Asymptotic and Structural Stability for a Linear 2D Discrete Roesser Model

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***Abstract*—Recently we have shown, by providing an explicit counterexample, that the structural stability of a linear 2D discrete Fornasini-Marchesini model is not equivalent to its asymptotic stability when dealing with boundary conditions on the positive axes. The main contribution of the present paper shows that this fact remains valid when dealing with linear 2D discrete Roesser models. Using the notion of equivalence in the sense of the algebraic analysis approach to linear systems theory, we recall that a Fornasini-Marchesini model can always be transformed into an equivalent Roesser model. We then prove that asymptotic stability is preserved by this particular equivalence transformation. We therefore deduce an example of a Roesser model which is asymptotically stable but not structurally stable.**

I. Introduction

Recently, a lot of attention has been devoted to stability questions in the field of multidimensional systems. It started with the study of nonlinear 2D systems with several authors introducing either new definitions of stability or extending existing Lyapunov theorems to the case of nonlinear 2D discrete Roesser models ([1], [2], [3]). Naturally, studies dealing with 2D continuous models followed ([4], [5], [6]). As the field mainly deals with two particular models, the Fornasini-Marchesini model [7] and the Roesser model [8], the stability definitions are usually derived for either of these models. Moreover, the 2D models can be either continuous, discrete or both, leading to a huge number of different definitions and stability criteria.

So far, three main ideas of stability have been extended from the 1D to the 2D case: the notions of asymptotic stability, exponential stability and, for linear models, structural stability. Here structural stability corresponds to the location of the roots of a "characteristic polynomial" as it will be explicitly recalled in the present paper. In this context, it is important to understand the relationships between the definitions and see the differences with the 1D case that has been heavily analyzed (see, for instance, [9, Chapter 4]).

Our goal with the present paper is to extend a recent result [10] that shows rather unexpected differences between the 1D and the 2D case. Indeed, in [10], we consider a linear 2D discrete Fornasini-Marchesini model and we show, that depending on the choice of the boundary conditions, structural stability and asymptotic stability are not equivalent contrary to a common belief that started with a misinterpretation of the work [7]. The idea of the current paper is to prove that this fact remains valid when we work with a linear 2D discrete Roesser model. More precisely, we show that, for a given Roesser model, asymptotic stability does not necessarily imply structural stability if we choose the boundary conditions on the positive axes.

In order to extend the result of [10], we will use the notion of equivalence in the sense of the algebraic analysis approach to linear systems theory as it has been developed in [11], [12]. The idea is then simple: we use the equivalence between models to transform the particular counterexample given in [10] into a Roesser model that is asymptotically stable but not structurally stable (see Proposition IV.2). Indeed, it has been shown in [11] that a Fornasini-Marchesini model can always be transformed into an equivalent Roesser model. Then the difficulty mainly lies in the boundary conditions when transforming one model into another. Interestingly enough, using the explicit transformations corresponding to an equivalence in the sense of algebraic analysis, we prove that the latter particular equivalence transformation of a Fornasini-Marchesini model into a Roesser model preserves the asymptotic stability (see Proposition IV.1).

The paper is organized as follows. After clarifying some notations used in the paper, we will shortly introduce the idea behind the equivalence in the sense of the algebraic analysis approach to linear systems theory in Section II. Then, in Section III we introduce the models considered in the sequel and we give the definitions of stability used. Finally, Section IV contains our main contributions, namely, Propositions IV.1 and IV.2.

Notations: We clarify here some notations used in the present paper. Let $N \in \mathbb{N}^*$.

- $\mathbb{R}^N$ is the vector space of dimension N over the field $\mathbb{R}$. It will be endowed with some norm $|\cdot|$. Moreover, $\mathbb{R}^{N\times N}$ refers the set of square matrices of dimension N. We will note $\|\cdot\|$ the *matrix* norm induced by $|\cdot|$ on this set. If $A,\, B \in \mathbb{R}^{N\times N}$, we then have: $\|A\,B\| \leq \|A\|\,\|B\|$.

