 \quad 0).$$

Hence, $E = \operatorname{coker}_D(.P) \cong \operatorname{im}_D(.Q) = D$, which proves that the left D-module E is free of rank 1, $(P^T \quad T^T)^T \in \mathrm{GL}_4(D)$ and $U = ((P^T \quad T^T)^T)^{-1}$ is such that $P\,U = (I_3 \quad 0)$. Theorem 22 then yields $M = D^{1\times 3}/(D^{1\times 3}\,R) \cong L = D/(D\,Q_2)$, where

$$Q_2 = (\partial^6 - t^2\,\partial^4 - 2\,\partial^3 + (2\,t^2+1)\,\partial + t^2)\,(\partial+1)\,\partial$$

is the last row of Q. Moreover, since Λ admits a left inverse, the left D-module $\ker_D(.Q_1)$ is stably free of rank 2, where Q_1 is the matrix formed by the three first rows of Q. We have $\ker_D(.Q_1) = D^{1\times 2}\,K$, where K is the full row rank matrix defined by:

$$K = \begin{pmatrix} \partial^2 & \partial+1 & 0 \\ 2\,\partial+1 & \partial^2\,(\partial+1) & \partial^2 \end{pmatrix}.$$

Computing a right inverse Q_3 of the matrix K, we obtain:

$$Q_3 = \begin{pmatrix} \partial^2 & -3\,\partial^2 - 2\,\partial + 1 \\ -\partial^3 + \partial^2 - \partial + 1 & 3\,\partial^3 - \partial^2 \\ \partial^4 - 2\,\partial - 2 & -3\,\partial^4 - 2\,\partial^3 + \partial^2 + 6\,\partial + 7 \end{pmatrix}.$$

Using Corollary 28, we then get:

$$\begin{aligned} W &= (Q_3 \quad Q_1) \\ &= \begin{pmatrix} \partial^2 & -3\,\partial^2 - 2\,\partial + 1 & \partial^2\,(\partial+1) \\ -\partial^3 + \partial^2 - \partial + 1 & (3\,\partial - 1)\,\partial^2 & -\partial^4 \\ \partial^4 - 2\,\partial - 2 & -3\,\partial^4 - 2\,\partial^3 + \partial^2 + 6\,\partial + 7 & \partial^5 + \partial^4 - 2\,\partial^2 - 3\,\partial - 1 \end{pmatrix}, \\ W^{-1} &= \begin{pmatrix} \partial^2 & \partial+1 & 0 \\ 2\,\partial+1 & \partial^2\,(\partial+1) & \partial^2 \\ -\partial+7 & 3\,\partial^2 + 2\,\partial - 2 & 3\,\partial - 1 \end{pmatrix}, \\ X = (R\,Q_3 \quad \Lambda) &= \begin{pmatrix} 1 & 0 & 0 \\ a & b & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad V = X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -a & 1 & -b \end{pmatrix}, \end{aligned}$$

$$\begin{cases} a = (\partial^6 - t^2\,\partial^4 - 2\,\partial^3 - \partial^2 + (2\,t^2+1)\,\partial + 2\,t^2)\,\partial, \\ b = -3\,\partial^7 - 2\,\partial^6 + (3\,t^2+1)\,\partial^5 + 2\,(3+t^2)\,\partial^4 - (t^2-4)\,\partial^3 \\ \qquad -(6\,t^2+5)\,\partial^2 - (7\,t^2+1)\,\partial + 1. \end{cases}$$

The matrix R is then equivalent to $V\,R\,W = \operatorname{diag}(I_2, Q_2)$. We note that the matrices Q_2, V and W are much simpler than the ones obtained in Example 5.3 of Levandovskyy and Schindelar (2011).

Indeed, using their Example 5.3, the OD operator Q_2 computed by SINGULAR (resp., JANET (Blinkov et al., 2003)), JACOBSON (Culianez, 2005) has also order 8 but contains 34 (resp., 146, 146) terms, and the matrix V computed by SINGULAR (resp., JANET, JACOBSON) has order 13 (resp., 10, 10) and a term of V contains 45 (resp., 183, 183) terms. More importantly, the matrices V and W computed by either SINGULAR, JANET and JACOBSON are unimodular over $B_1(\mathbb{Q})$, namely, $V, W \in \mathrm{GL}_3(B_1(\mathbb{Q}))$, i.e., they or their inverses contain singularities in t, whereas the ones obtained above are unimodular over $A_1(\mathbb{Q})$, i.e., $V, W \in \mathrm{GL}_3(A_1(\mathbb{Q}))$, and they and their inverses are free of singularities.

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