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- $\|\cdot\|_\infty$ corresponds to the supremum of all values of a sequence (with one or two indexes).
- $\overline{\mathbb{D}} = \{z \in \mathbb{C} / \, |z| \leq 1\}$ is the closed unit disc in the complex plane $\mathbb{C}$.
- I_N is the identity matrix of dimension N.

II. Algebraic analysis and equivalence

The algebraic analysis, or *behavioral*, approach to linear systems theory is a general mathematical framework to study a huge class of linear systems that can be discrete or continuous, determined, overdetermined, or underdetermined, ... For more details on the algebraic analysis approach, we refer to [13], [14], [15], [16], [17] and references therein.

Within the algebraic analysis approach, a linear system is represented by a matrix R with entries in a ring of functional operators that is in general non-commutative. This is simply due to the fact that any linear system of q functional equations in p unknown functions (or sequences) $\eta_1, \ldots, \eta_p$ can be written as:

$$R\,\eta = 0,$$

where R is a $q \times p$ matrix with entries in a ring of functional operators acting on the p unknown functions (or sequences) η_i's, and $\eta = (\eta_1 \ldots \eta_p)^T$.

In the present paper, we shall restrict ourselves to linear 2D discrete systems. In this context, a linear system is then given by a matrix of functional operators belonging to the commutative polynomial ring $D = \mathbb{R}[\sigma_1, \sigma_2]$[1] in two shift operators σ_1 and σ_2. Namely, $d = \sum_{k,l} p_{k,l}\, \sigma_1^k\, \sigma_2^l \in D$ with $p_{k,l} \in \mathbb{R}$ and $\sum_{k,l}$ is a finite sum, acts on a bivariate sequence $f(i,j)$ as $d.f(i,j) = \sum_{k,l} p_{k,l}\, f(i+k, j+l)$.

To the matrix $R \in D^{q \times p}$ defining a linear system, we then associate the finitely presented D-module ([18]):

$$M := D^{1\times p}/(D^{1\times q}\, R).$$

If $\mathcal{F}$ is a functional space having a left D-module structure, we consider the *linear system* (or *behavior*)

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\},$$

and we have Malgrange's isomorphism ([13]):

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}),$$

which shows that system properties of $\ker_{\mathcal{F}}(R.)$ can be studied by means of module properties of M (and $\mathcal{F}$).

Using this framework, equivalent linear systems naturally correspond to isomorphic modules.

Definition II.1. *Let $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$ be two matrices and $M = D^{1\times p}/(D^{1\times q}\, R)$, $M' = D^{1\times p'}/(D^{1\times q'}\, R')$ the associated D-modules. If $\mathcal{F}$ is a D-module, then the linear systems $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ are said to be* equivalent *if the D-modules M and M' are isomorphic [18], i.e., $M \cong M'$.*

By definition, this notion of equivalence has the advantage to preserve the invariants of the D-modules which has important consequences in the study of systems properties (see [16]). However, when stability issues are concerned, the preservation of stability properties by this kind of equivalence, is not so clear. This is mainly due to the fact that the algebraic analysis approach to linear systems theory makes no distinction between the different system variables that are all gathered in the vector η as above. Nevertheless, for stability issues, the input, state, and output variables do not play the same role. In [12], we have studied the preservation of the notion of structural stability by this kind of equivalence. In the present paper, we shall study the preservation of asymptotic stability in the particular case when we study a Fornasini-Marchesini model and an equivalent Roesser model ([11]). To achieve this, following the lines of [12], we shall take advantage of the fact that an equivalence transformation in the sense of algebraic analysis provides explicitly the (not necessarily static) transformations (change of variables) between the variables of the equivalent systems. This is recalled in the following lemma which effectively characterizes (homo)morphisms and isomorphisms between finitely presented D-modules in terms of matrices.

Lemma II.2 ([19])**.** *Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and consider the associated D-modules $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$.*

1) *The existence of a homomorphism $f \in \hom_D(M, M')$ is equivalent to the existence of two matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ satisfying the identity:*

$$R\,P = Q\,R'.$$

Then, the homomorphism $f \in \hom_D(M, M')$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in D^{1\times p}$, where $\pi : D^{1\times p} \to M$ and $\pi' : D^{1\times p'} \to M'$ denote the canonical projections onto M and M'.

2) *With the previous notations, f is an isomorphism, i.e., $M \cong M'$ if and only if there exist four matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$, and $Z' \in D^{p'\times q'}$ satisfying:*

$$R'\,P' = Q'\,R,\ P\,P' + Z\,R = I_p,\ P'\,P + Z'\,R' = I_{p'}.$$

In this case, the invertible changes of variables $\eta = P\,\eta'$ and $\eta' = P'\,\eta$ provide a one-to-one correspondence between solutions of $R\,\eta = 0$ and solutions of $R'\,\eta' = 0$, i.e., we have: $R\,\eta = 0 \Longleftrightarrow R'\,\eta' = 0$.

III. Models and stability

A. The equivalent models considered

Let $A,\ B \in \mathbb{R}^{N\times N}$. In this paper, we consider the following linear 2D discrete Fornasini-Marchesini model ([7]) of dimen-

[1] Note that as soon as symbolic computation aspects are concerned, the field $\mathbb{R}$ of real numbers should be replaced by the field $\mathbb{Q}$ of rational numbers.

sion N: $\forall\,(i,j)\in\mathbb{N}^2$,

$$x(i+1,j+1)=A\,x(i,j+1)+B\,x(i+1,j)\ \in\mathbb{R}^N. \quad (1)$$

In [11], it has been proved that (1) is equivalent in the sense of algebraic analysis to the following linear 2D discrete Roesser model ([8]): $\forall\,(i,j)\in\mathbb{N}^2$,

$$\begin{pmatrix} x^h(i+1,j)\\ x^v(i,j+1)\end{pmatrix}=\begin{pmatrix} A & AB\\ I_N & B\end{pmatrix}\begin{pmatrix} x^h(i,j)\\ x^v(i,j)\end{pmatrix}\ \in\mathbb{R}^{2N}, \quad (2)$$

which is obtained by setting $x^h(i,j)=x(i,j+1)-B\,x(i,j)$ and $x^v(i,j)=x(i,j)$.

Within the framework of Section II, the models (1) and (2) respectively correspond to:

$$R=(\sigma_1\sigma_2 I_N-\sigma_2 A-\sigma_1 B)\in D^{N\times N},\quad \eta=x,$$

$$R'=\begin{pmatrix}\sigma_1 I_N-A & -AB\\ -I_N & \sigma_2 I_N-B\end{pmatrix}\in D^{2N\times 2N},\quad \eta'=\begin{pmatrix} x^h\\ x^v\end{pmatrix}.$$

Moreover, using the notations of Lemma II.2, the one-to-one correspondence between solutions of $R\,\eta=0$ and solutions of $R'\,\eta'=0$ is explicitly given by the matrices:

$$P=\begin{pmatrix}0 & I_N\end{pmatrix}\in D^{N\times 2N}, \quad (3)$$

and

$$P'=\begin{pmatrix}\sigma_2 I_N-B\\ I_N\end{pmatrix}\in D^{2N\times N}. \quad (4)$$

We refer to [11], [12] for more details.

B. Asymptotic and structural stability

In the sequel, we shall consider stability properties of the Fornasini-Marchesini and Roesser models defined above. More precisely, we will focus on the notions of *asymptotic* and *structural* stability.

To be able to define clearly asymptotic stability, we should first choose boundary conditions. In general, there is neither an obvious nor a unique choice and there are papers explaining the reasons behind the choices that have been made: see, for instance, [7], [20], [3]. However, we must notice that this choice has crucial consequences for stability issues (see [10]). In the present paper, we will respectively consider the boundary conditions

$$\forall\,(i,j)\in\mathbb{N}^2,\quad x(0,j)=\Psi_1(j),\quad x(i,0)=\Psi_2(i), \quad (5)$$

for the model (1) and the boundary conditions

$$\forall\,(i,j)\in\mathbb{N}^2,\quad x^h(0,j)=\widetilde{\Psi}_1(j),\quad x^v(i,0)=\widetilde{\Psi}_2(i), \quad (6)$$

for the model (2), where Ψ_1, Ψ_2, $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ are one-index sequences in $\mathbb{R}^N$. This is enough to ensure the existence of a unique solution x to (1) (respectively $\widetilde{x}:=(x^h,x^v)$ to (2)) satisfying (5) (respectively (6)). The solutions x and $\widetilde{x}:=(x^h,x^v)$ are sequences with two indexes and go respectively to $\mathbb{R}^N$ and $\mathbb{R}^{2N}$.

For notational convenience, we will write Ψ to design the boundary conditions given by (5):

$$\forall\,(i,j)\in\mathbb{N}^2,\ \Psi(i,j)=(\Psi_1(j),\Psi_2(i))\,.$$

Note that $\Psi\to 0$ is equivalent to $\Psi_k\to 0$ for $k\in\{1,2\}$. Moreover, if the boundary conditions are bounded, we have that: $\|\Psi\|_\infty\le\|\Psi_1\|_\infty+\|\Psi_2\|_\infty$. Same remarks hold for $\widetilde{\Psi}$.

Definition III.1 ([3])**.** *The system* (1) *submitted to the boundary conditions* (5) *is said to be* asymptotically stable *if the solution* x

1) *is* stable*:* $\forall\,\varepsilon>0,\ \exists\,\delta_\varepsilon>0$ *such that*

$$\|\Psi\|_\infty\le\delta_\varepsilon\quad\Rightarrow\quad\|x\|_\infty\le\varepsilon. \quad (7)$$

2) *is* attractive*:*

$$\Psi\xrightarrow[\infty]{}0\quad\Rightarrow\quad x\xrightarrow[\infty]{}0. \quad (8)$$

There is an analogous definition for the Roesser model (2) submitted to the boundary conditions (6).

Note that the notation in (8) to design the convergence to zero of a sequence at infinity means in mathematical terms:

$$\forall\,\varepsilon>0,\ \exists\,N_\varepsilon\in\mathbb{N}\ /\ \forall\,i,\,j\ge N_\varepsilon,\ |x(i,j)|\le\varepsilon.$$

Contrary to asymptotic stability, the structural stability of a linear system is an algebraic notion that does not depend on the choice of boundary conditions:

Definition III.2 ([12])**.**

- *The system* (1) *is said to be* structurally stable *if:*

$$\forall\,(\lambda,\mu)\in\overline{\mathbb{D}}^2,\ \det\big(I_N-\lambda A-\mu B\big)\neq 0.$$

- *The system* (2) *is said to be* structurally stable *if:*

$$\forall\,(\lambda,\mu)\in\overline{\mathbb{D}}^2,\quad \det\left(I_{2N}-\begin{pmatrix}\lambda I_N & 0\\ 0 & \mu I_N\end{pmatrix}\begin{pmatrix}A & AB\\ I_N & B\end{pmatrix}\right)\neq 0.$$

From [12], we know that the structural stability of (1) is equivalent to the structural stability of (2). Note that here, this fact can also be verified directly by inspecting the determinants. The first contribution of the present paper (see Proposition IV.1 below) will be to prove that, in a similar way, the asymptotic stability of (1) is equivalent to the asymptotic stability of (2) when asymptotic stability is defined as in Definition III.1.

IV. Main Results

Our first contribution is given by the following proposition:

Proposition IV.1. *With the notations and definitions of the latter section III, the linear 2D discrete Fornasini-Marchesini model* (1) *is asymptotically stable if and only if the linear 2D discrete Roesser model* (2) *is asymptotically stable.*

Proof. We will proceed by double implication.

• First, let us suppose that (1) is asymptotically stable and we will show that (2) is also asymptotically stable. Let $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ be the boundary conditions of (2). We will note

$$\widetilde{x} := \left(x^h, x^v\right)$$

its unique solution satisfying those boundary conditions (i.e: (6) is fulfilled).

From (4), we can notice that:

$$\forall\,(i,j) \in \mathbb{N}^2,\ x^h(i,j) = x^v(i,j+1) - B\,x^v(i,j). \tag{9}$$

Then, we easily see[2] that x^v is solution to the supposed asymptotically stable Fornasini-Marchesini model (1) submitted to the boundary conditions: $\forall\,(i,j) \in \mathbb{N}^2$,

$$\begin{cases} \Psi_2(i) := x^v(i,0) = \widetilde{\Psi}_2(i), & \text{(10a)} \\ \Psi_1(j) := x^v(0,j) = B^j\widetilde{\Psi}_2(0) + \displaystyle\sum_{k=0}^{j-1} B^{j-1-k}\widetilde{\Psi}_1(k). & \text{(10b)} \end{cases}$$

(with the convention that the sum is reduced to zero if $j = 0$)

∘ Let us deal first with the stability feature of $\widetilde{x}$. By hypothesis, for any $\varepsilon > 0$, we can find a $\delta_\varepsilon > 0$ such that the solution x of (1) submitted to the boundary conditions Ψ verifies:

$$\|\Psi\|_\infty \le \delta_\varepsilon \quad \Rightarrow \quad \|x\|_\infty \le \varepsilon. \tag{11}$$

Let us fix an $\widetilde{\varepsilon} > 0$ and find a $\widetilde{\delta_\varepsilon} > 0$ such that

$$\|\widetilde{\Psi}\|_\infty \le \widetilde{\delta_\varepsilon} \quad \Rightarrow \quad \|\widetilde{x}\|_\infty \le \widetilde{\varepsilon}.$$

We begin by noticing with (10b) that for all $j \in \mathbb{N}^*$:

$$|\Psi_1(j)| = |x^v(0,j)| \le \underbrace{\left(\|B\|^j + \sum_{k=0}^{j-1}\|B\|^k\right)}_{:=\ v(j)} \|\widetilde{\Psi}\|_\infty.$$

From now on, we will consider the matrix norm given in the lemma A.1 (in order to have $\|B\| < 1$). Therefore, we see that the sequence v is bounded, so we can find a positive constant $M > 0$ such that the last inequality implies:

$$\|\Psi_1\|_\infty \le M\|\widetilde{\Psi}\|_\infty.$$

Thus, with (10a), we can deduce that:

$$\|\Psi\|_\infty \le (1+M)\,\|\widetilde{\Psi}\|_\infty. \tag{12}$$

On the other hand, we can see with (9) that:

$$\|\widetilde{x}\|_\infty \le (2+\|B\|)\,\|x^v\|_\infty. \tag{13}$$

Then, we choose $\varepsilon := (2+\|B\|)^{-1}\,\widetilde{\varepsilon}$ (which gives us access to a δ_ε) and $\widetilde{\delta_\varepsilon} := (1+M)^{-1}\,\delta_\varepsilon$. Finally we put all the pieces together:

$$\|\widetilde{\Psi}\|_\infty \le \widetilde{\delta_\varepsilon} \quad \overset{(12)}{\Rightarrow} \quad \|\Psi\|_\infty \le \delta_\varepsilon \quad \overset{(11)}{\Rightarrow} \quad \|x^v\|_\infty \le \varepsilon,$$

so we can conclude this part with (13):

$$\|\widetilde{x}\|_\infty \le (2+\|B\|)\,\varepsilon = \widetilde{\varepsilon},$$

i.e: (7) is satisfied.

∘ Now, let us see the attractivity feature of $\widetilde{x}$. In order to do so, we suppose that $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ (the boundary conditions to (2)) converge to zero and we will see that the solution $\widetilde{x}$ behaves similarly. We deduce from the lemma A.2 that the two sequences given by (10a) and (10b) go to zero at infinity. Thus, the attractivity of the model (1) implies that its solution, namely x^v, converges to zero too. We still have to check that $x^h \to 0$. This is straightforward with the following inequality (obtained from (9)):

$$\left|x^h(i,j)\right| \le |x^v(i,j+1)| + \|B\|\,|x^v(i,j)| \xrightarrow[\infty]{} 0.$$

Therefore (8) holds and we can conclude that (2) is asymptotically stable.

• Now, let us suppose that (2) is asymptotically stable. We begin with the stability condition (7) and we will see in a second time the attractivity (8).

We fix Ψ_1 and Ψ_2 boundary conditions to (1) and we will call x its unique solution satisfying those boundary conditions (i.e: (5) is fulfilled). Let us write for all $(i,j) \in \mathbb{N}^2$:

$$\begin{cases} x^h(i,j) := x(i,j+1) - B\,x(i,j), & \text{(14a)} \\ x^v(i,j) := x(i,j). & \text{(14b)} \end{cases}$$

Then, it is easy to check with (1) that the sequence $\widetilde{x} := (x^h, x^v)$, defined by (14a)-(14b), is solution to the supposed asymptotically stable Roesser model (2) under the following boundary conditions: $\forall\,(i,j) \in \mathbb{N}^2$,

$$\begin{cases} \widetilde{\Psi}_1(j) := x^h(0,j) = \Psi_1(j+1) - B\,\Psi_1(j), & \text{(15a)} \\ \widetilde{\Psi}_2(i) := x^v(i,0) = \Psi_2(i). & \text{(15b)} \end{cases}$$

∘ Let us look for stability first. For a fixed $\varepsilon > 0$ and for any $\widetilde{\varepsilon} > 0$, by hypothesis there is a $\widetilde{\delta_\varepsilon}$ such that:

$$\|\widetilde{\Psi}\|_\infty \le \widetilde{\delta_\varepsilon} \quad \Rightarrow \quad \|\widetilde{x}\|_\infty \le \widetilde{\varepsilon}. \tag{16}$$

On the other hand, we can deduce from (15a) and (15b) that:

$$\|\widetilde{\Psi}\|_\infty \le (2+\|B\|)\,\|\Psi\|_\infty. \tag{17}$$

Thus, we set $\widetilde{\varepsilon} := \varepsilon$ and $\delta_\varepsilon := (2+\|B\|)^{-1}\,\widetilde{\delta_\varepsilon}$ and we deduce that:

$$\|\Psi\|_\infty \le \delta_\varepsilon \quad \overset{(17)}{\Rightarrow} \quad \|\widetilde{\Psi}\|_\infty \le \widetilde{\delta_\varepsilon} \quad \underset{(14b)}{\overset{(16)}{\Rightarrow}} \quad \|x\| \le \|\widetilde{x}\|_\infty \le \varepsilon,$$

which concludes this part.

∘ As we did previously, to check the attractivity, we suppose that the initial conditions converge to zero and we prove that so does the solution. Then, with (15a):

$$\begin{aligned} |\widetilde{\Psi}_1(j)| &= |\Psi_1(j+1) - B\,\Psi_1(j)|\,, \\ &\le |\Psi_1(j+1)| + \|B\|\,|\Psi_1(j)| \to 0, \end{aligned}$$

[2] We use the first equation of (2) into the second one, then (9) will help to get rid of the remaining x^h-term.

and with (15b):

$$|\widetilde{\Psi}_2(i)| = |\Psi_2(i)| \to 0,$$

so, by hypothesis on the model (2), we deduce that $\widetilde{x} \to 0$. In particular: $x = x^v \to 0$, so (1) is attractive. The proof is complete. □

We already know from [10] that an asymptotically stable Fornasini-Marchesini model is not necessary structurally stable (however the converse is true). The main contribution of the present paper given in the following proposition shows that the latter situation remains the same if we deal with a Roesser model. Namely, using the notion of equivalence introduced in Section II together with Proposition IV.1, we provide a counterexample to the statement "an asymptotically stable Roesser model is structurally stable".

Proposition IV.2. *If a linear 2D discrete Roesser model is asymptotically stable in the sense of Definition III.1, then it does not imply that it is structurally stable in the sense of Definition III.2.*

Proof. Let us consider the particular scalar Fornasini-Marchesini model (1), where $A = B = 1/2$. We know from [10] that it is asymptotically stable but not structurally stable. From Section III (see also [11]), this particular Fornasini-Marchesini model is equivalent to the Roesser model given by: $\forall\,(i,j) \in \mathbb{N}^2$,

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} \in \mathbb{R}^2. \quad (18)$$

Now, on one hand Proposition IV.1 implies that the latter Roesser model (18) is asymptotically stable whereas on the other hand [12, Theorem 1] implies that (18) is not structurally stable, which can also be checked directly using Definition III.2 since, for $\lambda = \mu = 1$, we have $\det \begin{pmatrix} 1-\lambda/2 & -\lambda/4 \\ -\mu & 1-\mu/2 \end{pmatrix} = 0$. This ends the proof. □

V. Conclusions and perspectives

In our previous work [12], we have studied the preservation of structural stability by equivalence transformation in the sense of the algebraic analysis approach to linear systems theory. In the present paper, we first prove that asymptotic stability is preserved in a special case of equivalent models (Proposition IV.1). Using this we are then able to transform the counterexample given in [10] in order to provide an explicit Roesser model which is asymptotically stable when dealing with boundary conditions on the positive axes but not structurally stable (Proposition IV.2).

It would be interesting to see if the notion of attractivity is generally preserved when dealing with any two equivalent linear systems as it is the case for the particular Fornasini-Marchesini model and the equivalent Roesser model considered in this paper. Moreover the preservation of the notion of *exponential stability* by equivalence transformation is also an interesting question for future works.

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Appendix

Lemma A.1. *If* (1) *is asymptotically stable, there is a matrix norm* $\|\cdot\|$ *such that* $\|B\| < 1$ *and* $\|A\| < 1$.

Proof. It can be deduced from the following formula (which can be obtained by an induction over j, using (1)):

$$\forall\, j \in \mathbb{N}^*,\ x(1,j) = B^j \Psi_2(1) + \sum_{k=0}^{j-1} B^k A\, \Psi_1(j-k).$$

Indeed, as (1) is supposed to be asymptotically stable, then for any boundary conditions converging to zero, the solution must go to zero at infinity. So for $\Psi_1 = 0$ and for any $\Psi_2 \to 0$ such that $\Psi_2(1) \neq 0$ we have in particular:

$$x(1,j) = B^j \Psi_2(1) \to 0.$$

As $\Psi_2(1)$ can be chosen randomly, this implies the result (For more details about the last implication, please refer to the chapter 7 of [21] or the chapter 4 of [22]). Similarly, we can prove that $\|A\| < 1$. □

Lemma A.2. *If* (1) *is asymptotically stable and* $\widetilde{\Psi}_1 \to 0$*, then the sequence* Ψ_1*, given by* (10b)*, converges to* 0 *as well.*

Proof. We suppose that $B \neq 0$, otherwise the result would be obvious. Let $\varepsilon > 0$. By hypothesis, there is $N_\varepsilon^{(1)} \in \mathbb{N}$ such that for all $j \geq N_\varepsilon$, $|\widetilde{\Psi}_1(j)| \leq \varepsilon$. Moreover, thanks to the previous lemma A.1 we know that there is a matrix norm $\|\cdot\|$ such that $\|B\| < 1$.

For every $j \geq N_\varepsilon$ we have from (10b):

$$\begin{aligned} |\Psi_1(j)| &\leq \|B\|^j\, |\widetilde{\Psi}_2(0)| + \|\widetilde{\Psi}_1\|_\infty \sum_{k=0}^{N_\varepsilon^{(1)}-1} \|B\|^{j-1-k} \\ &\quad + \varepsilon \sum_{k=N_\varepsilon^{(1)}}^{j-1} \|B\|^{j-1-k}\, \|B\|\,, \\ &\leq \|B\|^j \underbrace{\left(|\widetilde{\Psi}_2(0)| + \|\widetilde{\Psi}_1\|_\infty \sum_{k=0}^{N_\varepsilon^{(1)}-1} \|B\|^{-(k+1)} \right)}_{:=\, C_{N_\varepsilon^{(1)}} \,<\, \infty} \\ &\quad + \varepsilon \underbrace{\left(\sum_{k \geq N_\varepsilon^{(1)}} \|B\|^{j-1-k} \right)}_{:=\, v(j)}. \end{aligned}$$

We can see that v is a sequence bounded by $(1 - \|B\|)^{-1}$. More, there is a $N_\varepsilon^{(2)} \in \mathbb{N}$ such that $\forall\, j \geq N_\varepsilon^{(2)}$, $\|B\|^j\, C_{N_\varepsilon^{(1)}} \leq \varepsilon$ (because $\|B\| < 1$). Finally, for all $j \geq \max\{N_\varepsilon^{(1)}, N_\varepsilon^{(2)}\}$ we have: $|\Psi_1(j)| \leq \varepsilon \left(1 + (1 - \|B\|)^{-1}\right)$, which suffices to conclude. □

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